

Bianchi Type-III and Kantowski-Sachs Universes with Wet Dark Fluid

Raghavendra Chaubey

Applied Mathematics, DST-Centre for Interdisciplinary Mathematical Sciences,
Banaras Hindu University, Varanasi, India

E-mail: rchaubey@bhu.ac.in

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Abstract

The Bianchi type-III and Kantowski-Sachs (KS) Universes filled with dark energy from a wet dark fluid has been considered. A new Equation of state for the dark energy component of the universe has been used. It is modeled on the Equation of state $p = \gamma(\rho - \rho_*)$ which can describe a liquid, for example water. The exact solutions to the corresponding field Equations are obtained in quadrature form. The solution for constant deceleration parameter have been studied in detail for power-law and exponential forms both. The case $\gamma = 0, \gamma = 1$ and $\gamma = \frac{1}{3}$ have been also analysed.

Keywords: Cosmological Universe, Cosmological Parameters, Wet Dark Fluid

1. Introduction

The nature of the dark energy component of the universe [1-3] remains one of the deepest mysteries of cosmology. There is certainly no lack of candidates: cosmological constant, quintessence [4-6], k-essence [7-9], phantom energy [10-12]. Modifications of the Friedmann Equation such as Cardassian expansion [13,14] as well as what might be derived from brane cosmology [15-17] have also been used to explain the acceleration of the universe. A particular case of the linear Equation of state has used in the cosmological context by Xanthopoulos [18], he considered space-times with two hypersurface orthogonal, spacelike, commuting killing fields.

In this work, we use Wet Dark Fluid (WDF) as a model for dark energy. This model is in the spirit of the generalized Chaplygin gas (GCG) [19], where a physically motivated Equation of state is offered with properties relevant for the dark energy problem. Here the motivation stems from an empirical Equation of state proposed by Tait [20] and Hayward [21] to treat water and aqueous solution. The Equation of state for WDF is very simple,

$$p_{WDF} = \gamma(\rho_{WDF} - \rho_*) \quad (1)$$

and is motivated by the fact that it is a good approximation for many fluids, including water, in which

the internal attraction of the molecules makes negative pressures possible. One of the virtues of this model is that the square of the sound speed, c_s^2 , which depends on $\partial p / \partial \rho$, can be positive (as opposed to the case of phantom energy, say), while still giving rise to cosmic acceleration in the current epoch.

We treat Equation (1) as a phenomenological Equation [22]. Holman *et al.* [23] have shown that this model can be made consistent with the most recent SNIa data [24], the WMAP results [25,26] as well as constraints coming from measurements of the matter power spectrum [27]. The parameters γ and ρ_* are taken to be positive and we restrict ourselves to $0 \leq \gamma \leq 1$. Note that if c_s denotes the adiabatic sound speed in WDF, then $\gamma = c_s^2$. (refer Babichev *et al.* [28]).

To find the WDF energy density, we use the energy conservation Equation

$$\dot{\rho}_{WDF} + 3H(p_{WDF} + \rho_{WDF}) = 0 \quad (2)$$

From Equation of state (1) and using $3H = \dot{V}/V$ in above Equation, we have

$$\rho_{WDF} = \frac{\gamma}{1+\gamma} \rho_a + \frac{C}{V^{(1+\gamma)}} \quad (3)$$

where C is a constant of integration. Here V is volume expansion.

WDF naturally includes two components: a piece that

behaves as a cosmological constant as well as a standard fluid with an Equation of state $p = \gamma\rho$. We can show that if we take $C > 0$, this fluid will not violate the strong energy condition $p + \rho \geq 0$:

$$p_{WDF} + \rho_{WDF} = (1 + \gamma)\rho_{WDF} - \gamma\rho_* = (1 + \gamma)\frac{C}{V^{(1+\gamma)}} \geq 0 \quad (4)$$

Chaubey and Chaubey *et al.* ([29,30]) have studied some anisotropic cosmological universes with wet dark fluid. In this paper we study the Bianchi type- III and Kantowski-Sachs Universes with matter term with dark energy treated as a Dark Fluid satisfying the Equation of state (1). The solution has been obtained in the quadrature form. The models with constant deceleration parameter have been studied in detail.

2. Basic Equation

2.1. Bianchi Type - III Universe

We take Bianchi type- III metric in form

$$ds^2 = dt^2 - a_1^2 dr^2 - a_2^2 [d\theta^2 + \sinh^2\theta d\phi^2] \quad (5)$$

where the metric functions a_1 and a_2 are functions of t only.

The Einstein field Equations for the metric (5) are written in the form

$$2\frac{\ddot{a}_2}{a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 - \frac{1}{a_2^2} = \kappa T_1^1. \quad (6)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} = \kappa T_2^2. \quad (7)$$

$$2\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 - \frac{1}{a_2^2} = \kappa T_0^0. \quad (8)$$

Here κ is the gravitational constant and overhead dot denotes differentiation with respect to t .

The energy-momentum tensor of the source is given by

$$T_i^j = (\rho_{WDF} + p_{WDF})u_i u^j - p_{WDF}\delta_i^j \quad (9)$$

where u^i is the flow vector satisfying

$$g_{ij}u^i u^j = 1. \quad (10)$$

In a co-moving system of coordinates, from Equation (9) we find

$$T_0^0 = \rho_{WDF}, T_1^1 = T_2^2 = -p_{WDF}. \quad (11)$$

Now using Equation (11) in Equations (6)-(8) we obtain

$$2\frac{\ddot{a}_2}{a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 - \frac{1}{a_2^2} = -\kappa p_{WDF} \quad (12)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} = -\kappa p_{WDF}. \quad (13)$$

$$2\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 - \frac{1}{a_2^2} = -\kappa p_{WDF}. \quad (14)$$

Let V be a function of t defined by

$$V = a_1 a_2^2 \quad (15)$$

Now adding three times Equation (14), two times Equation (13) in Equation (12), we get

$$\frac{\ddot{a}_1}{a_1} + 2\frac{\ddot{a}_2}{a_2} + 2\left(\frac{\dot{a}_2}{a_2} + 2\frac{\dot{a}_1\dot{a}_2}{a_1a_2}\right) - \frac{2}{a_2^2} = \frac{3\kappa}{2}(\rho_{WDF} - p_{WDF}). \quad (16)$$

From Equations (15) and (16) we have

$$\frac{\ddot{V}}{V} - \frac{2}{a_2^2} = \frac{3\kappa}{2}(\rho_{WDF} - p_{WDF}) \quad (17)$$

The conservational law for the energy-momentum tensor gives

$$\dot{\rho}_{WDF} = -\frac{\dot{V}}{V}(\rho_{WDF} + p_{WDF}) \quad (18)$$

Case 1: When $a_1 = V$

Then Equation (17) reduces to

$$\frac{\ddot{V}}{V} - 2 = \frac{3\kappa}{2}(\rho_{WDF} - p_{WDF}) \quad (19)$$

From Equations (18) and (19) we have

$$\dot{V} = \pm\sqrt{C_1 + (3\kappa\rho_{WDF} + 4)V^2} \quad (20)$$

with C_1 being an integration constant.

