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Enlarged Gradient Observability for Distributed Parabolic Systems: HUM Approach

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Abstract

This paper is focused on studying an important concept of the system analysis, which is the regional enlarged observability or constrained observability of the gradient for distributed parabolic systems evolving in the spatial domain Ω. We will explore an approach based on the Hilbert Uniqueness Method (HUM), which can reconstruct the initial gradient state between two prescribed functions f_1 and f_2 only in a critical subregion ω of Ω without the knowledge of the state. Finally, the obtained results are illustrated by numerical simulations.

Keywords

Distributed Systems, Parabolic Systems, Regional Enlarged Observability, Gradient Reconstruction, HUM Approach

1. Introduction

Control problem of partial differential equation (PDE) arises in many different contexts and engineering applications. A prototypical problem is that of observability, which is one of the most fundamental concepts in mathematical control theory, and has been the object of various works (see [1], [2], [3]), whose the aim is the possibility to reconstruct the initial state of the distributed system based on partial measurements taken on the system by means of tools called sensors. This concept depends on a very sensitive way of the class of PDE under consideration, in particular, the case of the heat and wave equations.

For distributed parameter systems, the concept of regional observability was introduced by El Jai *et al.* and interesting results have been obtained, whose target of interest is not fully the geometrical evolution domain Ω, but just in an internal subregion ω of Ω (see [4], [5]) or on a part of the boundary $\partial\Omega$ of Ω (see [6], [7]). Later the notion of regional gradient observability was developed (see [8]); it concerns the reconstruction of the initial state gradient only in a critical subregion of the system without the knowledge of the state. This concept finds its applications for many real problems. For example, the problem of estimating the energy exchanges between a casting plasma on a plane target which is perpendicular to the direction of the flow sub-diffusion process from measurements carried out by internal thermocouples.

Here we are interested in the concept of the regional enlarged observability of the gradient, which consists in reconstructing the initial gradient state between two prescribed profiles P_1 and P_2 only in a critical subregion interior of the evolution domain without the knowledge of the state. The introduction of this concept is motivated by many real problems. This is the case, for example, of the biological treatment of wastewater using a fixed bed bioreactor. The process has to regulate the substrate concentration of the bottom of the reactor between two prescribed levels. This concept was studied using two approaches where the first one is based on subdifferential techniques and the second one uses the Lagrangian multiplier method (see [9], [10]). In this work, we solve this problem using an extension of the Hilbert Uniqueness Method (HUM) developed by Lions (see [11], [12]).

The paper is structured as follows. Section 2 we recall the regional enlarged gradient observability of a linear parabolic system, then we give some definition and properties related to this notion. Section 3 concerns a reconstruction approach using an extension of the Hilbert Uniqueness Method. Section 4 we develop a numerical approach, which is illustrated by simulations that lead to some conjectures.

2. Problem Statement

Let Ω be an open bounded domain in $\mathbb{R}^n$ ($n = 1,2,3$), with a regular boundary $\partial\Omega$. For $T > 0$, let's consider $Q = \Omega \times [0,T]$ and $\Sigma = \partial\Omega \times [0,T]$. We consider the following system

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = Ay(x,t) & \text{in} \quad Q \\ y(x,0) = y_0(x) & \text{in} \quad \Omega \\ y(\xi,t) = 0 & \text{on} \quad \Sigma, \end{cases} \tag{1}$$

where A is a second-order linear differential operator with compact resolvent which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$. We assume that $y_0 \in H_0^1(\Omega)$ is unknown. The observation space is $\mathcal{O} = L^2(0,T;\mathbb{R}^q)$.

The measurements are obtained by the output function given by

$$z(t) = Cy(.,t), \quad t \in [0,T], \tag{2}$$

where C is called the observation operator, linear (possibly unbounded) depending on the structure and the number q of the considered sensors, with dense $\mathcal{S}$-invariant domain $D(C) \subseteq H_0^1(\Omega)$. One of the most popular examples equations with unbounded observation operator is a system of a linear partial differential equation which describes by pointwise sensors.

