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On the Automorphism Group of Distinct Weight Codes

Abdelfattah Haily, Driss Harzalla

Department of Mathematics, Faculty of Sciences, University Chouaib Doukkali, El Jadida, Morocco
Email: afhaily@yahoo.fr, drissHarzalla@yahoo.ca

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Abstract

In this work, we study binary linear distinct weight codes (DW-code). We give a complete classification of $[n,k]_2$-DW-codes and enumerate their equivalence classes in terms of the number of solutions of specific Diophantine Equations. We use the Q-extension program to provide examples.

Keywords

Distinct (Constante) Weigth Code, Automorphism Group, Extension Theorem of MacWilliams, Diophantine Equations

1. Preliminaries

One of the main objective of algebraic coding theory is to classify codes up to equivalence by using a list of invariants. The present work is following this way. We study here a class of linear binary codes whose all codewords have distinct weight and will give a classification theorems. Throughout this work all codes are linear binary codes. We call an $[n,k]_2$-binary code every k dimensional subspace $\mathcal{C}$ of $\mathbb{F}_2^n$. Recall also that the Hamming weight $wt(x)$ of vector x is defined to be the number of nonzero components of x. The minimum of weights where $x \neq 0$ is the minimal distance d of the code.

A Hamming isometry of $\mathbb{F}_2^n$ is a linear application $\sigma : \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that $wt(\sigma(x)) = wt(x)$, for every $x \in \mathbb{F}_2^n$. It is well known that in binary case, the isometries are merely the permutations of the coordinates, that is the elements of $\mathcal{S}_n$, the permutation group of $\{1,2,\cdots,n\}$.

Two codes $\mathcal{C}$ and $\mathcal{C}'$ are said to be equivalent if there exists an isometry σ of $\mathbb{F}_2^n$ such that $\sigma(\mathcal{C}) = \mathcal{C}'$. An automorphism of $\mathcal{C}$ is a Hamming isometry σ such that $\sigma(\mathcal{C}) = \mathcal{C}$. The automorphisms of $\mathcal{C}$ form a subgroup of $\mathcal{S}_n$ called the automorphism group of $\mathcal{C}$ and we denote it by $\mathrm{Aut}(\mathcal{C})$. Note also that the vector space $\mathbb{F}_2^n$ can be endowed with a product $(x_1,\cdots,x_n)\cdot(y_1,\cdots,y_n) = (x_1y_1,\cdots,x_ny_n)$, so that $(\mathbb{F}_2^n,+,\cdot)$ becomes a Boolean ring. Furthermore, $wt(x+y)=wt(x)+wt(y)-2wt(xy)$, for every $x,y\in\mathbb{F}_2^n$. The code $\mathcal{C}$ is said a constant-weight code (CW-code) if all nonzero codewords have the same weight. The dual of binary Hamming codes $H_2(m)$ are simplex codes Σ_m of parameters $\left[2^m-1,m\right]_2$. simplex codes Σ_m are constant weight code (CW-code).

Any permutation of the columns of a k by n binary matrix G which maps the rows of G into rows of the same matrix, is called an automorphism of the binary matrix G [1]. The set of all automorphisms of G is a subgroup of the symmetric group S_n and we denote it by $\mathrm{Aut}(G)$. More treatment of linear codes can be found in the book [2].

Ideally, we would like the rate $R=\frac{k}{n}$ to be high, in order to be able to send a large number of errors. The rate of a DW-code approch zero very quickly when the code length increase: $\frac{k}{n}\le\frac{1}{n}\left[\frac{\ln(n+1)}{\ln(2)}\right]\searrow 0$ as shown in **Figure 1** where $R=\frac{k}{n}$ and $r(k)=\frac{k}{2^k-1}$, so $R\le r(k)$.

It is more convenient to use the DW-codes in the construction of other codes by using some technic of construction and not to use it alone.

2. Distinct Weight Codes

Definition 1 *A linear binary code* $\mathcal{C}$ *of length* n *is said to be a Distinct Weight Code*, (*in short*: *DW-code*), *if the weight mapping*: $\mathrm{wt}:\mathcal{C}\to\{0,1,\cdots,n\}$, *is one to one, that is* $x=y$ *whenever* $wt(x)=wt(y)$, $\forall x,y\in\mathcal{C}$.

The simplest example of such codes are the repetition codes. Later we shall give more nontrivial examples. Let $\mathcal{C}$ a DW-code of length n and dimension k. Since the number of element of $\mathcal{C}$ is 2^k, then we have $2^k\le n+1$. In the sequel we fix our interest to the extreme case $2^k=n+1$, in which we give a construction.

Proposition 2 *Let* k *such that* $2^k\le n+1$. *Then every family* $u_1,\cdots,u_k\in\mathbb{F}_2^n$ *of words such that* $wt(u_r)=2^{r-1}$ *is linearly independent.*

Proof. Suppose on the contrary that $u_1,\cdots,u_k$ are not linearly independent, then we have a linear combination $\sum_{i=1}^{k}\alpha_i u_i=0$, where some α_i is nonzero. Let r be the maximal integer such that $\alpha_r\ne 0$. Then $\alpha_r=1$, and $u_r=\sum_{i=1}^{r-1}\alpha_i u_i$. Now taking the weights leads to:

$$2^{r-1}=wt(u_r)\le\sum_{i=1}^{r-1}wt(\alpha_i u_i)\le\sum_{i=1}^{r-1}2^{i-1}=2^{r-1}-1$$

a contradiction. □

Now we give a construction of a $\left[2^k-1,k\right]$ DW-code.

Let k be a nonzero integer and $n=2^k-1$. Take $(e_1,\cdots,e_n)$ the canonical basis of $\mathbb{F}_2^n$. Put $c_r=\sum_{i=2^{r-1}}^{2^r-1}e_i$, then clearly $wt(c_r)=2^{r-1}$. By the proposition 2, the code-words $c_1,c_1,\cdots,c_k$ are linearly independent and generate a $[n,k]$ linear code that we denotes by $\mathcal{D}(k)$. It also seen that $c_i c_j=0$, whenever $i\ne j$. This

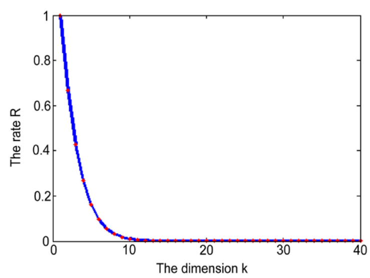

Figure 1. $R\le r(k)$ where $r(k)\searrow 0$.

implies that $wt\left(\sum_{i=1}^{k}\alpha_i c_i\right)=\sum_{i=1}^{k}wt(\alpha_i c_i)=\sum_{i=1}^{k}\alpha_i 2^{i-1}$.