Rewriting (18) in the form

$$\frac{\dot{\rho}}{\rho_{WDF} + p_{WDF}} = -\frac{\dot{V}}{V}. \quad (21)$$

and taking into account that the pressure and the energy density obeying an equation of state of type

$p_{WDF} = f(\rho_{WDF})$, we conclude that ρ_{WDF} and p_{WDF} , hence the right hand side of the Equation (17) is a function of V only.

$$\dot{V} = \frac{3\kappa}{2}(\rho_{WDF} - p_{WDF})V + 2V \equiv F(V). \quad (22)$$

From the mechanical point of view Equation (22) can be interpreted as Equation of motion of a single particle with unit mass under the force $F(V)$. Then

$$\dot{V} = \sqrt{2[\mathcal{E} - U(V)]}. \quad (23)$$

Here \mathcal{E} can be viewed as energy and $U(V)$ as the potential of the force F . Comparing the Equations (20)

and (23) we find $\mathcal{E} = \frac{C_1}{2}$ and

$$U(V) = -\left[\frac{3}{2}\kappa\rho_{WDF} + 4\right]V^2. \quad (24)$$

Finally, we write the solution to the Equation (20) in quadrature form

$$\int \frac{dV}{\sqrt{C_1 + (3\kappa\rho_{WDF} + 4)V^2}} = t + t_0. \quad (25)$$

where the integration constant t_0 can be taken to be zero, since it only gives a shift in time.

From Equations (3) and (25) we obtain

$$\int \frac{dV}{\sqrt{\left(\frac{3\kappa\gamma}{1+\gamma}\rho_* + 4\right)V^2 + 3\kappa CV^{(1-\gamma)} + C_1}} = t + t_0. \quad (26)$$

Case 2: When $a_2 = \sqrt{V}$

Then Equation (17) reduces to

$$\frac{\ddot{V}}{V} - \frac{2}{V} = \frac{3\kappa}{2}(\rho_{WDF} - P_{WDF}) \quad (27)$$

After simplification, we get

$$\int \frac{dV}{\sqrt{\frac{3\kappa\gamma}{1+\gamma}\rho_* V^2 + 3\kappa CV^{(1-\gamma)} + 4V + C_1}} = t + t_0. \quad (28)$$

2.2. Kantowski-Sachs Universe

We take Kantowski-Sachs metric in form

$$ds^2 = dt^2 - a_1^2 dr^2 - a_2^2 [d\theta^2 + \sin^2\theta d\phi^2] \quad (29)$$

where the metric functions a_1 and a_2 are functions of t only.

The Einstein field Equations for the metric (29) are written in the form

$$2\frac{\ddot{a}_2}{a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 + \frac{1}{a_2^2} = \kappa T_1^1. \quad (30)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} = \kappa T_2^2. \quad (31)$$

$$2\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 + \frac{1}{a_2^2} = \kappa T_0^0 \quad (32)$$

Here κ is the gravitational constant and overhead dot denotes differentiation with respect to t .

Now using Equations (9)-(11) in Equations (30)-(32) we obtain

$$2\frac{\ddot{a}_2}{a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 + \frac{1}{a_2^2} = -\kappa P_{WDF}. \quad (33)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} = -\kappa P_{WDF}. \quad (34)$$

$$2\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 + \frac{1}{a_2^2} - \kappa P_{WDF}. \quad (35)$$

Let V be a function of t defined by

$$V = a_1 a_2^2 \quad (36)$$

Now adding three times Equation (35), two times Equation (34) in Equation (33), we get

$$\frac{\ddot{a}_1}{a_1} + 2\frac{\ddot{a}_2}{a_2} + 2\left(\frac{\dot{a}_2^2}{a_2^2} + 2\frac{\dot{a}_1\dot{a}_2}{a_1a_2}\right) + \frac{2}{a_2^2} = \frac{3\kappa}{2}(\rho_{WDF} - P_{WDF}) \quad (37)$$

From Equations (36) and (37) we have

$$\frac{\ddot{V}}{V} + \frac{2}{a_2^2} = \frac{3\kappa}{2}(\rho_{WDF} - P_{WDF}). \quad (38)$$

Case 1: When $a_1 = V$

Then Equation (38) reduces to

$$\frac{\ddot{V}}{V} + 2 = \frac{3\kappa}{2}(\rho_{WDF} - P_{WDF}). \quad (39)$$

After simplification, we get

$$\int \frac{dV}{\sqrt{\left(\frac{3\kappa\gamma}{1+\gamma}\rho_* - 4\right)V^2 + 3\kappa CV^{(1-\gamma)} + C_1}} = t + t_0. \quad (40)$$

Case 2 When $a_2 = \sqrt{V}$

Then Equation (38) reduces to

$$\frac{\ddot{V}}{V} + \frac{2}{V} = \frac{3\kappa}{2}(\rho_{WDF} - P_{WDF}) \quad (41)$$

After simplification, we get

$$\int \frac{dV}{\sqrt{\frac{3\kappa\gamma}{1+\gamma}\rho_* V^2 + 3\kappa CV^{(1-\gamma)} - 4V + C_1}} = t + t_0. \quad (42)$$

3. Some Particular Cases

3.1. Bianchi Type - III Universe

Case 1: When $a_1 = V$

Case I $\gamma = 0$ (Dust Universe)

Equation (26) reduces to

$$\int \frac{dV}{\sqrt{4V^2 + 3\kappa CV + C_1}} = t \quad (43)$$

which gives

$$V = \frac{1}{2}\sqrt{C_1 - \frac{9\kappa^2 C^2}{16}} \sinh(2t) - \frac{3\kappa C}{8},$$

when

$$C_1 > \frac{9\kappa^2 C^2}{16} \quad (44)$$

$$V = \left(e^{2t} - \frac{3\kappa C}{8}\right),$$

when

$$C_1 = \frac{9\kappa^2 C^2}{16} \tag{45}$$

$$V = \frac{1}{2} \sqrt{\frac{9\kappa^2 C^2}{16} - C_1} \cosh(2t) - \frac{3\kappa C}{8},$$

when

$$C_1 < \frac{9\kappa^2 C^2}{16} \tag{46}$$

We consider these subcases separately.