Moreover, the system (1) is autonomous the output function can be expressed by

$$z(t) = CS(t)y_0 = K(t)y_0, t \in [0,T], \tag{3}$$

where $K : H_0^1(\Omega) \to \mathcal{O}$ is linear operator. To obtain the adjoint operator of K, we have

Case 1. ***C*** **is bounded (e.g. zone sensors)**

We denote $C : H_0^1(\Omega) \to \mathbb{R}^q$, and C^* its adjoint. We get that the adjoint operator of K can be given by

$$K^* : \mathcal{O} \to H_0^1(\Omega)$$
$$z^* \to \int_0^T S^*(t)^* z^*(t)\mathrm{d}t.$$

Case 2. ***C*** **is unbounded (e.g. pointwise sensors)**

In this case, we have $C : D(C) \subseteq H_0^1(\Omega) \to \mathbb{R}^q$, with C^* denote its adjoint. Based on the works (see [13], [14], [15]), to state our results, we have to make the following assumptions:

(H_1) $CS(t)$ can be extended to a bounded linear operator $\overline{CS(t)}$ in $\mathcal{L}(H_0^1(\Omega), \mathcal{O})$;

(H_2) $(CS)^*$ exists and $(CS)^* = S^*C^*$.

Extend K by $K(t)y_0 = \overline{CS(t)}y_0$, with $K \in \mathcal{L}(H_0^1(\Omega), \mathcal{O})$. Then the adjoint operator of K can be defined as

$$K^* : D(K^*) \subseteq \mathcal{O} \to H_0^1(\Omega)$$
$$z^* \to \int_0^T S^*(t)C^*z^*(t)\mathrm{d}t.$$

For ω a subregion of Ω with a positive Lebesgue measure, let χ_ω be the restriction function defined by

$$\chi_\omega : (L^2(\Omega))^n \to (L^2(\omega))^n$$
$$y \to \chi_\omega y = y_\omega,$$

with the adjoint χ_ω^* given by

$$\chi_\omega^* y = \begin{cases} y & \text{in} \quad \omega \\ 0 & \text{in} \quad \Omega \setminus \omega. \end{cases}$$

Let's consider the operator

$$\nabla : H_0^1(\Omega) \to (L^2(\Omega))^n$$
$$y \to \nabla y = \left(\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n}\right).$$

Its adjoint is given by

$$\nabla^* : (L^2(\Omega))^n \to H_0^1(\Omega)$$
$$y \to \nabla^* y = v,$$

where v is the solution of the following Dirichlet problem

$$\begin{cases} \Delta v = -\operatorname{div}(y) & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4}$$

We recall that a sensor is conventionally defined by a couple (D, f), where D is its spatial support represented by a nonempty part of $\overline{\Omega}$ and f is the spatial distribution of the information on the support D. Then the output function (2) can be written in the following form

$$z(t) = \int_D y(x,t) f(x)\,\mathrm{d}x. \tag{5}$$

A sensor may be pointwise (internal or boundary) if $D = \{b\}$ with $b \in \overline{\Omega}$ and $f = \delta(b - .)$, where δ is the Dirac mass concentrated in b, and the sensor is then denoted by (b, δ_b). In this case, the operator C is unbounded and the output function (2) can be written in the form

$$z(t) = y(b,t). \tag{6}$$

We also recall that the system (1) together with the output (2) is said to be exactly (respectively weakly) gradient observable in ω if $\operatorname{Im}(\chi_\omega \nabla K^*) = (L^2(\omega))^n$ (respectively $\overline{\operatorname{Im}(\chi_\omega \nabla K^*)} = (L^2(\omega))^n$). For more details, we refer the reader to (see [8]).

Let $(\alpha_i(.))_{i=1}^n$ and $(\beta_i(.))_{i=1}^n$ be two functions defined in $(L^2(\omega))^n$ such that $\alpha_i(.) \le \beta_i(.)$ a.e. in ω for all $1 \le i \le n$. Throughout the paper we set

$$[\alpha(.), \beta(.)] = \left\{ (y_1, \cdots, y_n) \in (L^2(\omega))^n \,\middle|\, \alpha_i(.) \le y_i(.) \le \beta_i(.) \quad \text{a.e. in } \omega \ \forall i \in \{1, \cdots, n\} \right\}.$$

Definition 1. *The system* (1) *together with the output* (2) *is said to be* $[\alpha(.), \beta(.)]$*-gradient observable in* ω *if*

$$(\operatorname{Im} \chi_\omega \nabla K^*) \cap [\alpha(.), \beta(.)] \ne \varnothing.$$

Definition 2. *The sensor* (D, f) *is said to be* $[\alpha(.), \beta(.)]$*-gradient strategic in* ω *if the observed system is* $[\alpha(.), \beta(.)]$*-gradient observable in* ω.

Remark 1.

