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Linear Problems of Optimal Control of Fuzzy Maps

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Abstract: In the present paper, we show the some properties of the fuzzy R-solution of the control linear fuzzy differential inclusions and research the optimal time problems for it.

Keywords: fuzzy differential inclusions, control problems

1. Introduction

The first study of differential equations with multivalued right-hand sides was performed by A. Marchaud [1,2] and S.C. Zaremba [3]. In early sixties, T. Wazewski [4,5], A.F. Filippov [6] obtained fundamental results on existence and properties of the differential equations with multivalued right-hand sides (differential inclusions). One of the most important results of these articles was an establishment of the relation between differential inclusions and optimal control problems, that promoted to develop the differential inclusion theory [7].

Considering of the differential inclusions required to study properties of multivalued functions, i.e. an elaboration the whole tool of mathematical analysis for multivalued functions [8–10].

Simultaneously there were appeared papers [11–14] which used Hukuhara derivative [9,10] of multivalued function for investigation of differential equations with multivalued right-hand sides and solutions.

In works [15,16] annotate of an R-solution for differential inclusion is introduced as an absolutely continuous multivalued function. Various problems for the R- solution theory were regarded in [17–22].

The basic idea for a development of an equation for R-solutions (integral funnels) is contained in [23].

Here the equation was considered as a particular case of an approximated equation in a metric space. Approximated equations make possible to exclude an differentiation operation and there by to avoid linearity requirement for a solution space and to consider differential equations in linear metric spaces, equations with multivalued solutions and dynamical systems in nonlinear metric spaces by unified positions [14,20,23,24].

Therefore, an approximated equation is a quasidifferential equation for determination of the dynamical system in a metric space.

The theory of mutational equations in metric spaces, which deals with multivalued trajectories (pipers) and trajectories in nonlinear spaces has been developed in [25].

As well as in [23] it is taken an approach that does not use a derivative in explicit form for description in nonlinear metric spaces, while in [25] analogous results are obtained by construction "differential calculus" in nonlinear metric spaces.

Moreover in [23] quasidifferential equations were considered for locally compact spaces, while in [20] for complete metric space .

In the last years there has been forming new approach to control problems of dynamic systems, which foundation on analysis of trajectory bundle but not separate trajectories. The section of this bundle in any instant is some set and it is necessary to describe the evolution of this set. Obtaining and research dynamic equations of sets there is important problem in this case. The metric space of sets with the Hausdorff metric is natural space for description dynamic of sets. In theory of multivalued maps definitions on derivative as for single-valued maps is impossible because space of sets is nonlinear. This bound possibility description dynamic sets by differential equations. Therefore, the control differential equations with set of initial conditions [26–28], the control differential inclusions [29–40], the control differential equations with Hukuhara derivative [14] and the control quasidifferential equations [14,40,41] use for it.

In recent years, the fuzzy set theory introduced by Zadeh [42] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical, differential equations, and engineering sciences. Recently there have been new advances in the theory of fuzzy differential equations [43–55] and inclusions [50,56–59] as well as in the theory of control fuzzy differential equations [60–62] and

inclusions [63–65].

In this article we consider the some properties of the fuzzy R-solution of the control linear fuzzy differential inclusions and research the optimal time problems for it.

2. The Control Differential inclusions

2.1. The Fundamental Definitions and Designations

Let $comp(R^n)(conv(R^n))$ be a set of all nonempty (convex) compact subsets from the space R^n,

$$h(A, B) = \min_{r \geq 0} \{ S_r(A) \supset B, S_r(B) \supset A \}$$

be Hausdorff distance between sets A and B, $S_r(A)$ is r-neighborhood of set A.

Let E^n be the set of all $u : R^n \to [0,1]$ such that u satisfies the following conditions:

1) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

2) u is fuzzy convex, that is,
$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$$

3) for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$;

4) u is upper semicontinuous,

5) $[u]^0 = cl\{x \in R^n : u(x) > 0\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space. For $0 < \alpha \leq 1$, denote

$$[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}.$$

Then from 1)-4), it follows that the α-level set $[u]^\alpha \in conv(R^n)$ for all $0 \leq \alpha \leq 1$.