Case I (a) when $C_1 = \frac{9\kappa^2 C^2}{16}$

From Equations (15) and (45), we get

$$a_1(t) = e^{2t} - \frac{3\kappa C}{8} \tag{47}$$

$$a_2(t) = 1 \tag{48}$$

From Equation (3) and (45) we have

$$\rho_{WDF} = C \left(e^{2t} - \frac{3\kappa C}{8} \right)^{-1} \tag{49}$$

and from Equation (1) and (49) we get

$$p_{WDF} = 0 \tag{50}$$

The physical quantities of observational interest in cosmology are the expansion scalar θ , the mean anisotropy parameter A , the shear scalar σ^2 and the deceleration parameter q . They are defined as [31,32],

$$\theta = 3H. \tag{51}$$

$$A = \frac{1}{3} \sum_{i=1}^3 \left(\frac{\Delta H_i}{H} \right)^2. \tag{52}$$

$$\sigma^2 = \frac{1}{2} \left(\sum_{i=1}^3 H_i^2 - 3H^2 \right) = \frac{3}{2} AH^2. \tag{53}$$

$$q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1. \tag{54}$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{2e^{2t}}{\left(e^{2t} - \frac{3\kappa C}{8} \right)} \tag{55}$$

$$A = 2 \tag{56}$$

$$\sigma^2 = \frac{4e^{4t}}{3 \left(e^{2t} - \frac{3\kappa C}{8} \right)^2} \tag{57}$$

$$q = \frac{9\kappa C}{8} e^{-2t} - 1 \tag{58}$$

For large t , the shear dies out.

Case I (b) when $C_1 > \frac{9\kappa^2 C^2}{16}$

Then for small t (i.e. near singularity $t = 0$),

$$\sinh(2t) \approx 2t \tag{59}$$

Then Equation (44) reduces to

$$V = \sqrt{C_1 - \frac{9\kappa^2 C^2}{16} t} - \frac{3\kappa C}{8} \tag{60}$$

From Equations (15) and (60), we get

$$a_1(t) = \sqrt{C_1 - \frac{9\kappa^2 C^2}{16} t} - \frac{3\kappa C}{8} \tag{61}$$

$$a_2(t) = 1 \tag{62}$$

From Equations (3) and (60), we have

$$\rho_{WDF} = C \left[\sqrt{C_1 - \frac{9\kappa^2 C^2}{16} t} - \frac{3\kappa C}{8} \right]^{-1} \tag{63}$$

and from Equations (1) and (63) we get

$$p_{WDF} = 0 \tag{64}$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{\sqrt{C_1 - \frac{9\kappa^2 C^2}{16}}}{\sqrt{C_1 - \frac{9\kappa^2 C^2}{16} t} - \frac{3\kappa C}{8}} \tag{65}$$

$$A = 2 \tag{66}$$

$$\sigma^2 = \frac{C_1 - \frac{9\kappa^2 C^2}{16}}{3 \left[\sqrt{C_1 - \frac{9\kappa^2 C^2}{16} t} - \frac{3\kappa C}{8} \right]^2} \tag{67}$$

$$q = 2 \tag{68}$$

For large t , the shear dies out.

Case I (c) when $C_1 < \frac{9\kappa^2 C^2}{16}$

Then for small t (i.e. near singularity $t = 0$),

$$\cosh(2t) \approx 1 + 4t^2 \tag{69}$$

Then Equation (46) reduces to

$$V = 2 \left(\sqrt{\frac{9\kappa^2 C^2}{16} - C_1} \right) t^2 + \frac{1}{2} \sqrt{\frac{9\kappa^2 C^2}{16} - C_1} - \frac{3\kappa C}{8} \tag{70}$$

From Equations (15) and (70), we get

$$a_1(t) = 2 \left(\sqrt{\frac{9\kappa^2 C^2}{16} - C_1} \right) t^2 + \frac{1}{2} \sqrt{\frac{9\kappa^2 C^2}{16} - C_1} - \frac{3\kappa C}{8} \tag{71}$$

$$a_2(t) = 1 \tag{72}$$

From Equations (3) and (70) we have

$$\rho_{WDF} = C \left[2 \left(\sqrt{\frac{9\kappa^2 C^2}{16} - C_1} \right) t^2 + \frac{1}{2} \sqrt{\frac{9\kappa^2 C^2}{16} - C_1} - \frac{3\kappa C}{8} \right]^{-1} \tag{73}$$

and from Equations (1) and (73) we get

$$p_{WDF} = 0 \tag{74}$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{4 \sqrt{\frac{9\kappa^2 C^2}{16} - C_1} t}{2 \left(\sqrt{\frac{9\kappa^2 C^2}{16} - C_1} \right) t^2 + \frac{1}{2} \sqrt{\frac{9\kappa^2 C^2}{16} - C_1} - \frac{3\kappa C}{8}} \tag{75}$$

$$A = 2 \tag{76}$$

$$\sigma^2 = \frac{\left(\frac{9\kappa^2 C^2}{4} - 4C_1 \right) t^2}{3 \left[2 \left(\sqrt{\frac{9\kappa^2 C^2}{16} - C_1} \right) t^2 + \frac{1}{2} \sqrt{\frac{9\kappa^2 C^2}{16} - C_1} - \frac{3\kappa C}{8} \right]^2} \tag{77}$$

$$q = \frac{\frac{1}{2} - \frac{3}{2} \sqrt{\frac{9\kappa^2 C^2}{16} - C_1} - \frac{9\kappa C}{8}}{4 \left(\sqrt{\frac{9\kappa^2 C^2}{16} - C_1} \right) t^2} \tag{78}$$

For large t , the shear dies out.