- *If the system* (*1*) *together with the output* (*2*) *is* $[\alpha(.), \beta(.)]$*-gradient observable in* ω_1 *then it is* $[\alpha(.), \beta(.)]$*-gradient observable in any subregion* $\omega_2 \subset \omega_1$.
- *If the system* (1) *together with the output* (2) *is exactly gradient observable in* ω *then it is* $[\alpha(.), \beta(.)]$*-gradient observable in* ω.

Proposition 1. *We have the equivalence between the following statements.*

1. *The system* (1) *together with the output* (2) *is* $[\alpha(.), \beta(.)]$*-gradient observable in* ω.
2. $Ker(K\nabla^* \chi_\omega^*) \cap [\alpha(.), \beta(.)] = \{0\}$.

Proof. (1) $\Rightarrow$ (2)

We shall show that

$$\operatorname{Im}\left(\chi_\omega \nabla K^*\right) \cap \left[\alpha(.), \beta(.)\right] \neq \varnothing \quad \Rightarrow \quad Ker\left(K\nabla^* \chi_\omega^*\right) \cap \left[\alpha(.), \beta(.)\right] = \{0\}$$

Suppose that

$$Ker\left(K\nabla^* \chi_\omega^*\right) \cap \left[\alpha(.), \beta(.)\right] \neq \{0\}$$

Let's consider $y \in Ker\left(K\nabla^* \chi_\omega^*\right) \cap \left[\alpha(.), \beta(.)\right]$, such that $y \neq 0$. Then $y \in Ker\left(K\nabla^* \chi_\omega^*\right)$ and $y \in \left[\alpha(.), \beta(.)\right]$. We have $Ker\left(K\nabla^* \chi_\omega^*\right) = \operatorname{Im}\left(\chi_\omega \nabla K^*\right)^{\perp}$, thus $y \in \operatorname{Im}\left(\chi_\omega \nabla K^*\right)^{\perp}$ such that $y \neq 0$.

Therefore $y \notin \operatorname{Im}\left(\chi_\omega \nabla K^*\right)$.

Then

$$Ker\left(K\nabla^* \chi_\omega^*\right) \cap \left[\alpha(.), \beta(.)\right] \subset \left(L^2(\omega)\right)^n \setminus \operatorname{Im}\left(\chi_\omega \nabla K^*\right).$$

Hence

$$\operatorname{Im}\left(\chi_\omega \nabla K^*\right) \subset \left(L^2(\omega)\right)^n \setminus Ker\left(K\nabla^* \chi_\omega^*\right) \cup \left(L^2(\omega)\right)^n \setminus \left[\alpha(.), \beta(.)\right].$$

We have

$$\operatorname{Im}\left(\chi_\omega \nabla K^*\right) \subset \left(L^2(\omega)\right)^n \setminus Ker\left(K\nabla^* \chi_\omega^*\right),$$

accordingly

$$\operatorname{Im}\left(\chi_\omega \nabla K^*\right) \cap Ker\left(K\nabla^* \chi_\omega^*\right) = \varnothing,$$

then

$$\operatorname{Im}\left(\chi_\omega \nabla K^*\right) \cap \operatorname{Im}\left(\chi_\omega \nabla K^*\right)^{\perp} = \varnothing \quad (\text{Absurd}).$$

Since

$$\operatorname{Im}\left(\chi_\omega \nabla K^*\right) \subset \left(L^2(\omega)\right)^n \setminus \left[\alpha(.), \beta(.)\right]$$

we have

$$\operatorname{Im}\left(\chi_\omega \nabla K^*\right) \cap \left[\alpha(.), \beta(.)\right] = \varnothing \quad (\text{Absurd}).$$

Consequently

$$Ker\left(K\nabla^* \chi_\omega^*\right) \cap \left[\alpha(.), \beta(.)\right] = \{0\}.$$

(2) $\Rightarrow$ (1)

We shall show that

$$Ker\left(K\nabla^* \chi_\omega^*\right) \cap \left[\alpha(.), \beta(.)\right] = \{0\} \quad \Rightarrow \quad \operatorname{Im}\left(\chi_\omega \nabla K^*\right) \cap \left[\alpha(.), \beta(.)\right] \neq \varnothing$$

Suppose that

$$Ker\left(K\nabla^* \chi_\omega^*\right) \cap \left[\alpha(.), \beta(.)\right] = \{0\}.$$