A generator matrix of $\mathcal{D}(k)$ looks like:

$$G_k=\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&\cdots&0&0\\0&1&1&0&0&0&0&0&0&0&\cdots&0&0\\0&0&0&1&1&1&1&0&0&0&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&0&0&0&0&0&1&\cdots&1&1\end{pmatrix}$$

Proposition 3 *The* $\left[2^k-1,k\right]$*-code* $\mathcal{D}(k)$ *is a* DW*-code.*

Proof. Since the cardinal of $\mathcal{D}(k)$ is $2^k=n+1$, it suffices to show that wt: $\mathcal{D}(k)\to\{0,1,\cdots,n\}$ is onto. Let $r\in\{0,1,\cdots,n\}$, then r can be written $r=\sum_{i=1}^{k}\alpha_i 2^{i-1}$ in the base 2, where $\alpha_i\in\{0,1\}$. Set $x=\sum_{i=1}^{k}\alpha_i c_i$, then $wt(x)=\sum_{i=1}^{k}\alpha_i 2^{i-1}=r$. □

Up an equivalence we have the following result:

Theorem 4 *There exists only one distinct weight* $\left[2^k-1,k\right]$*-code, moreover such code is Boolean subring of* $\left(\mathbb{F}_2^n,+,\cdot\right)$.

Proof. Let $\mathcal{C}$ be such a code and take code-words $u_1,u_2,\cdots,u_k$, each u_i has weight 2^{i-1}. These are linearly independent and form a basis of $\mathcal{C}$. Next we show that $u_s u_r=0$, $\forall s<r$. Otherwise, there exists a least integer r such that $u_s u_r\neq 0$ for some $s<r$. Since $wt(u_s u_r)\le wt(u_s)$, one have $u_s u_r=\sum_{i=1}^{r-1}\alpha_i u_i$. Multiplying by u_s yields $u_s u_r=u_s$. Now consider the word $c=u_r+u_s$, $wt(c)=wt(u_r)+wt(u_s)-2wt(u_s u_r)=wt(u_r)+wt(u_s)-2wt(u_s)=2^{r-1}-2^{s-1}$ On the other hand, if we consider $h=u_s+u_{s+1}+\cdots+u_{r-1}$, then $wt(h)=\sum_{i=s-1}^{r-2}2^i=2^{r-1}-2^{s-1}$. Thus $u_s+u_{s+1}+\cdots+u_{r-1}=u_r+u_s$ hence $u_{s+1}+\cdots+u_{r-1}+u_r=0$ a contradiction. This means that $u_r u_s=0$, if $r\neq s$. Since $u_r^2=u_r$, and $u_1,\cdots,u_k$ is a basis of $\mathcal{C}$, then $\mathcal{C}$ is a Boolean ring.

Now we define a linear mapping $f:\mathcal{C}\to\mathcal{D}(k)$ by $f(u_i)=c_i$. Then, $f\left(\sum_{i=1}^{k}\alpha_i u_i\right)=\sum_{i=1}^{k}\alpha_i c_i$. If $x=\sum_{i=1}^{k}\alpha_i u_i$, then $wt(f(x))=wt\left(f\left(\sum_{i=1}^{k}\alpha_i u_i\right)\right)=wt\left(\sum_{i=1}^{k}\alpha_i c_i\right)=\sum_{i=1}^{k}\alpha_i 2^{i-1}=wt(x)$. This implies that f is an isometry between $\mathcal{C}$ and $\mathcal{D}(k)$, and by the extension theorem of MacWilliams, see [3] or [4], there exists a permutation $\sigma\in\mathcal{S}_n$, such that $\sigma(\mathcal{C})=\mathcal{D}(k)$. □

Example 5 $k=3$ ***and*** $n=7$ $\left(2^3-1=7\right)$

By using the software Q-extension, see [5] we show, up to equivalence, that among six equivalence classes the unique DW-code C_3 of parametters $[7,3,1]_2$ is the code of generator matrix $G_3=\begin{pmatrix}0000100\\1110010\\0001001\end{pmatrix}$. It is clear that it is equivallent to the code $\mathcal{D}(3)$ of generator matrix $G_3'=\begin{pmatrix}1000000\\0110000\\0001111\end{pmatrix}$. Just swap the second and third rows and then apply the permutation $\sigma=(1,5)(2,4)(3,7)$.

Theorem 6 *Let* $2^k=n+1$, *Diophantine equations* $n=t_1+t_2+\cdots+t_k$ *for which*

(1) $t_1<t_2<t_3<\cdots<t_k$

(2) $t_i\neq\sum_{j\in(I\setminus\{i\})}\varepsilon_j t_j$, $\forall i=1,2,\cdots,k$, $\forall I\subseteq\{1,2,\cdots,k\}$, $\forall\varepsilon_j=\pm1$,

have a unique solution which is the k-uplet $(t_1, t_2, \cdots, t_k) = (1, 2, 2^2, 2^3, \cdots, 2^{k-1})$.

Proof. $(1, 2, 2^2, 2^3, \cdots, 2^{k-1})$ is clearly a solution of the Diophantine equation which satisfies the conditions (1). Assume that $2^i = \sum_{j \in (I \setminus \{i\})} \varepsilon_j 2^j$ for some i and I, then $2^i + \sum_{i \in K^-} 2^j = \sum_{i \in K^+} 2^j \quad (< 2^k - 1)$ where $K^+ = \{ j \in I \setminus \{i\} / \varepsilon_j = 1 \}$ and $K^- = \{ j \in I \setminus \{i\} / \varepsilon_j = -1 \}$. We can assume without loss of generality that $\{2^j / j \in K^+\} \cap \{2^j / j \in K^-\} = \varnothing$. So by the uniqueness of Development of any integer less than or equal $2^k - 1$ in binary basis, the equality $2^i + \sum_{i \in K^-} 2^j = \sum_{i \in K^+} 2^j$ leads to a contradiction. So the solution $(1, 2, 2^2, 2^3, \cdots, 2^{k-1})$ satisfies the conditions (2).