Case II $\gamma = 1$ (Zeldovich Fluid)

Equation (26) reduces to

$$\int \frac{dV}{\sqrt{\left(\frac{3\kappa}{2} \rho_* + 4 \right) V^2 + 3\kappa C + C_1}} = t \tag{79}$$

which gives

$$V = \frac{\sqrt{3\kappa C + C_1} \sinh \left(\sqrt{\frac{3\kappa}{2} \rho_* + 4} \right) t}{\sqrt{\frac{3\kappa}{2} \rho_* + 4}} \tag{80}$$

Then for small t (i.e. near singularity $t = 0$),

$$\sinh \left(\sqrt{\frac{3\kappa}{2} \rho_* + 4} \right) t \approx \left(\sqrt{\frac{3\kappa}{2} \rho_* + 4} \right) t \tag{81}$$

Then Equation (80) reduces to

$$V = \left(\sqrt{3\kappa C + C_1} \right) t \tag{82}$$

From Equations (15) and (82), we get

$$a_1(t) = \left(\sqrt{3\kappa C + C_1} \right) t \tag{83}$$

$$a_2(t) = 1 \tag{84}$$

From Equations (3) and (82) we have

$$\rho_{WDF} = C \left[\left(\sqrt{3\kappa C + C_1} \right) t \right]^{-1} \tag{85}$$

and from Equations (1) and (85) we get

$$p_{WDF} = 0 \tag{86}$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{1}{t} \tag{87}$$

$$A = 2 \tag{88}$$

$$\sigma^2 = \frac{1}{3t^2} \tag{89}$$

$$q = 2 \tag{90}$$

For large cosmic time, the shear dies out and ρ , $p \rightarrow 0$ and the model reduces to vacuum.

Case III $\gamma = \frac{1}{3}$ (Radiation)

For $C_1 = 0$, Equation (26) reduces to

$$\int \frac{dV}{\sqrt{\left(\frac{3\kappa}{4} \rho_* + 4 \right) V^2 + 3\kappa C V^{2/3}}} = t \tag{91}$$

which gives

$$V = \left[\frac{\sqrt{12\kappa C} \sinh \left(\frac{\sqrt{3\kappa \rho_* + 16}}{3} \right) t}{\sqrt{3\kappa \rho_* + 16}} \right]^{3/2} \tag{92}$$

Then for small t (i.e. near singularity $t = 0$),

$$\sinh \left(\frac{\sqrt{3\kappa \rho_* + 16}}{3} \right) t \approx \frac{\sqrt{3\kappa \rho_* + 16}}{3} t \tag{93}$$

Then Equation (92) reduces to

$$V = \left[\frac{2\sqrt{3\kappa C}}{3} t \right]^{3/2} \tag{94}$$

From Equations (15) and (94), we get

$$a_1(t) = \left[\frac{2\sqrt{3\kappa C}}{3} t \right]^{3/2} \tag{95}$$

$$a_2(t) = 1 \tag{96}$$

From Equations (3) and (96) we have

$$\rho_{WDF} = C \left[\frac{2\sqrt{3\kappa C}}{3} t \right]^{-3/2} \tag{97}$$

and from Equations (1) and (97) we get

$$p_{WDF} = 0 \tag{98}$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{3}{2t} \quad (99)$$

$$A = 2 \quad (100)$$

$$\sigma^2 = \frac{3}{4t^2} \quad (101)$$

$$q = 1 \quad (102)$$

For large cosmic time, the shear dies out and ρ , $p \rightarrow 0$ and the model reduces to vacuum.

Case 2: When $a_2 = \sqrt{V}$

Case I: $\gamma = 0$ (Dust Universe)

Equation (28) reduces to

$$\int \frac{dV}{\sqrt{\left(\frac{3}{2}\kappa C + 4\right)V + C_1}} = t \quad (103)$$

which gives

$$V = \frac{\left(\frac{3\kappa C}{4} + 2\right)^2 t^2 - C_1}{\left(\frac{3\kappa C}{2} + 4\right)} \quad (104)$$

From Equations (15) and (104), we get

$$a_1(t) = 1 \quad (105)$$

$$a_2(t) = \left[\frac{\left(\frac{3\kappa C}{4} + 2\right)^2 t^2 - C_1}{\left(\frac{3\kappa C}{2} + 4\right)} \right]^{1/2} \quad (106)$$

From Equations (3) and (104) we have

$$\rho_{WDF} = C \left[\frac{\left(\frac{3\kappa C}{4} + 2\right)^2 t^2 - C_1}{\left(\frac{3\kappa C}{2} + 4\right)} \right]^{-1} \quad (107)$$

and from Equations (1) and (107) we get

$$p_{WDF} = 0 \quad (108)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{2\left(\frac{3\kappa C}{4} + 2\right)^2 t}{\left(\frac{3\kappa C}{4} + 2\right)^2 t^2 - C_1} \quad (109)$$

$$A = \frac{1}{2} \quad (110)$$

$$\sigma^2 = \frac{1}{3} \left[\frac{\left(\frac{3\kappa C}{4} + 2\right)^2 t}{\left(\frac{3\kappa C}{4} + 2\right)^2 t^2 - C_1} \right]^2 \quad (111)$$

$$q = \frac{1}{2} + \frac{C_1}{\left(\frac{3\kappa C}{4} + 2\right)^2 t^2} \quad (112)$$

For large cosmic time, the shear dies out.

Case II $\gamma = 1$ (Zeldovich Fluid)

Equation (28) reduces to

$$\int \frac{dV}{\sqrt{\frac{3}{4}\rho_* V^2 + 4V + \left(\frac{3}{2}\kappa C + C_1\right)}} = t \quad (113)$$

which gives

$$V = \frac{\sqrt{6\rho_*(3\kappa C + 2C_1) - 64}}{3\rho_*} \sinh\left(\frac{\sqrt{3\rho_*}}{2}t\right) - \frac{8}{3\rho_*},$$

when

$$\rho_* > \frac{32}{3(3\kappa C + 2C_1)} \quad (114)$$

$$V = \left(\frac{2}{\sqrt{3\rho_*}} e^{\frac{\sqrt{3\rho_*}}{2}t} - \frac{8}{3\rho_*} \right),$$

when

$$\rho_* = \frac{32}{3(3\kappa C + 2C_1)} \quad (115)$$

$$V = \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)}}{3\rho_*} \cosh\left(\frac{\sqrt{3\rho_*}}{2}t\right) - \frac{8}{3\rho_*},$$

when

$$\rho_* < \frac{32}{3(3\kappa C + 2C_1)} \quad (116)$$

We consider these subcases separately.