Theorem 1 (Negoita and Ralescu [66]). *If $u \in E^n$, then*

1) $[u]^\alpha \in conv(R^n)$ *for all* $\alpha \in [0,1]$;

2) $[u]^\alpha \subset [u]^\beta$ *for* $0 \leq \alpha < \beta \leq 1$;

3) *If* $\{\alpha_k\} \subset [0,1]$ *is a decreasing sequence converging to* $\alpha > 0$ *then* $[u]^\alpha = \bigcap_{k \geq 1} [u]^{\alpha_k}$

Conversely, if $\{A^\alpha : 0 \leq \alpha \leq 1\}$ *is a family of convex compact subsets of* R^n *satisfying* 1)-3)*, then* $[u]^\alpha = A^\alpha$ *for* $0 < \alpha \leq 1$ *and* $[u]^0 = \overline{\bigcap_{0 < \alpha \leq 1} A^\alpha} \subset A^0$.

If $g : R^n \times R^n \to R^n$ is a function, then using Zadeh's extension principle we can extend $\tilde{g}$ to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u, v)(z) = \sup_{z = g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that

$$[\tilde{g}(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)$$

for all $u, v \in E^n$, $0 \leq \alpha \leq 1$ and continuous function g.

Further, we have

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $k \in R$.

Define $D : E^n \times E^n \to [0, \infty)$ by the relation

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric defined in $comp(R^n)$.

Then D is a metric in E^n.

Further we know that [67]:

1) (E^n, D) is a complete metric space,

2) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$,

3) $D(\lambda u, \lambda v) = |\lambda| D(u, v)$ for all $u, v \in E^n$ and $\lambda \in R$.

It can be proved that

$$D(u + v, w + z) \leq D(u, w) + D(v, z)$$

for $u, v, w, z \in E^n$.

Definition 1. A mapping $F : [0, T] \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued map $F_\alpha : [0, T] \to conv(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.

Definition 2. A mapping $F : [0, T] \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \leq h(t)$ for every $x(t) \in F_0(t)$.

Definition 3. The integral of a fuzzy mapping $F : [0, T] \to E^n$ is defined levelwise by $\left[\int_0^T F(t)dt \right]^\alpha$ $= \int_0^T F_\alpha(t)dt$. The set of all $\int_0^T f(t)dt$ such that $f : [0, T] \to R^n$ is a measurable selection for F_α for all $\alpha \in [0,1]$.

Definition 4. A strongly measurable and integrably bounded mapping $F : [0, T] \to E^n$ is said to be integrable over $[0, T]$ if $\int_0^T F(t)dt \in E^n$.

Note that if $F : [0, T] \to E^n$ is strongly measurable and integrably bounded, then F is integrable. Further if

$F : [0,T] \to E^n$ is continuous, then it is integrable.

Theorem 2. [43]. *Let $F, G : [0,T] \to E^n$ be integrable and $c \in [0,T]$, $\lambda \in R$. Then*

a) $\int\limits_0^T F(t)dt = \int\limits_0^c F(t)dt + \int\limits_c^T F(t)dt$;

b) $\int\limits_0^T F(t) + G(t)dt = \int\limits_0^T F(t)dt + \int\limits_0^T G(t)dt$;

c) $\int\limits_0^T \lambda F(t)dt = \lambda \int\limits_0^T F(t)dt$;

d) $D(F,G)$ *is integrable;*

e) $D\left(\int\limits_0^T F(t)dt, \int\limits_0^T G(t)dt \right) \le \int\limits_0^T D(F(t), G(t))dt$.

Consider the following control linear fuzzy differential inclusions

$$\dot{x} \in A(t)x + G(t,w), \ x(t_0) = x_0, \qquad (1)$$

and the following nonlinear fuzzy differential inclusions

$$\dot{x} \in F(t,x,w), \quad x(t_0) = x_0,, \qquad (2)$$

where $\dot{x}$ means dx / dt; $t \in R_+$ is the time; $x \in R^n$ is the state; $w \in R^m$ is the control; $A(t)$ is (n×n)- dimensional matrix-valued function; $G : R_+ \times R^m \to E^n$, $F : R_+ \times R^n \times R^m \to E^n$ are the set-valued functions.

Let

$$W : R_+ \to conv(R^m) \qquad (3)$$

be the measurable multivalued map.