Conversely, Let $(t_1, t_2, \cdots, t_k)$ a solution of the equation $n = t_1 + t_2 + \cdots + t_k$ satisfying (1)-(2). We can take d_i, $i = 1, 2, \cdots, k$ elements of F_2^n such that $wt(d_i) = t_i$ and $d_1 = \left(\underbrace{1 \cdots 1}_{t_1} 0 \cdots 0 \right)$,

$d_2 = \left(\underbrace{0 \cdots 0}_{t_1} \underbrace{1 \cdots 1}_{t_2} 0 \cdots 0 \right)$, $\cdots$, $d_k = \left(\underbrace{0 \cdots 0}_{t_1} \underbrace{0 \cdots 0}_{t_2} 0 \cdots 0 \underbrace{1 \cdots 1}_{t_k} \right)$, d_i, $i = 1, 2, \cdots, k$ are linearly independent. The condition (2) means that the code of generator matrix $G = \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix}$ is a dw-code. On after Theorem 1.3, the condition (1) implies that there exists an invertible k by k matrix $A = (a_{i,j})_{i,j}$ and a permutation matrix P_σ such that $AG_k P_\sigma = G$ where $\sigma \in S_n$ and $G_k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 1 & 1 \end{pmatrix}$

is the generator matrix of the code $\mathcal{D}(k)$. It is clear that G is of the form:

$$\begin{pmatrix} a_{1,\sigma(1)} & a_{1,\sigma(2)} & a_{1,\sigma(2)} & \cdots & a_{1,\sigma(k)} & a_{1,\sigma(k)} & \cdots & a_{1,\sigma(k)} \\ a_{2,\sigma(1)} & a_{2,\sigma(2)} & a_{2,\sigma(2)} & \cdots & a_{2,\sigma(k)} & a_{2,\sigma(k)} & \cdots & a_{2,\sigma(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k,\sigma(1)} & a_{k,\sigma(2)} & a_{k,\sigma(2)} & \cdots & a_{k,\sigma(k)} & a_{k,\sigma(k)} & \cdots & a_{k,\sigma(k)} \end{pmatrix}$$

where $a_{i,j} = 0$ or 1, $\forall i, j$ and $\forall j = 1, 2, 3, \cdots, k$ we have $t_j = \sum_{i=1}^{i=k} 2^{i-1} a_{j,\sigma(i)}$. So we have $2^k - 1 = \sum_{j=1}^{j=k} t_j = \sum_{j=1}^{j=k} \left(\sum_{i=1}^{i=k} 2^{i-1} a_{j,\sigma(i)} \right) = \sum_{i=1}^{i=k} 2^{i-1} \left(\sum_{j=1}^{j=k} a_{j,\sigma(i)} \right)$, and then we have $\sum_{j=1}^{j=k} a_{j,\sigma(i)} = 1$, $\forall i = 1, 2, \cdots, k$ by the uniqueness of development of $2^k - 1$ in binary basis. By (1) we have $\sum_{i=1}^{i=k} 2^{i-1} a_{1,\sigma(i)} < \sum_{i=1}^{i=k} 2^{i-1} a_{2,\sigma(i)} < \cdots < \sum_{i=1}^{i=k} 2^{i-1} a_{k,\sigma(i)}$, then we have: $\forall i, j$, $a_{i,\sigma(j)} = \delta_i^j$ (Kronecker symbol).

Since $t_j = \sum_{i=1}^{i=k} 2^{i-1} a_{j,\sigma(i)}$, we have $\forall j = 1, 2, \cdots, k$, $t_j = 2^{j-1}$ and finally we have $(t_1, t_2, t_3, \cdots, t_k) = (1, 2, 2^2, 2^3, \cdots, 2^{k-1})$. □

Remark 7 *Without the conditions* (1) *and* (2), *Diophantine equations have* $C_{2^k-2}^{k-1}$ *different solutions. For all*

$k \geq 3$ *note that there is no DW-self-dual code. Indeed, if not, we will have* $2^k - 1 \leq n = 2k$ *, wich is impossible.*

3. Classification and Automorphism Group of DW-Codes

3.1. Automorphism Group: The General Case

We consider, without loss of generality, that a generator matrix of a DW-code has no zero columns. Indeed, if this is the case, the zero columns are omitted and we consider the obtained DW-code. This assumption is made in the entier paper. We study the automorphism group of DW-codes. We first notice the following:

Proposition 8 *Let* $(u_1, u_2, \cdots, u_k)$ *any basis of an* $[n,k]$ *DW-code. Then*

$\mathrm{Aut}(\mathcal{C}) = \{\sigma \in \mathcal{S}_n \mid \sigma(u_i) = u_i, \forall i = 1, \cdots, k\}$.

Moreover, if G any generator matrix of $\mathcal{C}$, then σ is an automorphism of $\mathcal{C}$, if and only if, σ is an automorphism of the binary matrix G.

Proof. Clear.

Proposition 9 *The automorphism group of any DW-code is nontrivial of even order.*

Proof. Let G be a generator matrix of a DW $[n,k]$-code $\mathcal{C}$. We may suppose that all columns of G are nonzero. The n columns of G are taken among a set of $2^k - 1$ columns. Suppose that all columns of G are distincts, since $n \geq 2^k - 1$, then the columns of G are the $n = 2^k - 1$ distinct nonzero vectors of $\mathbb{F}_2^k$ and $\mathcal{C}$ will be the simplex code, which is clearly not DW. This contradiction shows that at least 2 columns of G are identical. Now the transposition of these two columns gives an automorphism of $\mathcal{C}$.□

We deduce that the dual code $\mathcal{C}^{\perp}$ of a DW-code has a non-trivial automorphism group and has minimum distance $d^{\perp} = 2$.