$$\text{Case II (a) } \rho_* = \frac{32}{3(3\kappa C + 2C_1)}$$

Then

$$a_1(t) = 1 \quad (117)$$

$$a_2(t) = \left(\frac{2}{\sqrt{3\rho_*}} e^{\frac{\sqrt{3\rho_*}}{2}t} - \frac{8}{3\rho_*} \right)^{1/2} \quad (118)$$

From Equations (3) and (115), we have

$$\rho_{WDF} = \frac{\rho_*}{2} + C \left(\frac{2}{\sqrt{3\rho_*}} e^{\frac{\sqrt{3\rho_*}}{2}t} - \frac{8}{3\rho_*} \right)^{-2} \quad (119)$$

and from Equations (1) and (119), we get

$$P_{WDF} = -\frac{\rho_*}{2} + C \left(\frac{2}{\sqrt{3\rho_*}} e^{\frac{\sqrt{3\rho_*}t}{2}} - \frac{8}{3\rho_*} \right)^{-2} \quad (120)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{\frac{\sqrt{3\rho_*}}{2} e^{\frac{\sqrt{3\rho_*}t}{2}}}{\frac{e^{\frac{\sqrt{3\rho_*}t}{2}}}{4} - \sqrt{3\rho_*}} \quad (121)$$

$$A = \frac{1}{2} \quad (122)$$

$$\sigma^2 = \frac{1}{12} \left[\frac{\frac{\sqrt{3\rho_*}}{2} e^{\frac{\sqrt{3\rho_*}t}{2}}}{\frac{e^{\frac{\sqrt{3\rho_*}t}{2}}}{4} - \sqrt{3\rho_*}} \right]^2 \quad (123)$$

$$q = \frac{4\sqrt{3}}{\sqrt{\rho_*}} e^{\frac{\sqrt{3\rho_*}t}{2}} - 1 \quad (124)$$

The model has no singularity.

Case II (b) $\rho_* > \frac{32}{3(3\kappa C + 2C_1)}$

Then for small t (i.e. near singularity $t = 0$),

$$\sinh\left(\frac{\sqrt{3\rho_*}}{2}t\right) \approx \frac{\sqrt{3\rho_*}}{2}t \quad (125)$$

Then Equation (114) reduces to

$$V = \sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}t - \frac{8}{3\rho_*}} \quad (126)$$

Then

$$a_1(t) = 1 \quad (127)$$

$$a_2(t) = \left[\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}t - \frac{8}{3\rho_*}} \right]^{1/2} \quad (128)$$

From Equations (3) and (126), we have

$$\rho_{WDF} = \frac{\rho_*}{2} + C \left[\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}t - \frac{8}{3\rho_*}} \right]^{-2} \quad (129)$$

and from Equations (1) and (129), we get

$$P_{WDF} = -\frac{\rho_*}{2} + C \left[\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}t - \frac{8}{3\rho_*}} \right]^{-2} \quad (130)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}t - \frac{8}{3\rho_*}}}{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}t - \frac{8}{3\rho_*}} \quad (131)$$

$$A = \frac{1}{2} \quad (132)$$

$$\sigma^2 = \frac{1}{12} \left[\frac{\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}t - \frac{8}{3\rho_*}}}{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}t - \frac{8}{3\rho_*}} \right]^2 \quad (133)$$

$$q = 2 \quad (134)$$

The model has no singularity.

Case II (c) $\rho_* < \frac{32}{3(3\kappa C + 2C_1)}$

Then for small t (i.e. near singularity $t = 0$),

$$\cosh\left(\frac{\sqrt{3\rho_*}}{2}t\right) \approx 1 + \frac{3\rho_*}{4}t^2 \quad (135)$$

Then Equation (116) reduces to

$$V = \sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)} - 8}{3\rho_*}} \quad (136)$$

Then $a_1(t) = 1$ (137)

$$a_2(t) = \left[\left[\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)} - 8}{3\rho_*}} \right]^{1/2} \right] \quad (138)$$

From Equations (3) and (136), we have

$$\rho_{WDF} = \frac{\rho_*}{2} + C \left[\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)} - 8}{3\rho_*}} \right]^{-2} \quad (139)$$

and from Equations (1) and (139), we get

$$P_{WDF} = -\frac{\rho_*}{2} + C \left[\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)} - 8}{3\rho_*}} \right]^{-2} \quad (140)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{\sqrt{16 - \frac{3}{2}\rho_*(3\kappa C + 2C_1)}t}{\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)} - 8}{3\rho_*}}} \quad (141)$$

$$A = \frac{1}{2} \quad (142)$$

$$\sigma^2 = \frac{1}{12} \left[\frac{\sqrt{16 - \frac{3}{2}\rho_*(3\kappa C + 2C_1)}t}{\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)} - 8}{3\rho_*}}} \right]^2 \quad (143)$$

$$q = \frac{1}{2} \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)} - 8}{\sqrt{16 - \frac{3}{2}\rho_*(3\kappa C + 2C_1)}t^2} \quad (144)$$

The model has no singularity.

3.2. Kantowski-Sachs Universe

Case 1: When $a_1 = V$

Case I $\gamma = 0$ (Dust Universe)

Equation (2.40) reduces to

$$\int \frac{dV}{\sqrt{-4V^2 + 3\kappa CV + C_1}} = t \quad (145)$$

which gives

$$V = \frac{1}{2} \sqrt{C_1 + \frac{9\kappa^2 C^2}{16} \sin^2(2t) + \frac{3\kappa C}{8}} \quad (146)$$

From Equations (36) and (146), we get

$$a_1(t) = \frac{1}{2} \sqrt{C_1 + \frac{9\kappa^2 C^2}{16} \sin^2(2t) + \frac{3\kappa C}{8}} \quad (147)$$

$$a_2(t) = 1 \quad (148)$$

From Equations (3) and (146) we have

$$\rho_{WDF} = C \left[\frac{1}{2} \sqrt{C_1 + \frac{9\kappa^2 C^2}{16} \sin^2(2t) + \frac{3\kappa C}{8}} \right]^{-1} \quad (149)$$

and from Equations (1) and (149) we get

$$p_{WDF} = 0 \quad (150)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{\sqrt{4C_1 + \frac{9\kappa^2 C^2}{4}} \cos(2t)}{\sqrt{C_1 + \frac{9\kappa^2 C^2}{16} \sin^2(2t) + \frac{3\kappa C}{8}}} \quad (151)$$

$$A = 2 \quad (152)$$

$$\sigma^2 = \frac{\left[4C_1 + \frac{9\kappa^2 C^2}{4} \right] \cos^2(2t)}{3 \left[\sqrt{C_1 + \frac{9\kappa^2 C^2}{16} \sin^2(2t) + \frac{3\kappa C}{8}} \right]^2} \quad (153)$$

$$q = 3 \sec^2(2t) + \frac{9\kappa C}{4 \sqrt{C_1 + \frac{9\kappa^2 C^2}{16}}} \sec(2t) \tan(2t) - 1 \quad (154)$$