Let's consider

$$y \in Ker\left(K\nabla^* \chi_\omega^*\right) \cap \left[\alpha(.), \beta(.)\right],$$

then

$$y \in Ker\left(K\nabla^* \chi_\omega^*\right) \text{ and } y \in \left[\alpha(.), \beta(.)\right], \text{ such that } y = 0.$$

We have

$$Ker\left(K\nabla^{*}\chi_{\omega}^{*}\right)=\operatorname{Im}\left(\chi_{\omega}\nabla K^{*}\right)^{\perp}, \text{ so } y\in\operatorname{Im}\left(\chi_{\omega}\nabla K^{*}\right)^{\perp} \text{ such that } y=0,$$

hence

$$y\in\operatorname{Im}\left(\chi_{\omega}\nabla K^{*}\right) \text{ and } y\in\left[\alpha(.),\beta(.)\right].$$

Thus

$$\operatorname{Im}\left(\chi_{\omega}\nabla K^{*}\right)\cap\left[\alpha(.),\beta(.)\right]\neq\varnothing,$$

which shows that the system (1) together with the output (2) is $\left[\alpha(.),\beta(.)\right]$-gradient observable in ω.

3. HUM Approach

In this section, we present an approach that allows the reconstruction of the initial gradient state in $\left[\alpha(.),\beta(.)\right]$. The approach constitutes an extension of the Hilbert Uniqueness Method developed by Lions (see [11]) to the case of regional enlarged observability of the gradient. Let the initial state gradient decomposed in the following form

$$y_0=\begin{cases} y_0^1 & \text{in } \left[\alpha(.),\beta(.)\right] \\ y_0^2 & \text{in } \left(L^2(\Omega)\right)^n\setminus\left[\alpha(.),\beta(.)\right] \end{cases} \tag{7}$$

In the sequel our object is the reconstruction of the component y_0^1 in $\left[\alpha(.),\beta(.)\right]$, let G be defined by

$$G=\left\{g\in\left(L^2(\Omega)\right)^n \middle| g=0 \text{ in } \left(L^2(\Omega)\right)^n\setminus\left[\alpha(.),\beta(.)\right]\right\}\cap\left\{\nabla f\,|\,f\in H_0^1(\Omega)\right\}. \tag{8}$$

For $\phi_0\in H_0^1(\Omega)$, we consider the system

$$\begin{cases} \dfrac{\partial\phi(x,t)}{\partial t}=A\phi(x,t) & \text{in } Q \\ \phi(x,0)=\phi_0(x) & \text{in } \Omega \\ \phi(\xi,t)=0 & \text{on } \Sigma \end{cases} \tag{9}$$

which admits a unique solution $\phi\in L^2\left(0,T;H_0^1(\Omega)\right)\cap C\left(\Omega\times[0,T]\right)$ (see [16]). Let us go further in the state reconstruction by considering various types of sensors.

3.1. Pointwise Sensors

In this case, the output function is given by

$$z(t)=y(b,t),\ t\in[0,T] \tag{10}$$

where $b\in\Omega$ denote the given location of the sensor.

For $\tilde{\phi}_0\in G$, there exists a unique $\phi_0\in H_0^1(\Omega)$ such that $\tilde{\phi}_0=\nabla\phi_0$. Then we consider the semi-norm on G be defined by

$$\tilde{\phi}_0\mapsto\left\|\tilde{\phi}_0\right\|_G=\left[\int_0^T\left(\sum_{k=1}^{n}\frac{\partial\phi}{\partial x_k}(b,t)\right)^2 \mathrm{d}t\right]^{\frac{1}{2}}, \tag{11}$$

where ϕ the solution of (9). We consider the following retrograde system

$$\begin{cases} -\dfrac{\partial \psi(x,t)}{\partial t} = A^{*}\psi(x,t) + \sum_{k=1}^{n} \dfrac{\partial \phi}{\partial x_k}(b,t)\delta(x-b) & \text{in } Q \\ \psi(x,T) = 0 & \text{in } \Omega \\ \dfrac{\partial \psi(\xi,t)}{\partial \nu_{A^{*}}} = 0 & \text{on } \Sigma, \end{cases} \tag{12}$$

which admits a unique solution $\psi \in L^2\left(0,T;H_0^1(\Omega)\right)$ (see [16]).