Definition 5. Set *LW* of all single-valued branches of the multivalued map $W(\cdot)$ is the set of the possible controls.

Obviously, the control fuzzy differential inclusion (2) turns into the ordinary fuzzy differential inclusion

$$\dot{x} \in \Phi(t,x), \ x(t_0) = x_0, \qquad (4)$$

if the control $\tilde{w}(\cdot) \in LW$ is fixed and $\Phi(t,x) \equiv F(t, x, \tilde{w}(t))$

The fuzzy differential inclusions (3) has the fuzzy R-solution, if right-hand side of the fuzzy differential inclusion (3) satisfies some conditions [59].

Let $X(t)$ denotes the fuzzy R-solution of the differential inclusion (3), then $X(t,w)$ denotes the fuzzy R-solution of the control differential inclusion (2) for the fixed $w(\cdot) \in LW$.

Definition 6. The set

$$Y(T) = \{ X(T,w) : w(\cdot) \in LW \}$$

be called the attainable set of the fuzzy system (2).

2.2. The some properties of the R-solution

In this section, we consider the some properties of the R-solution of the control fuzzy differential inclusion (1). Let the following condition is true.

Condition A:
A1. $A(\cdot)$ is measurable on $[t_0, T]$;

A2. The norm $\|A(t)\|$ of the matrix $A(t)$ is integrable on $[t_0, T]$;

A3. The multivalued map $W : [t_0, T] \to conv(R^m)$ is measurable on $[t_0, T]$;

A4. The fuzzy map $G : R_+ \times R^m \to E^n$ satisfies the conditions
1) measurable in *t*;
2) continuous in *w*;

A5. There exist $v(\cdot) \in L_2[t_0, T]$ and $l(\cdot) \in L_2[t_0, T]$ such that

$$|W(t)| \le v(t), \ |G(t,w)| \le l(t)$$

almost everywhere on $[t_0, T]$.

A6. The set $Q(t) = \{ G(t, w(t)) : w(\cdot) \in LW \}$ is compact and convex for almost every $[t_0, T]$, i.e. $Q(t) \in conv(E^n)$.

Theorem 3. *Let the condition A is true.*
Then for every $w(\cdot) \in LW$ there exists the fuzzy R- solution $X(\cdot, w)$ such that

1). *the fuzzy map $X(\cdot, w)$ has form*

$$X(t,w) = \Phi(t)x_0 + \Phi(t)\int\limits_{t_0}^t \Phi^{-1}(s)G(s, w(s))ds,$$

where $t \in [t_0, T]$; $\Phi(t)$ is Cauchy matrix of the differential equation $\dot{x} = A(t)x$;

2). $X(t,w) \in E^n$ *for every $t \in [t_0, T]$;*

3). *the fuzzy map $X(\cdot, w)$ is the absolutely continuous fuzzy map on $[t_0, T]$.*

Proof. The proof is easy consequence of the [32,37,40,59] and theorem 1.

Theorem 4. *Let the condition A is true.*
Then the attainable set Y(T) is compact and convex.

Proof. The proof is easy consequence of the [32,37, 40,59] and theorem 1.

We obtained the basic properties of the fuzzy R- solution of systems (1). Now, we consider the some control

fuzzy problems.

3. The Optimal Time Problems

Consider the following optimal control problem: it is necessary to find the minimal time T and the control $w^*(\cdot) \in LW$ such that the fuzzy R-solution of system (1) satisfies one of the conditions:

$$X(T, w^*) \cap S_k \neq \varnothing, \qquad (5)$$

$$X(T, w^*) \subset S_k, \qquad (6)$$

$$X(T, w^*) \supset S_k, \qquad (7)$$

where $S_k \in E^n$ is the terminal set.

Clearly, these time optimal problems are different from the ordinary time optimal problem by that here control object has the volume.