Case II $\gamma = 1$ (Zeldovich Fluid)

Equation (40) reduces to

$$\int \frac{dV}{\sqrt{\left(\frac{3\kappa}{2}\rho_* - 4\right)V^2 + 3\kappa C + C_1}} = t \quad (155)$$

which gives

$$V = \sqrt{\frac{3\kappa C + C_1}{\frac{3\kappa}{2}\rho_* - 4}} \sinh\left(\sqrt{\frac{3\kappa}{2}\rho_* - 4}\right)t \quad (156)$$

Then for small t (i.e. near singularity $t = 0$),

$$\sinh\left(\sqrt{\frac{3\kappa}{2}\rho_* - 4}\right)t \approx \left(\sqrt{\frac{3\kappa}{2}\rho_* - 4}\right)t \quad (157)$$

Then Equation (156) reduces to

$$V = \left(\sqrt{3\kappa C + C_1}\right)t \quad (158)$$

From Equations (36) and (158), we get

$$a_1(t) = \left(\sqrt{3\kappa C + C_1}\right)t \quad (159)$$

$$a_2(t) = 1 \quad (160)$$

From Equations (3) and (158) we have

$$\rho_{WDF} = C \left[\left(\sqrt{3\kappa C + C_1}\right)t \right]^{-1} \quad (161)$$

and from Equations (1) and (161) we get

$$p_{WDF} = 0 \quad (162)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{1}{t} \quad (163)$$

$$A = 2 \quad (164)$$

$$\sigma^2 = \frac{1}{3t^2} \quad (165)$$

$$q = 2 \quad (166)$$

For large cosmic time, the shear dies out and ρ , $p \rightarrow 0$ and the model reduces to vacuum.

Case III $\gamma = \frac{1}{3}$ (Radiation)

For $C_1 = 0$, Equation (40) reduces to

$$\int \frac{dV}{\sqrt{\left(\frac{3\kappa}{4}\rho_* - 4\right)V^2 + 3\kappa CV^{2/3}}} = t \quad (167)$$

which gives

$$V = \left[\sqrt{\frac{12\kappa C}{3\kappa\rho_* - 16}} \sinh\left(\frac{\sqrt{3\kappa\rho_* - 16}}{3}t\right) \right]^{3/2} \quad (168)$$

Then for small t (i.e. near singularity $t = 0$),

$$\sinh\left(\frac{\sqrt{3\kappa\rho_* - 16}}{3}t\right) \approx \frac{\sqrt{3\kappa\rho_* - 16}}{3}t \quad (169)$$

Then Equation (168) reduces to

$$V = \left[\frac{2\sqrt{3\kappa C}}{3}t \right]^{3/2} \quad (170)$$

From Equations (36) and (170), we get

$$a_1(t) = \left[\frac{2\sqrt{3\kappa C}}{3}t \right]^{3/2} \quad (171)$$

$$a_2(t) = 1 \quad (172)$$

From Equations (3) and (170) we have

$$\rho_{WDF} = C \left[\frac{2\sqrt{3\kappa C}}{3}t \right]^{-3/2} \quad (173)$$

and from Equations (1) and (173) we get

$$p_{WDF} = 0 \quad (174)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{3}{2t} \quad (175)$$

$$A = 2 \quad (176)$$

$$\sigma^2 = \frac{3}{4t^2} \quad (177)$$

$$q = 1 \quad (178)$$

For large cosmic time, the shear dies out and ρ , $p \rightarrow 0$ and the model reduces to vacuum.

Case 2: When $a_2 = \sqrt{V}$

Case I $\gamma = 0$ (Dust Universe)

Equation (42) reduces to

$$\int \frac{dV}{\sqrt{\left(\frac{3}{2}\kappa C - 4\right)V + C_1}} = t \quad (179)$$

which gives

$$V = \frac{\left(\frac{3\kappa C}{4} - 2\right)^2 t^2 - C_1}{\left(\frac{3\kappa C}{2} - 4\right)} \quad (180)$$

From Equations (36) and (180), we get

$$a_1(t) = 1 \quad (181)$$

$$a_2(t) = \left[\frac{\left(\frac{3\kappa C}{4} - 2\right)^2 t^2 - C_1}{\left(\frac{3\kappa C}{2} - 4\right)} \right]^{1/2} \quad (182)$$

From Equations (3) and (180) we have

$$\rho_{WDF} = C \left[\frac{\left(\frac{3\kappa C}{4} - 2\right)^2 t^2 - C_1}{\left(\frac{3\kappa C}{2} - 4\right)} \right]^{-1} \quad (183)$$

and from Equations (1) and (183) we get

$$p_{WDF} = 0 \quad (184)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{2\left(\frac{3\kappa C}{4} - 2\right)^2 t}{\left(\frac{3\kappa C}{4} - 2\right)^2 t^2 - C_1} \quad (185)$$

$$A = \frac{1}{2} \quad (186)$$

$$\sigma^2 = \frac{1}{3} \left[\frac{\left(\frac{3\kappa C}{4} - 2\right)^2 t}{\left(\frac{3\kappa C}{4} - 2\right)^2 t^2 - C_1} \right]^2 \quad (187)$$

$$q = \frac{1}{2} + \frac{C_1}{\left(\frac{3\kappa C}{4} - 2\right)^2 t^2} \quad (188)$$

For large cosmic time, the shear dies out.