Let the operator Λ be defined by

$$\begin{aligned} \Lambda : G &\to G^{*} \\ \tilde{\phi}_0 &\to \Lambda\tilde{\phi}_0 = \mathcal{P}\left(\Psi(0)\right) \end{aligned} \tag{13}$$

where $\mathcal{P} = \chi_\omega^{*}\chi_\omega$ and $\Psi(0) = \left(\psi(0),\cdots,\psi(0)\right)$.

Let's consider the system

$$\begin{cases} -\dfrac{\partial \overline{z}(x,t)}{\partial t} = A^{*}\overline{z}(x,t) + \sum_{k=1}^{n} \dfrac{\partial y}{\partial x_k}(b,t)\delta(x-b) & \text{in } Q \\ \overline{z}(x,T) = 0 & \text{in } \Omega \\ \dfrac{\partial \overline{z}(\xi,t)}{\partial \nu_{A^{*}}} = 0 & \text{on } \Sigma, \end{cases} \tag{14}$$

If $\tilde{\phi}_0$ is chosen such that $\overline{z}(0) = \psi(0)$ in ω, then the system (14) could be seen as an adjoint of the system (1) and our problem of the enlarged gradient observability is to solve the equation

$$\Lambda\tilde{\phi}_0 = \mathcal{P}\left(\overline{Z}(0)\right), \tag{15}$$

where $\overline{Z}(0) = \left(\overline{z}(0),\cdots,\overline{z}(0)\right)$, with $\overline{z}$ is the solution of the system (14).

Proposition 2. *If the system* (1) *together with the output* (2) *is* $\left[\alpha(.),\beta(.)\right]$*-gradient observable in* ω, *then the equation* (15) *admits a unique solution* $\tilde{\phi}_0 \in G$, *which coincides with the initial gradient state* y_0^1 *to be observed in* $\left[\alpha(.),\beta(.)\right]$.

Proof. **1.** Firstly, we show that if the system (1) together with the output (2) is $\left[\alpha(.),\beta(.)\right]$-gradient observable in ω, then (11) defines a norm on G.

Let's consider $\tilde{\phi}_0 \in G$, we have

$$\left\|\tilde{\phi}_0\right\|_G = 0 \Rightarrow \sum_{i=1}^{\infty} \mathrm{e}^{\lambda_i t}\left\langle \phi_0, \varphi_i \right\rangle \sum_{k=1}^{n} \frac{\partial \phi_i}{\partial x_k}(b) = 0 \quad \text{a.e. in } [0,T].$$

Then

$$\left\langle \phi_0, \varphi_i \right\rangle \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial x_k}(b) = 0, \quad \forall i.$$

Since the observed system is $\left[\alpha(.),\beta(.)\right]$-gradient observable in ω, we obtain $\sum_{k=1}^{n} \dfrac{\partial \varphi_i}{\partial x_k}(b) \neq 0, \quad \forall i.$

Then $\langle \phi_0, \varphi_i \rangle = 0$, hence $\phi_0 = 0$. Consequently $\tilde{\phi}_0 = 0$. Thus (11) is a norm.

2. Now let us prove that (15) has a unique solution. Equation (15) admits a unique solution if the operator Λ is an isomorphism.

Indeed, multiplying (12) by $\frac{\partial \phi}{\partial x_k}$ and integrating the result over Q, we obtain

$$\left\langle \frac{\partial \phi}{\partial x_k}(x,t), \frac{\partial \psi}{\partial t}(x,t) \right\rangle_{L^2(Q)} = \left\langle \frac{\partial \phi}{\partial x_k}(x,t), -A^* \psi(x,t) \right\rangle_{L^2(Q)} - \left\langle \frac{\partial \phi}{\partial x_k}(x,t), \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\delta(x-b) \right\rangle_{L^2(Q)}.$$

which gives

$$\left[\left\langle \frac{\partial \phi}{\partial x_k}(x,t), \psi(x,t) \right\rangle_{L^2(\Omega)} \right]_0^T - \left\langle \frac{\partial}{\partial x_k}\left(\frac{\partial \phi}{\partial t}(x,t)\right), \psi(x,t) \right\rangle_{L^2(Q)}$$
$$= \left\langle \frac{\partial \phi}{\partial x_k}(x,t), -A^* \psi(x,t) \right\rangle_{L^2(Q)} - \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\,\mathrm{d}t.$$

With the initial condition, we have

$$\left\langle \frac{\partial \phi}{\partial x_k}(x,0), \psi(x,0) \right\rangle_{L^2(\Omega)} = \left\langle A \frac{\partial \phi}{\partial x_k}(x,t), \psi(x,t) \right\rangle_{L^2(Q)}$$
$$- \left\langle \frac{\partial \phi}{\partial x_k}(x,t), A^* \psi(x,t) \right\rangle_{L^2(Q)} - \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\,\mathrm{d}t$$