We consider the general case $2^k < n+1$. The action of automorphism group $\mathrm{Aut}(\mathcal{C})$ on the set $\Omega = \{c_1, c_2, \cdots, c_n\}$ of columns of a generator matrix G defined by: $\sigma(c_i) = c_{\sigma(i)}$ for all σ in $\mathrm{Aut}(\mathcal{C})$ and c_i in Ω, splits all the columns of G into disjoint orbits. The orbits $O_1, O_2, \cdots, O_f$ each formed of a single column, they are the columns fixed by the group $\mathrm{Aut}(\mathcal{C})$. We set $f = 0$ if no orbit is formed of a single column and then it is clear that $0 \leq f < 2^k - 1$ since since $\mathrm{Aut}(\mathcal{C})$ can not be trivial. The $r(\geq 1)$ other orbits are $O_{t_1}, O_{t_2}, \cdots, O_{t_r}$, $|O_{t_i}| \geq 2$, $i = 1, 2, \cdots, r$. We set $O_i = \{c_i\}$, $i = 1, 2, \cdots, f$ if $f \geq 1$ and $O_{t_i} = \{c_i^{(1)}, c_i^{(2)}, \cdots, c_i^{(t_i)}\}$, $i = 1, 2, \cdots, r$, therefore, we have precisely $0 \leq f \leq 2^k - 1 - r$.

Up to equivalence, we can consider that the code $\mathcal{C}$ is of generator matrix

$$G = \left(\underbrace{c_1^{(1)}, c_1^{(2)}, \cdots, c_1^{(t_1)}}_{O_{t_1}}, \cdots, \underbrace{c_r^{(1)}, c_r^{(2)}, \cdots, c_r^{(t_r)}}_{O_{t_r}}, \underbrace{c_1}_{O_1}, \underbrace{c_2}_{O_2}, \cdots, \underbrace{c_f}_{O_f} \right) = \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix}$$

$d_j = \left(d_j^{(i)}\right)_i$ rows of G, such that $2 \leq t_1 \leq t_2 \leq \cdots \leq t_r$.

Since $\mathcal{C}$ is a DW-code, then for each $j = 1, 2, \cdots, k$ for each $\sigma \in \mathrm{Aut}(\mathcal{C})$ we have $\sigma(d_j) = d_j$ $\forall j = 1, 2, \cdots, k$. So $\forall i = 1, 2, \cdots, n$ we have $d_j^{(\sigma(i))} = d_j^{(i)}$. We therefore deduce that: $\forall i = 1, 2, \cdots, r$, $c_i^{(1)} = c_i^{(2)} = \cdots = c_i^{(t_i)}$. So each orbit O_{t_i} consists of $t_i \geq 2$ equal columns.

The following theorem legitimate the idea of giving a definition to the 3-tuple $(f, k, (t_1, t_2, \cdots, t_r))$ which we call signature of the DW-code and we denote $\mathrm{sign}(\mathcal{C}) = (f, k, (t_1, t_2, \cdots, t_r))$. We give here the full classification of such a code in several cases.

Theorem 10 *If two DW-codes* $\mathcal{C}$ *and* $\mathcal{C}'$ *are equivalent then they have the same signatures:* $\mathrm{sign}(\mathcal{C}) = \mathrm{sign}(\mathcal{C}')$.

Proof. Let $\mathcal{C}$ and $\mathcal{C}'$ two equivalent DW-codes of parameters $[n,k]$. So $\exists \sigma \in S_n$ such as $\sigma(\mathcal{C}) = \mathcal{C}'$.

We have $\mathrm{Aut}(\mathcal{C}') = \sigma \mathrm{Aut}(\mathcal{C}) \sigma^{-1}$. Let G be a generator matrix of the code $\mathcal{C}$. Under the action of the automorphism group $\mathrm{Aut}(\mathcal{C})$ we can assume that G is of the form $G = (O_{t_1}, \cdots, O_{t_r}, O_1, O_2, \cdots, O_f)$ where $O_{t_i} = \{c_{s_{i-1}+1}, \cdots, c_{s_i}\}$, $s_i = \sum_{j=1}^{j=i} t_j \quad \forall i = 1, 2, \cdots, t_r$ and $O_i = \{c_{s_r+i}\} \quad \forall i = 1, 2, \cdots, f$.

So we have $O_{t_i} = \{c_{s_{i-1}+1}, \cdots, c_{s_i}\} = \{c_{\sigma(s_i)} / \sigma \in \mathrm{Aut}(\mathcal{C})\} = \{c_{\sigma^{-1}\rho\sigma(s_i)} / \rho \in \mathrm{Aut}(\mathcal{C}')\}$ and then $\sigma(O_{t_i}) = \{c_{\rho\sigma(s_i)} / \rho \in \mathrm{Aut}(\mathcal{C}')\} = O'_{t_i}$ which is an orbit of the column $c_{\sigma(s_i)}$ under the action of $\mathrm{Aut}(\mathcal{C}')$ on the generator matrix $G' = \sigma(G)$ of the code $\mathcal{C}'$. similarly we have $\sigma(O_i) = \{c_{\rho\sigma(s_r+i)} / \rho \in \mathrm{Aut}(\mathcal{C}')\} = O'_i$ which is an orbit of the column $c_{\sigma(s_r+i)}$ under the action of $\mathrm{Aut}(\mathcal{C}')$ on the generator matrix G'. Thus G and G' have the same number of ponctual orbits, the same number of non-ponctual orbits and the two orbits O_{t_i} and O'_{t_i} on the one hand and O_i and O'_i on the other hand have the same number of columns. we conclude that the two codes C and C' have the same Signature. □

3.2. Classification

3.2.1. Case 1: $f = 0$ and $k = r$

We have

$$G = \left(\underbrace{c_1^{(1)}, c_1^{(2)}, \cdots, c_1^{(t_1)}}_{O_{t_1}}, \cdots, \underbrace{c_k^{(1)}, c_k^{(2)}, \cdots, c_k^{(t_k)}}_{O_{t_k}} \right) = \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix}$$

Theorem 11 *If $\mathcal{C}$ is an $[n,k]_2$ DW-code without punctual orbits $(f = 0)$ and if the number of non punctual orbits is equal to the dimension of the DW-code $(r = k)$ then the code $\mathcal{C}$ is equivallent to a DW-code of generator matrix*

$$G = G_k(t_1, t_2, \cdots, t_k) = \begin{pmatrix} \underbrace{1\cdots1}_{t_1}000\cdots0\cdots0 \\ 0\cdots0\underbrace{1\cdots1}_{t_2}\cdots0\cdots0 \\ \vdots \\ 000000\cdots00\underbrace{1\cdots1}_{t_k} \end{pmatrix}$$

whith $2 \le t_1 < t_2 < \cdots < t_k$ and $t_1 = d$ is the minimal distance of C_k.