Case II $\gamma = 1$ (Zeldovich Fluid)

Equation (42) reduces to

$$\int \frac{dV}{\sqrt{\frac{3}{4}\rho_* V^2 - 4V + \left(\frac{3}{2}\kappa C + C_1\right)}} = t \quad (189)$$

which gives

$$V = \frac{\sqrt{6\rho_*(3\kappa C + 2C_1) - 64}}{3\rho_*} \sinh\left(\frac{\sqrt{3}\rho_* t}{2}\right) + \frac{8}{3\rho_*},$$

when

$$\rho_* > \frac{32}{3(3\kappa C + 2C_1)} \quad (190)$$

$$V = \left(\frac{2}{\sqrt{3\rho_*}} e^{\frac{\sqrt{3}\rho_* t}{2}} + \frac{8}{3\rho_*} \right),$$

when

$$\rho_* = \frac{32}{3(3\kappa C + 2C_1)} \quad (191)$$

$$V = \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1)}}{3\rho_*} \cosh\left(\frac{\sqrt{3}\rho_* t}{2}\right) + \frac{8}{3\rho_*},$$

when

$$\rho_* < \frac{32}{3(3\kappa C + 2C_1)} \quad (192)$$

We consider these subcases separately.

$$\text{Case II (a) } \rho_* = \frac{32}{3(3\kappa C + 2C_1)}$$

Then

$$a_1(t) = 1 \quad (193)$$

$$a_2(t) = \left(\frac{2}{\sqrt{3\rho_*}} e^{\frac{\sqrt{3}\rho_* t}{2}} + \frac{8}{3\rho_*} \right)^{1/2} \quad (194)$$

From Equations (3) and (191), we have

$$\rho_{WDF} = \frac{\rho_*}{2} + C \left(\frac{2}{\sqrt{3\rho_*}} e^{\frac{\sqrt{3}\rho_* t}{2}} + \frac{8}{3\rho_*} \right)^{-2} \quad (195)$$

and from Equations (1) and (195), we get

$$p_{WDF} = -\frac{\rho_*}{2} + C \left(\frac{2}{\sqrt{3\rho_*}} e^{\frac{\sqrt{3}\rho_* t}{2}} + \frac{8}{3\rho_*} \right)^{-2} \quad (196)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{\frac{\sqrt{3\rho_*}}{2} e^{\frac{\sqrt{3}\rho_* t}{2}}}{e^{\frac{\sqrt{3}\rho_* t}{2}} + \frac{4}{\sqrt{3\rho_*}}} \quad (197)$$

$$A = \frac{1}{2} \quad (198)$$

$$\sigma^2 = \frac{1}{12} \left[\frac{\frac{\sqrt{3\rho_*}}{2} e^{\frac{\sqrt{3}\rho_* t}{2}}}{e^{\frac{\sqrt{3}\rho_* t}{2}} + \frac{4}{\sqrt{3\rho_*}}} \right]^2 \quad (199)$$

$$q = \frac{4\sqrt{3}}{\sqrt{\rho_*}} e^{-\frac{\sqrt{3}\rho_* t}{2}} - 1 \quad (200)$$

The model has no singularity.

$$\text{Case II (b) } \rho_* > \frac{32}{3(3\kappa C + 2C_1)}$$

Then for small t (i.e. near singularity $t = 0$),

$$\sinh\left(\frac{\sqrt{3}\rho_* t}{2}\right) \approx \frac{\sqrt{3}\rho_* t}{2} \quad (201)$$

Then Equation (190) reduces to

$$V = \sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*} t} + \frac{8}{3\rho_*} \quad (202)$$

Then

$$a_1(t) = 1 \quad (203)$$

$$a_2(t) = \left[\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*} t} + \frac{8}{3\rho_*} \right]^{1/2} \quad (204)$$

From Equations (3) and (202), we have

$$\rho_{WDF} = \frac{\rho_*}{2} + C \left[\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*} t} + \frac{8}{3\rho_*} \right]^{-2} \quad (205)$$

and from Equations (1) and (205), we get

$$p_{WDF} = -\frac{\rho_*}{2} + C \left[\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*} t} + \frac{8}{3\rho_*} \right]^{-2} \quad (206)$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}}}{\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*} t} + \frac{8}{3\rho_*}} \quad (207)$$

$$A = \frac{1}{2} \quad (208)$$

$$\sigma^2 = \frac{1}{12} \left[\frac{\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*}}}{\sqrt{\frac{(3\kappa C + 2C_1)}{2} - \frac{16}{3\rho_*} t} + \frac{8}{3\rho_*}} \right]^2 \quad (209)$$

$$q = 2 \quad (210)$$

The model has no singularity.

Case II (c) $\rho_* < \frac{32}{3(3\kappa C + 2C_1)}$

Then for small t (i.e. near singularity $t = 0$),

$$\cosh\left(\frac{\sqrt{3\rho_*}t}{2}\right) \approx 1 + \frac{3\rho_*}{4}t^2 \tag{211}$$

Then Equation (192) reduces to

$$V = \sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1) + 8}}{3\rho_*}} \tag{212}$$

Then

$$a_1(t) = 1 \tag{213}$$

$$a_2(t) = \left[\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1) + 8}}{3\rho_*}} \right]^{1/2} \tag{214}$$

From Equations (3) and (212), we have

$$\rho_{WDF} = \frac{\rho_*}{2} + C \left[\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1) + 8}}{3\rho_*}} \right]^{-2} \tag{215}$$

and from Equations (1) and (215), we get

$$p_{WDF} = -\frac{\rho_*}{2} + C \left[\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1) + 8}}{3\rho_*}} \right]^{-2} \tag{216}$$

With the use of Equations (51) - (54) we can express the physical quantities as

$$\theta = \frac{\sqrt{16 - \frac{3}{2}\rho_*(3\kappa C + 2C_1)t}}{\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1) + 8}}{3\rho_*}}} \tag{217}$$

$$A = \frac{1}{2} \tag{218}$$

$$\sigma^2 = \frac{1}{12} \left[\frac{\sqrt{16 - \frac{3}{2}\rho_*(3\kappa C + 2C_1)t}}{\sqrt{4 - \frac{3}{8}\rho_*(3\kappa C + 2C_1)t^2 + \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1) + 8}}{3\rho_*}}} \right]^2 \tag{219}$$

$$q = \frac{1}{2} - \frac{\sqrt{64 - 6\rho_*(3\kappa C + 2C_1) + 8}}{\sqrt{16 - \frac{3}{2}\rho_*(3\kappa C + 2C_1)t^2}} \tag{220}$$

The model has no singularity.