Using the Green formula, we obtain

$$\left\langle \frac{\partial \phi}{\partial x_k}(x,0), \psi(x,0) \right\rangle_{L^2(\Omega)} = \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\,\mathrm{d}t.$$

Hence

$$\sum_{k=1}^{n} \left\langle \frac{\partial \phi}{\partial x_k}(x,0), \psi(x,0) \right\rangle_{L^2(\Omega)} = \sum_{k=1}^{n} \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\,\mathrm{d}t.$$

Thus

$$\langle \tilde{\phi}_0, \Lambda \tilde{\phi}_0 \rangle = \int_0^T \left(\sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t) \right)^2 \mathrm{d}t.$$

Then

$$\langle \tilde{\phi}_0, \Lambda \tilde{\phi}_0 \rangle = \|\tilde{\phi}_0\|_G^2.$$

We deduce that Λ is an isomorphism, consequently the equation (15) has a unique solution $\tilde{\phi}_0 \in G$ which corresponds to the initial state observed in $[\alpha(.), \beta(.)]$.

3.2. Zonal Sensors

Let us come back to the system (1) and suppose that the measurements are given by an internal zone sensor defined by (D, f), with $D \subset \Omega$ and $f \in L^2(D)$.

The system is augmented with the output function

$$z(t)=\int_D y(x,t)f(x)\mathrm{d}x. \tag{16}$$

In this case, we consider the system (9), G is given by (8), and we define a semi-norm on G by

$$\left\|\tilde{\phi}_0\right\|_G=\left[\int_0^T\left(\sum_{k=1}^n\left\langle\frac{\partial\phi}{\partial x_k}(t),f\right\rangle_{L^2(D)}\right)^2\mathrm{d}t\right]^{\frac{1}{2}}. \tag{17}$$

With the system

$$\begin{cases}-\dfrac{\partial\psi(x,t)}{\partial t}=A^*\psi(x,t)+\displaystyle\sum_{k=1}^n\left\langle\frac{\partial\phi}{\partial x_k}(t),f\right\rangle_{L^2(D)}\chi_D f(x) & \text{in } Q\\ \psi(x,T)=0 & \text{in } \Omega\\ \dfrac{\partial\psi(\xi,t)}{\partial\nu_{A^*}}=0 & \text{on } \Sigma,\end{cases} \tag{18}$$

we introduce the operator

$$\begin{aligned}&\Lambda:G\to G^*\\ &\tilde{\phi}_0\to\Lambda\tilde{\phi}_0=\mathcal{P}(\Psi(0)),\end{aligned} \tag{19}$$

where $\mathcal{P}=\chi_\omega^*\chi_\omega$ and $\Psi(0)=(\psi(0),\cdots,\psi(0))$.

Let's consider the system

$$\begin{cases}-\dfrac{\partial\overline{z}(x,t)}{\partial t}=A^*\overline{z}(x,t)+\displaystyle\sum_{k=1}^n\left\langle\frac{\partial y}{\partial x_k}(t),f\right\rangle_{L^2(D)}\chi_D f(x) & \text{in } Q\\ \overline{z}(x,T)=0 & \text{in } \Omega\\ \dfrac{\partial\overline{z}(\xi,t)}{\partial\nu_{A^*}}=0 & \text{on } \Sigma.\end{cases} \tag{20}$$

If $\tilde{\phi}_0$ is chosen such that $\overline{z}(0)=\psi(0)$ in ω, then the system (20) can be seen as an adjoint of the system (1) and our problem of the enlarged gradient observability is to solve the equation

$$\Lambda\tilde{\phi}_0=\mathcal{P}(\overline{Z}(0)), \tag{21}$$

where $\overline{Z}(0)=(\overline{z}(0),\cdots,\overline{z}(0))$, with $\overline{z}$ the solution of the system (20).

Proposition 3. *If the system* (1) *together with the output* (2) *is* $[\alpha(.),\beta(.)]$-*gradient observable in* ω, *then the equation* (21) *has a unique solution* $\tilde{\phi}_0\in G$, *which coincides with the initial gradient state* y_0^1 *observed in* $[\alpha(.),\beta(.)]$.

Proof. The proof is similar to the pointwise case.