Proof. After a series of permutations and elementary operations on rows of G we can make the first line of the first orbit formed only by ones and all other rows are null $\begin{matrix} \overbrace{11\cdots1}^{O_{t_1}} \\ 00\cdots0 \\ \vdots \\ 00\cdots0 \end{matrix}$ all other bits of the first row of the generator matrix are zero. Otherwise the first line of another orbit O_{t_s} will be formed only by 1 s. And a series of permutations and elementary row operations can make null all the other rows of this orbit so $O_{t_1} \cap O_{t_s} \neq \varnothing$. This is a contradiction since two orbits are disjoint. We obtain a generator matrix of an equivalent code denoted by the same sign $G_k = \begin{pmatrix} 11\cdots1 & | & 00\cdots0 \\ 0 & | & \\ \vdots & | & G_k^0 \\ 0 & | & \end{pmatrix}$. It is easy to see that G_k^0 is a generator matrix of a DW-code without punctual orbits $\left(f_{G_k^0}=0\right)$ and the number of orbits is equal to the dimension of this DW-code $\left(r_{G_k^0}=k-1\right)$ and This allows for reasoning by induction. We obtain a generator matrix of an equivalent code

$$G_k\left(t_1,t_2,\cdots,t_k\right)=\begin{pmatrix}\underbrace{1\cdots1}_{t_1}000\cdots0\cdots0\\ 0\cdots0\underbrace{1\cdots1}_{t_2}\cdots0\cdots0\\ \vdots\\ 000000\cdots00\underbrace{1\cdots1}_{t_k}\end{pmatrix}.$$

It is clear that we have $wt\left(d_1\right)=t_1=d$, $wt\left(d_2\right)=t_2$, $\cdots$, $wt\left(d_k\right)=t_k$ and $2\le t_1<t_2<\cdots<t_k$, ($t_1\ge2$ since $t_1=1$ implies the existence of a punctual orbit) . □

Remark 12 *In this case, up to equivalence, each* $[n,k]_2$ *DW-code admits the system* $\{d_1,d_2,\cdots,d_k\}$ *as orthogonal basis:* $d_1=\left(\underbrace{1\cdots1}_{t_1}0\cdots0\right)$, $d_2=\left(\underbrace{0\cdots0}_{t_1}\underbrace{1\cdots1}_{t_2}0\cdots0\right)$, $\cdots$, $d_2=\left(\underbrace{0\cdots0}_{t_1}\underbrace{0\cdots0}_{t_2}0\cdots0\underbrace{1\cdots1}_{t_k}\right)$ *such as* $wt\left(d_1\right)=t_1=d$, $wt\left(d_2\right)=t_2,\cdots$, $wt\left(d_k\right)=t_k$ *and* $2\le t_1<t_2<\cdots<t_k$.

Example 13 *Consider the* $[15,3,2]_2$ *DW-code of generator matrix*
$\begin{pmatrix}100000000000100\\011111111000010\\000000000111001\end{pmatrix}$. It is equivallente to the code of generator matrix $\begin{pmatrix}110000000000000\\001111000000000\\000000111111111\end{pmatrix}$
$t_1=2$, $t_2=4$, $t_3=9$, $f=0$, $r=k=3$

Corollary 14 *Let two* $[n,k]_2$ *DW-codes* $\mathcal{C}$ *and* $\mathcal{C}'$ *without punctual orbits and the number of their orbits is equal to their dimension. Then the codes* $\mathcal{C}$ *and* $\mathcal{C}'$ *are equivallent if and only if* $\mathrm{sign}(\mathcal{C})=\mathrm{sign}(\mathcal{C}')$. □

The converse of Theorem 11 is true under an additional condition.

Theorem 15 *Let* $\mathcal{C}$ *an* $[n,k]_2$ *code of generator matrix* $G_k\left(t_1,t_2,\cdots,t_k\right)$ (*as in the last remark*) *. If*:

(1) $2\le t_1<t_2<\cdots<t_k$

(2) $\forall I\subseteq\{1,2,\cdots,k\}$, $\forall i=1,2,\cdots,k$, $\forall\varepsilon_j=\pm1$ we have $t_i\ne\sum_{j\in(I\setminus\{i\})}\varepsilon_j t_j$

then $\mathcal{C}$ is a DW-code of minimum distance $d=t_1$ for which $f=0$ and $r=k$.

Proof. Clear.

Corollary 16 *The number of equivalence classes of* $[n,k,d]_2$ *DW-codes such as* $f=0$, $k=r$ *and* $2^k<n+1$ *equals the number of solutions* $\left(t_1,t_2,t_3,\cdots,t_k\right)$ *of the Diophantine equations* $n=t_1+t_2+t_3+\cdots+t_k$ *satisfying the following conditions*

(1) $d=t_1<t_2<t_3<\cdots<t_k$

(2) $t_i\ne\sum_{j\in(I\setminus\{i\})}\varepsilon_j t_j$, $\forall i=1,2,\cdots,k$, $\forall\varepsilon_j=\pm1$, $\forall I\subseteq\{1,2,\cdots,k\}$.