4. Models with Constant Deceleration Parameter

Case I Power-Law

Here we take

$$V = at^b, \tag{221}$$

where a and b are constants,

Here we discuss three interesting cases

Case I (a) When $a_1 = \sqrt{V}$

From (221), we get

$$a_1(t) = a^{1/2}t^{b/2} \tag{222}$$

$$a_2(t) = a^{1/4}t^{b/4} \tag{223}$$

From (3) and (221), we have

$$\rho_{WDF} = \frac{\gamma}{1+\gamma}\rho_* + \frac{C}{a^{(1+\gamma)}}t^{-(1+\gamma)b} \tag{224}$$

and from (1) and (224), we get

$$p_{WDF} = \gamma \left(\frac{C}{a^{(1+\gamma)}}t^{-(1+\gamma)b} - \frac{1}{1+\gamma}\rho_* \right) \tag{225}$$

With the use of Equations (51) - (54) we can express the physical quantities as

$$\theta = \frac{b}{t} \tag{226}$$

$$A = \frac{1}{8} \tag{227}$$

$$\sigma^2 = \frac{1}{48} \frac{b^2}{t^2} \tag{228}$$

$$q = \frac{3}{b} - 1 \tag{229}$$

Case I (b) When $a_1 = V$

From (221), we get

$$a_1(t) = at^b \tag{230}$$

$$a_2(t) = 1 \tag{231}$$

From (3) and (221), we have

$$\rho_{WDF} = \frac{\gamma}{1+\gamma} \rho_* + \frac{C}{a^{(1+\gamma)}} t^{-(1+\gamma)b} \tag{232}$$

and from (1) and (232), we get

$$p_{WDF} = \gamma \left(\frac{C}{a^{(1+\gamma)}} t^{-(1+\gamma)b} - \frac{1}{1+\gamma} \rho_* \right) \tag{233}$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \frac{b}{t} \tag{234}$$

$$A = 2 \tag{235}$$

$$\sigma^2 = \frac{b^2}{3t^2} \tag{236}$$

$$q = \frac{3}{b} - 1 \tag{237}$$

Case I (c) When $a_1 = V^2$

From (221), we get

$$a_1(t) = a^2 t^{2b} \tag{238}$$

$$a_2(t) = a^{-1/2} t^{-b/2} \tag{239}$$

From (3) and (221), we have

$$\rho_{WDF} = \frac{\gamma}{1+\gamma} \rho_* + \frac{C}{a^{(1+\gamma)}} t^{-(1+\gamma)b} \tag{240}$$

and from (1) and (240), we get

$$p_{WDF} = \gamma \left(\frac{C}{a^{(1+\gamma)}} t^{-(1+\gamma)b} - \frac{1}{1+\gamma} \rho_* \right) \tag{241}$$

With the use of Equations (51) - (54) we can express the physical quantities as

$$\theta = \frac{b}{t} \tag{242}$$

$$A = \frac{25}{2} \tag{243}$$

$$\sigma^2 = \frac{25}{12} \frac{b^2}{t^2} \tag{244}$$

$$q = \frac{3}{b} - 1 \tag{245}$$

For large t , the shear dies out and model has no singularity.

Case II Exponential-Type

Here we take

$$V = \alpha e^{\beta t}, \tag{246}$$

where α and β are constants.

Here we discuss three interesting cases

Case II (a) When $a_1 = \sqrt{V}$

From (246), we get

$$a_1(t) = \alpha^{1/2} e^{\frac{\beta t}{2}} \tag{247}$$

$$a_2(t) = \alpha^{1/4} e^{\frac{\beta t}{4}} \tag{248}$$

From (3) and (246), we have

$$\rho_{WDF} = \frac{\gamma}{1+\gamma} \rho_* + \frac{C}{\alpha^{(1+\gamma)}} e^{-(1+\gamma)\beta t} \tag{249}$$

and from (1) and (249), we get

$$p_{WDF} = \gamma \left(\frac{C}{\alpha^{(1+\gamma)}} e^{-(1+\gamma)\beta t} - \frac{1}{1+\gamma} \rho_* \right) \tag{250}$$

With the use of Equations (51)-(54) we can express the physical quantities as

$$\theta = \beta \tag{251}$$

$$A = \frac{1}{8} \tag{252}$$

$$\sigma^2 = \frac{1}{48} \beta^2 \tag{253}$$

$$q = -1 \tag{254}$$

Case II (b) When $a_1 = V$

From (246), we get

$$a_1(t) = \alpha e^{\beta t} \tag{255}$$

$$a_2(t) = 1 \tag{256}$$

From (3) and (246), we have

$$\rho_{WDF} = \frac{\gamma}{1+\gamma} \rho_* + \frac{C}{\alpha^{(1+\gamma)}} e^{-(1+\gamma)\beta t} \tag{257}$$

and from (1) and (257), we get

$$p_{WDF} = \gamma \left(\frac{C}{\alpha^{(1+\gamma)}} e^{-(1+\gamma)\beta t} - \frac{1}{1+\gamma} \rho_* \right) \tag{258}$$

With the use of Equations (51) - (54) we can express the physical quantities as

$$\theta = \beta \tag{259}$$

$$A = 2 \tag{260}$$

$$\sigma^2 = \frac{\beta^2}{3} \quad (261)$$

$$q = -1 \quad (262)$$

Case II (c) When $a_1 = V^2$

From (246), we get

$$a_1(t) = \alpha^2 e^{2\beta t} \quad (263)$$

$$a_2(t) = \alpha^{-1/2} e^{\frac{-\beta t}{2}} \quad (264)$$

From (3) and (246), we have

$$\rho_{WDF} = \frac{\gamma}{1+\gamma} \rho_* + \frac{C}{\alpha^{(1+\gamma)}} e^{-(1+\gamma)\beta t} \quad (265)$$

and from (1) and (265), we get

$$p_{WDF} = \gamma \left(\frac{C}{\alpha^{(1+\gamma)}} e^{-(1+\gamma)\beta t} - \frac{1}{1+\gamma} \rho_* \right) \quad (266)$$

With the use of Equations (51) - (54) we can express the physical quantities as

$$\theta = \beta \quad (267)$$

$$A = \frac{25}{2} \quad (268)$$

$$\sigma^2 = \frac{25}{12} \beta^2 \quad (269)$$

$$q = -1 \quad (270)$$

The model has no singularity.

5. Conclusions

The Bianchi type-III and Kantowski-Sachs (KS) universes have been considered for a new Equation of state for the Dark Energy component of the universe (known as dark wet fluid). The solution has been obtained in quadrature form. The models with constant deceleration parameter have been discussed in detail. The behaviour of the models for large time have been analyzed.

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