4. Numerical Approach

We consider the system (1) observed by a pointwise sensor located in $b\in\Omega$. In the previous section, it was shown that the regional enlarged observability of the initial gradient state in $[\alpha(.),\beta(.)]$ is equivalent, in all cases, to solving the equation

$$\Lambda\tilde{\phi}_0=\mathcal{P}(\overline{Z}(0)), \tag{22}$$

The numerical approximation of (22) is realized when one can have a basis $(\bar{\varphi}_i)_{i\in\mathbb{N}}$ of $(L^2(\Omega))^n$ and the idea is to calculate the components Λ_{ij} of the operator Λ.

Then we approximate the solution of (22) by the linear system

$$\sum_{j=1}^{N}\Lambda_{ij}\tilde{\phi}_{0j}=\bar{Z}_i \quad \text{for} \quad i=1,\cdots,N, \tag{23}$$

where N is the order of approximation and $\bar{Z}_i$ are the components of $\mathcal{P}(\bar{Z}(0))$ $(i=1,\cdots,N)$ in the basis considered.

Let $(\varphi_i)_{i\in\mathbb{N}}$ be a complete set of the eigenfunctions of the operator A in $H_0^1(\Omega)$, which is orthonormal in $L^2(\Omega)$. We also consider a basis of $(L^2(\Omega))^n$ denoted by $(\bar{\varphi}_i)_{i\in\mathbb{N}}$. Then the components Λ_{ij} are the solutions of the following equation, for a pointwise sensor

$$\begin{cases}\sum_{i,j=1}^{\infty}\left\langle\left(\frac{\partial\varphi_k}{\partial x_1},\cdots,\frac{\partial\varphi_k}{\partial x_n}\right),\bar{\varphi}_i\right\rangle\times\left\langle\left(\frac{\partial\varphi_l}{\partial x_1},\cdots,\frac{\partial\varphi_l}{\partial x_n}\right),\bar{\varphi}_j\right\rangle\Lambda_{ij}\\ =\frac{e^{(\lambda_k+\lambda_l)T}-1}{\lambda_k+\lambda_l}\sum_{m,p=1}^{n}\frac{\partial\varphi_k}{\partial x_m}(b)\frac{\partial\varphi_l}{\partial x_p}(b)\\ k,l=1,\cdots,\infty.\end{cases} \tag{24}$$

In the case of a zonal sensor (D,f), we obtain

$$\begin{cases}\sum_{i,j=1}^{\infty}\left\langle\left(\frac{\partial\varphi_k}{\partial x_1},\ldots,\frac{\partial\varphi_k}{\partial x_n}\right),\bar{\phi}_i\right\rangle\times\left\langle\left(\frac{\partial\varphi_l}{\partial x_1},\ldots,\frac{\partial\varphi_l}{\partial x_n}\right),\bar{\varphi}_j\right\rangle\Lambda_{ij}\\ =\frac{e^{(\lambda_k+\lambda_l)T}-1}{\lambda_k+\lambda_l}\sum_{m,p=1}^{n}\left\langle\frac{\partial\varphi_k}{\partial x_m},f\right\rangle_{L^2(D)}\times\left\langle\frac{\partial\varphi_l}{\partial x_p},f\right\rangle_{L^2(D)},\\ k,l=1,\cdots,\infty.\end{cases} \tag{25}$$

Then, we have the following algorithm :

Algorithm.

Step 1: The subregion ω, the location of the sensor *b*.

Choose the function $y_0\in[\alpha(.),\beta(.)]$.

Threshold accuracy ε.

Step 2: Repeat

▷ Solve the system (9) to obtain ϕ.

▷ Solve the system (14) to obtain $\bar{z}$.

▷ Solve the equation (23) to obtain $\tilde{\phi}_0$.

Until $\left\|y_0-\tilde{\phi}_0\right\|^2_{(L^2(\omega))^n}<\varepsilon$.

Step 3: The solution $\tilde{\phi}_0$ corresponds to the initial gradient state to be observed in $[\alpha(.),\beta(.)]$.

5. Simulation Results

Here, we present a numerical example which illustrates the previous algorithm. The obtained results are related to the initial gradient state and the sensors location.