Proof. Let the application that maps each equivalence class represented by the matrix $G_k\left(t_1,t_2,\cdots,t_k\right)$ to the *t*-tuple $\left(t_1,t_2,\cdots,t_k\right)$ solution of the Diophantine equation as described in Theorem 11. This application is clearly a bijection between the set of equivalence classes and the set of solutions of the Diophantine equation satisfying conditions (1) and (2). □

3.2.2. Case 2: $f\ne0$ and $k=r$

Theorem 17 *If* $\mathcal{C}$ *is an* $[n,k]_2$ *DW-code with* f *punctual orbits* $(f\ne0)$ *and if the number of non punctual orbits is equal to the dimension of the DW-code* $(r=k)$ *then the code* C *is equivallent to the*

DW-code of generator matrix $G = \begin{pmatrix} \overbrace{1\cdots1}^{t_1} & 0\cdots0 & \cdots & 0\cdots0 & | & \overbrace{c_1c_2\cdots c_f}^{\text{fixed columns}} \\ 0\cdots0 & \underbrace{1\cdots1}_{t_2} & \cdots & 0\cdots0 & | & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & 0\cdots0 & | & \vdots \\ \vdots & \vdots & \vdots & \vdots & | & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & 0\cdots0 & | & \vdots \\ 0\cdots0 & 0\cdots0 & \cdots & \underbrace{1\cdots1}_{t_k} & | & \vdots \end{pmatrix}$ *with* $1 \le f \le 2^k - 1 - k$ *and* $2 \le t_1 \le t_2 \le \cdots \le t_k$. □

Example 18 *Consider the* $[15,3,4]_2$ *DW-code* $\mathcal{C}_1$ *of generator matrix*

$G_1 = \begin{pmatrix} 111 & 0000000 & 000 & 10 \\ 111 & 1111111 & 000 & 01 \\ 111 & 0000000 & 111 & 01 \end{pmatrix}$. It is equivallente to the code $\mathcal{C}_2$ of generator matrix

$G_2 = \begin{pmatrix} 111 & 000 & 0000000 & \overbrace{10}^{F} \\ 000 & 111 & 0000000 & 11 \\ 000 & 000 & 1111111 & 11 \end{pmatrix}$ $t_1 = 3$, $t_2 = 3$, $t_3 = 7$, $f = 2$, $r = k = 3$

We have $\text{sign}(\mathcal{C}_1) = \text{sign}(\mathcal{C}_2)$ since $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivallente.

The converse of this theorem is true under an additional condition. Let $\mathcal{C}$ an $[n,k]_2$ of generator matrix $\begin{pmatrix} \underbrace{1\cdots1}_{t_1} & & | & c_1c_2\cdots c_f \\ & \ddots & | & \vdots \\ & & \underbrace{1\cdots1}_{t_k} \; | & \vdots \end{pmatrix}$ with: $1 \le f \le 2^k - 1 - k$ and $2 \le t_1 \le t_2 \le \cdots \le t_k$. $c_1, c_2, \cdots, c_f$ are f different columns which are also different from all unitary columns $(1,0,\cdots,0)^{\mathrm{T}}$, $(0,1,0,\cdots,0)^{\mathrm{T}}$, $\cdots$, $(0,\cdots,0,1)^{\mathrm{T}}$.

For each $I \subseteq \{1,2,\cdots,k\}$, denote by $\omega_{I,\epsilon}$ the weight of the sum of all the jth rows where $j \in I \setminus \{i\}$ of the $k \times f$ matrix

$$A = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \varepsilon_k \end{pmatrix} \begin{pmatrix} c_1, & c_2, & \cdots & ,c_f \end{pmatrix} \quad \varepsilon_j = \pm 1, \quad \varepsilon = (\varepsilon_j)_j$$

For all $i = 1,2,\cdots,k$ denote by α_i the weight of the ith row of the $k \times f$ matrix $(c_1, c_2, \cdots, c_f)$. So by setting the numbers $\rho_{I,i,\epsilon} = \omega_{I,\epsilon} - \alpha_i$ we have the following result, let $\mathcal{C}$ an $[n,k]$ code of generator matrix G as described in theorem 17 we have:

Theorem 19 *If for all* $I \subseteq \{1,2,\cdots,k\}$, *for all* $i \in I$ *and for all* $\epsilon = (\varepsilon_j)_j$, $\varepsilon_j = \pm 1$ *we have*

$t_i \ne \left(\sum_{j \in (I \setminus \{i\})} \varepsilon_j t_j \right) + \rho_{I,i,\epsilon}$ *then the code* $\mathcal{C}_k$ *is a DW-code.* □

Let C_k an $[n,k]_2$ DW-code such as $2^k < n+1$, $r = k$ and $1 \le f \le 2^k - 1 - k$. We have $n(f) = C^f_{2^k - 1 - k}$ different way to the choice of f fixed columns. For each value of f and for the s_f-th choice of f fixed columns we denote by $N(f,s_f)$, $1 \le s_f \le C^f_{2^k-1-k}$ the number of solutions of the Diophantine equations $n-f=t_1+t_2+t_3+\cdots+t_k$ which satisfy the following conditions

(1) $2 \le t_1 \le t_2 \le t_3 \le \cdots \le t_k$

(2) $t_i \ne \left(\sum_{j\in(I\setminus\{i\})} \varepsilon_j t_j\right)+\rho_{I,i,\epsilon}$ $\forall i=1,2,\cdots,k$, $\forall I \subseteq \{1,2,\cdots,k\}$, $\forall \epsilon=(\varepsilon_j)_j$, $\varepsilon_j=\pm 1$

So we have the following result.

Theorem 20

1) The number of equivalence classes of $[n,k]$ DW-codes with $2^k < n+1$ and a given f such that $1 \le f \le 2^k-1-k$ and $k=r$ equals the number $\sum_{s_f=1}^{s_f=n(f)} N(f,s_f)$.

2) The number of equivalence classes of $[n,k]_2$ DW-codes with $2^k < n+1$, $f \ne 0$ and $k=r$ equals the number $\sum_{f=1}^{f=2^k-1-k}\left(\sum_{s_f=1}^{s_f=n(f)} N(f,s_f)\right)$.□

Example 21 *By using the result of the last theorem and the Q-extension software, We show that there exist Only* 4 $[11,3]_2$ *DW-code up to equvallence verifying* $r=k=3$. *Indeed* $1 \le f \le 2^3-1-3=4$ *and we have*:

- For $f=1$ the set of possible columns taken in the following order are:

c_1	c_2	c_3	c_4
1	1	0	1
1	0	1	1
0	1	1	1

$$N(1,1)=N(1,3)=N(1,4)=1,\quad N(1,2)=0$$

So the number of DW-codes with $2^k < n+1$, $f=1$ and $k=r$ is $\sum_{i=1}^{i=4} N(1,i)=3$.