Definition 6. We shall say that the pair $\left(w^*(\cdot), X(\cdot, w^*)\right)$ satisfies the maximum principle on $[t_0, T]$, if there exists the vector-function $\psi(\cdot)$, which is the solution of the system

$$\dot\psi \in -A^T(t)\psi, \ \psi(T) \in S_1(0)$$

and the following conditions are true
1) the maximum condition

$$C\left(\left[G(t, w^*(t))\right], \psi(t)\right) = \max_{w \in W(t)} C\left(\left[G(t, w)\right], \psi(t)\right)$$

almost everywhere on $[t_0, T]$;
2) the transversal condition:
a) in the case (5):

$$C\left(\left[X(T, w^*)\right], \psi(T)\right) = -C\left(\left[S_k\right], -\psi(T)\right);$$

b) in the case (6): for all $\alpha \in [0,1]$

$$C\left(\left[X(T, w^*)\right]^\alpha, \psi(T)\right) \leq C\left(\left[S_k\right]^\alpha, \psi(T)\right)$$

and there exists $\beta \in [0,1]$ such that

$$C\left(\left[X(T, w^*)\right]^\beta, \psi(T)\right) = C\left(\left[S_k\right]^\beta, \psi(T)\right);$$

c) in the case (7): for all $\beta \in [0,1]$

$$C\left(\left[X(T, w^*)\right]^\alpha, -\psi(T)\right) \leq C\left(\left[S_k\right]^\alpha, -\psi(T)\right)$$

and there exists $\beta \in [0,1]$ such that

$$C\left(\left[X(T, w^*)\right]^\beta, -\psi(T)\right) = C\left(\left[S_k\right]^\beta, -\psi(T)\right).$$

Clearly, that there cases of the transversal condition of the maximum principle correspond to the three cases of the time optimal problems.

Theorem 5. (necessary optimal condition). *Let the condition A are true and the pair* $\left(T, w^*(\cdot)\right)$ *is optimality. Then the pair* $\left(w^*(\cdot), X(\cdot, w^*)\right)$ *satisfies the maximum principle on* $[t_0, T]$.

Proof. The proof is easy consequence of the [32,37, 40].

Example. Consider the following control linear fuzzy differential inclusions

$$\dot x \in \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + w + F, \ x(0) = 0,$$

where $x = (x_1, x_2)^T$ is the state; $w = (w_1, w_2)^T \in W = S_1(0)$ is the control; $F \in E^2$ is the fuzzy set, where

$$\upsilon(f) = \begin{cases} 1 - 4f_1^2 - 9f_2^2, & 4f_1^2 + 9f_2^2 \leq 1 \\ 0, & 4f_1^2 + 9f_2^2 > 1 \end{cases}.$$

Consider the following optimal control problem: it is necessary to find the minimal time T and the control $w^*(\cdot) \in LW$ such that the fuzzy R-solution of system satisfies of the conditions:

$$X(T, w^*) \cap S_k \neq \varnothing$$

where $S_k \in E^2$ is the terminal set such, that

$$\sigma(x) = \begin{cases} \sqrt{1 - (x_1 - 2\pi)^2 - (x_2 - 1)^2}, & x \in Q, \ x_2 \geq 1 \\ \sqrt{1 - (x_1 - 2\pi)^2} & x \in Q, \ -1 < x_2 < 1 \\ \sqrt{1 - (x_1 - 2\pi)^2 - (x_2 + 1)^2} & x \in Q, \ x_2 \leq -1 \\ 0 & x \notin Q \end{cases}$$

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2 : \begin{array}{c} 2\pi - 1 \leq x_1 \leq 2\pi + 1 \\ \sqrt{1 - (x_1 - 2\pi)^2} - 1 \leq x_2 \leq \\ \leq \sqrt{1 - (x_1 - 2\pi)^2} + 1 \end{array} \right\}.$$

Obviously, the optimal pair $T = 2\pi$ and $w^*(t) = (\cos(t), -\sin(t))$ satisfy of the conditions of the theorem 5:

1) $\left(w^*(t), \psi(t)\right) = C(W, \psi(t))$ for a.e. $t \in [0, 2\pi]$;

2) $C\left(\left[X(T, w^*)\right], \psi(T)\right) = -C\left(\left[S_k\right], -\psi(T)\right),$

where $\psi(t) = (\cos(t), -\sin(t))^T$ for a.e. $t \in [0, 2\pi]$,

$$\left[X(T, w^*)\right] = (T\cos(T), -T\sin(T))^T = (2\pi, 0)^T,$$
$$\left[S_k\right] = \left\{(x_1, x_2)^T : x_1 = 2\pi, \ -1 \leq x_2 \leq 1\right\}.$$

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