- For $f=2$ the set of possible columns taken in the following order are:

1th $[c_1,c_2]$	2th $[c_1,c_3]$	3th $[c_1,c_4]$	4th $[c_2,c_3]$	5th $[c_2,c_4]$	6th $[c_3,c_4]$
11	10	11	10	11	01
10	11	11	01	01	11
01	01	01	11	11	11

$$N(2,1)=N(2,2)=N(2,3)=N(2,4)=N(2,5)=N(2,6)=0$$

So there is no DW-codes with $2^k < n+1$, $f=2$ and $k=r$ since $\sum_{i=1}^{i=6} N(2,i)=0$.

- For $f=3$ the set of possible columns taken in the following order are:

1th $[c_1c_2c_3]$	2th $[c_1c_2c_4]$	3th $[c_1c_3c_4]$	4th $[c_2c_3c_4]$
110	111	101	101
101	101	111	011
011	011	011	111

$$N(3,1)=N(3,2)=N(3,3)=N(3,4)=0$$

So there is no DW-codes avec $2^k < n+1$, $f=3$ and $k=r$ since $\sum_{i=1}^{i=4} N(3,i)=0$.

- For $f = 4$ the set of possible columns taken in the following order are:

$$\overbrace{[c_1c_2c_3c_4]}^{\text{1th}}$$
$$1101$$
$$1011$$
$$0111$$

$$N(4,1)=0,\ N(4,2)=0,\ N(4,3)=1,\ N(4,4)=0.$$

So there is one DW-codes such as $2^k < n+1$, $f = 3$ and $k = r$ since $\sum_{i=1}^{i=1} N(4,i) = 1$.

We deduce that there is only four $[11,3]_2$ DW-codes, among 98 equivalence classes, satisfying $r = k = 3$ since

$$\sum_{f=1}^{f=4}\left(\sum_{i=1}^{i=n(f)} N(f,i)\right) = \sum_{i=1}^{i=4} N(1,i) + \sum_{i=1}^{i=6} N(2,i) + \sum_{i=1}^{i=4} N(3,i) + \sum_{i=1}^{i=1} N(4,i) = 4$$

3.2.3. Case 3: $f = 0$ and $k \neq r$

We have necessarily $k < r$.

Theorem 22 *If $\mathcal{C}$ is an $[n,k]_2$ DW-code without punctual orbits $(f = 0)$ and if the number of non punctual orbits is different from the dimension of the DW-code $(r \neq k)$ then*

$k < r$ and the code $\mathcal{C}$ is equivallent to the DW-code of generator matrix

$$\begin{pmatrix} \underbrace{1\cdots1}_{t_1} & & & | & \overbrace{*\ldots*}^{t_{k+1}} & \cdots & \overbrace{*\ldots*}^{t_r} \\ & \ddots & & | & \vdots & \ddots & \vdots \\ & & \underbrace{1\cdots1}_{t_k} & | & \vdots & \cdots & \vdots \end{pmatrix}$$

with $2 \le t_1 \le t_2 \le \cdots \le t_k$. □

Example 23 *The $[15,3,4]_2$ DW-code of generator matrix $\begin{pmatrix} 111000000000100 \\ 000111111110010 \\ 000111100001001 \end{pmatrix}$ is equivallent to the code of generator matrix* $\begin{pmatrix} 11000000000 & 1111 \\ 00111100000 & 0000 \\ 00000011111 & 1111 \end{pmatrix}$ $f = 0$, $k = 3$, $r = 4$, $t_1 = 2$, $t_2 = 4$, $t_3 = 5$, $t_4 = 4$. □

In this case two DW-codes with the same signature are not necessarily equivalent as shown in the following example:

Example 24 *Let $\mathcal{C}_1$ the DW-code of generator matrix G_1 and $\mathcal{C}_2$ the DW-code of generator matrix G_2 such as $\mathcal{C}_1$ and $\mathcal{C}_2$ are not equivalent and*

$$G_1 = \begin{pmatrix} 110000000000001 \\ 111111111100000 \\ 000000000011110 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 110000000000001 \\ 111111111100000 \\ 110000000011110 \end{pmatrix}.$$

We have $\operatorname{sign}(\mathcal{C}_1) = \operatorname{sign}(\mathcal{C}_2) = (1,3,(2,4,8))$.

3.2.4. Case 4: $f \neq 0$ and $k \neq r$

We can have two cases $k < r$ or $r < k$

Theorem 25 *If $\mathcal{C}$ is an $[n,k]_2$ DW-code with f punctual orbits $(f \neq 0)$ and if the number of non punctual orbits is greater than the dimension of the DW-code $(r > k)$ then the code $\mathcal{C}$ is equivallent to the DW-code of generator matrix*

$$\begin{pmatrix} \underbrace{1\cdots1}_{t_1} & & \overbrace{c_1\cdots c_f}^{\text{fixed columns}} & \overbrace{*\ldots*}^{t_{k+1}} & \cdots & \overbrace{*\ldots*}^{t_r} \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & \underbrace{1\cdots1}_{t_k} & \vdots & \vdots & \cdots & \vdots \end{pmatrix}$$

with $2 \leq t_1 \leq t_2 \leq \cdots \leq t_k$. □

Example 26 *The* $[15,3,4]_2$ *DW-code of generator matrix* $\begin{pmatrix} 111000000000100 \\ 111111111110010 \\ 111110000001001 \end{pmatrix}$ *is equivallent to the code of generator matrix* $\begin{pmatrix} 1100000 & 1111111 & 0 \\ 0011000 & 1111111 & 1 \\ 0000111 & 0000000 & 0 \end{pmatrix}$ $t_1 = 2$, $t_2 = 2$, $t_3 = 3$, $t_4 = 7$, $f = 1$, $k = 3$, $r = 4$

Theorem 27 *If C is an $[n,k]_2$ DW-code with f punctual orbits $(f \neq 0)$ and if the number of non punctual orbits is lower than the dimension of the DW-code $(r < k)$ then the code C is equivallent to the DW-code of generator matrix*

$$\begin{pmatrix} \underbrace{1\cdots1}_{t_1} & & & \overbrace{c_1c_2\cdots c_f}^{\text{fixed columns}} \\ & \ddots & & \vdots \\ & & \underbrace{1\cdots1}_{t_r} & \vdots \\ 000 & \cdots & 000 & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 000 & \cdots & 000 & \vdots \end{pmatrix}$$

with $2 \leq t_1 \leq t_2 \leq \cdots \leq t_r$. □

Example 28 *The* $[15,3,4]_2$ *DW-code of generator matrix* $\begin{pmatrix} 110000000000100 \\ 111111110000010 \\ 100000001111001 \end{pmatrix}$ *is equivallent to the code of generator matrix* $\begin{pmatrix} 111110000000 & 100 \\ 000001111111 & 110 \\ 000000000000 & 111 \end{pmatrix}$ $t_1 = 5$, $t_2 = 7$, $f = 3$, $k = 3$, $r = 2$. □

Remark 29 *Self-orthogonality.*

A code which is equivalent to a self-orthogonal code is also self-orthogonal. The property of self- orthogonality is then an invariant of the equivalence of codes. We then have the following points:

- If $f \neq 0$ or ($f = 0$ and $r \neq k$)then, up to equivalence, a generator matrix of the code is of the form $G = [TD]$ where D is not an empty submatrix. If the code is self-othogonal then $GG^{\mathrm{T}} = 0$. So $[TD][TD]^{\mathrm{T}} = 0$ and then we have:
  - $f \neq 0$ and $r > k$ and then $DD^{\mathrm{T}} = \operatorname{diag}(t_1[2], t_2[2], \cdots, t_k[2])$

- $f \neq 0$ and $r < k$ then $DD^{\mathrm{T}} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ where $B = \mathrm{diag}\left(t_1[2], t_2[2], \cdots, t_r[2]\right)$.
- $f \neq 0$ and $r = k$ then $DD^{\mathrm{T}} = \mathrm{diag}\left(t_1[2], t_2[2], \cdots, t_k[2]\right)$
- finally $f = 0$ and $r \neq k$ and then $DD^{\mathrm{T}} = \mathrm{diag}\left(t_1[2], t_2[2], \cdots, t_k[2]\right)$

- If $f = 0$ and $r = k$ the code C is self-orthogonal if and only if $t_i \equiv 0[\mathrm{mod}\, 2]$ for all $i = 1, 2, \cdots, k$. □

3.3. Determination of the Automorphism Group

Theorem 30 *The automorphism group of a* $[n,k]$ *DW-code of signature* $\left(f, k, \left(t_1, t_2, \cdots, t_r\right)\right)$ is isomorphic to the group direct product $\prod_{i=1}^{r} \mathcal{S}_{t_i}$.

Proof. Let G be a generator matrix of the code $\mathcal{C}$. We can assume that G is of the form $G = \left(O_{t_1}, \cdots, O_{t_r}, O_1, O_2, \cdots, O_f\right)$ where $O_{t_i} = \left\{c_{s_{i-1}+1}, \cdots, c_{s_i}\right\}$, $s_i = \sum_{j=1}^{j=i} t_j$ $\forall i = 1, 2, \cdots, t_r$ and $O_i = \left\{c_{s_r+i}\right\}$ $\forall i = 1, 2, \cdots, f$.

For $i = 1, \cdots, r$, let $E_i = \left\{s_{i-1}+1, \cdots, s_i\right\}$, let $E_{s_r+i} = \left\{s_r + i\right\}$ for all $i = 1, 2, \cdots, f$. Clearly, the subsets $E_1, \cdots, E_n$ form a partition of $\{1, 2, \cdots, n\}$. Now let $G_i = \left\{\sigma \in S_n / \sigma(E_i) \subset E_i \text{ and } \forall j \neq i, \sigma(x) = x, \forall x \in E_j\right\}$.

Clearly the G_i are subgroups of $\mathcal{S}_n$ and each is isomorphic to $\mathcal{S}_{t_i}$ and $G_i = \{id\}$ for all $i = s_r + 1, \cdots, s_r + f = n$. Since forall $\sigma \in G_i$, $\sigma\left(c_j\right) = c_j$, it follows that the G_i are subgroups of $\mathrm{Aut}(\mathcal{C})$. Now we are going to show that $\mathrm{Aut}(\mathcal{C})$ is the inner direct product $G_1 G_2 \cdots G_k$

If $i \neq j$, and $\sigma \in G_i$, $\tau \in G_j$, then $\sigma\tau = \tau\sigma$.

Let $\sigma_1 \sigma_1 \cdots \sigma_k = I$, then applying this equality to each E_i yields $\sigma_i = I$, $\forall i$.

Now let $\sigma \in \mathrm{Aut}(\mathcal{C})$. Since $\sigma(c_i) = c_i$, the E_i are globally invariant under σ. Let σ_i the permutation defined by $\sigma_i(x) = \sigma(x)$, if $x \in E_i$, and $\sigma_i(x) = x$ elswhere. Then it is clear that $\sigma = \sigma_1 \sigma_2 \cdots \sigma_r$, and this finishes the proof. □

Example 31

- Consider the $[15,3,2]_2$ DW-code of generator matrix $\begin{pmatrix} 100000000000100 \\ 011111111000010 \\ 000000000111001 \end{pmatrix}$. It is equivallente to the code of generator matrix $\begin{pmatrix} 110000000000000 \\ 001111000000000 \\ 000000111111111 \end{pmatrix}$ $t_1 = 2$, $t_2 = 4$, $t_3 = 9$, $f = 0$, $r = k = 3$,

  $\mathrm{Aut}(\mathcal{C}) = S_2 \cdot S_4 \cdot S_9$ and $\left|\mathrm{Aut}(C)\right| = (2!) \times (4!) \times (9!)$

- Consider the $[15,3,5]_2$ DW-code of generator matrix $\begin{pmatrix} 111100000000100 \\ 111111111111010 \\ 111011100000001 \end{pmatrix}$. It is equivallente to the code of generator matrix $\begin{pmatrix} 111000000000 & 110 \\ 000111000000 & 111 \\ 000000111111 & \underbrace{101}_{\text{fixed}} \end{pmatrix}$ $t_1 = 3$, $t_2 = 3$, $t_3 = 6$, $f = 3$, $r = k = 3$

  $\mathrm{Aut}(\mathcal{C}) = S_2 \cdot S_4 \cdot S_9$ and $\left|\mathrm{Aut}(C)\right| = (2!) \times (4!) \times (9!)$

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