Let's consider the following one-dimensional system in $\Omega = [0,1]$ excited by a pointwise sensor

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} = 0.01 \frac{\partial^2 y(x,t)}{\partial x^2} & \text{in } \Omega \times [0,T] \\ y(x,0) = y_0(x) & \text{in } \Omega \\ y(0,t) = y(1,t) = 0 & \text{in } [0,T], \end{cases} \tag{26}$$

augmented with the output function

$$z(t) = y(b,t), b \in \Omega. \tag{27}$$

The initial gradient state to be reconstructed is

$$y_0(x) = x(2x-1)(x-1).$$

We take $b = 0.18$ and $T = 2$, with

$$\alpha(x) = \left(x - \frac{1}{7}\right)\left(\frac{1}{2}x - \frac{2}{3}\right) \quad \text{and} \quad \beta(x) = \left(x - \frac{2}{3}\right)\left(\frac{1}{30}x - \frac{1}{4}\right).$$

Applying the previous algorithm, we obtain the following results:

- For $\omega =]0.15, 0.70[$

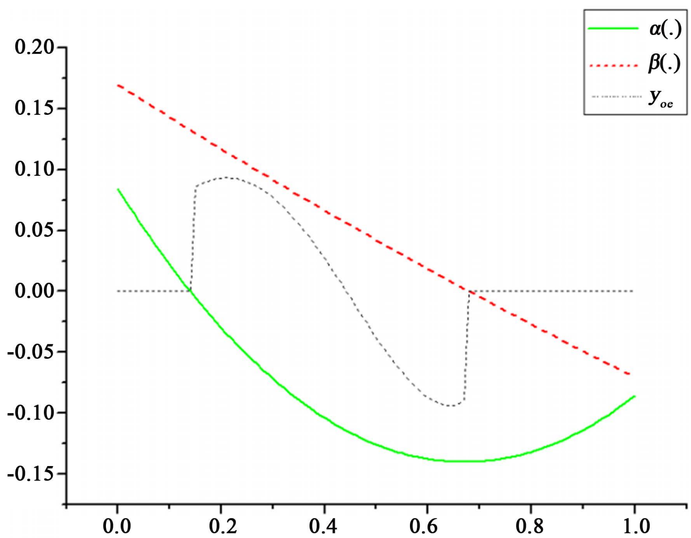

Figure 1. The estimated initial gradient state y_{0e}.

Figure 1 shows that the initial gradient state estimated y_{0e} is between $\alpha(.)$ and $\beta(.)$ in $\omega =]0.15, 0.70[$, then the location of the sensor is $[\alpha(.), \beta(.)]$-gradient strategic in ω.

The initial gradient state y_{oe} is estimated with a reconstruction error

$$y\|_0 - y_{oe}\|^2 = 2.84 \times 10^{-3}$$

- If the sensor is located in $b = 0.32$, we obtain the **Figure 2**.

Figure 2 is showing that the initial gradient state estimated y_{0e} is not between $\alpha(.)$ and $\beta(.)$ in $\omega =]0.15, 0.70[$, this means that the location of the sensor is not $[\alpha(.), \beta(.)]$-gradient strategic in ω.

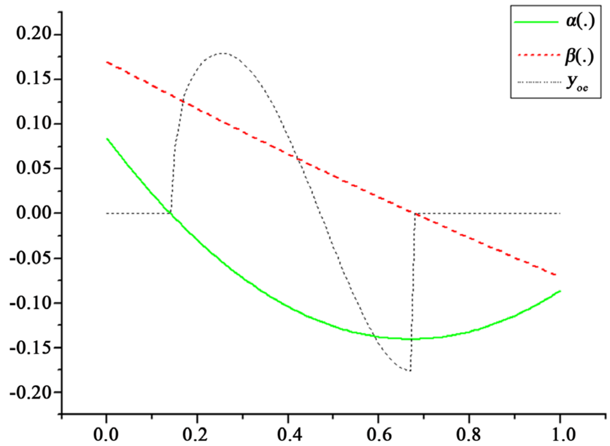

Figure 2. The estimated initial gradient state y_{0e}.

Here numerically we study the dependence of the gradient reconstruction error with respect to the subregion area of ω. We have the following **Table 1**.

Table 1. Relation between the subregion and the reconstruction error.

Subregion	The reconstruction error
]0.15, 0.85[	3.68×10^{-1}
]0.2, 0.7[	1.11×10^{-1}
]0.35, 0.65[	6.03×10^{-2}
]0.10, 0.35[	2.44×10^{-2}
]0.20, 0.30[	6.19×10^{-3}

Table 1 shows how the reconstruction error grows with respect to the subregion area.

The following simulation results show the evolution of the observed gradient error with respect to the sensor location.

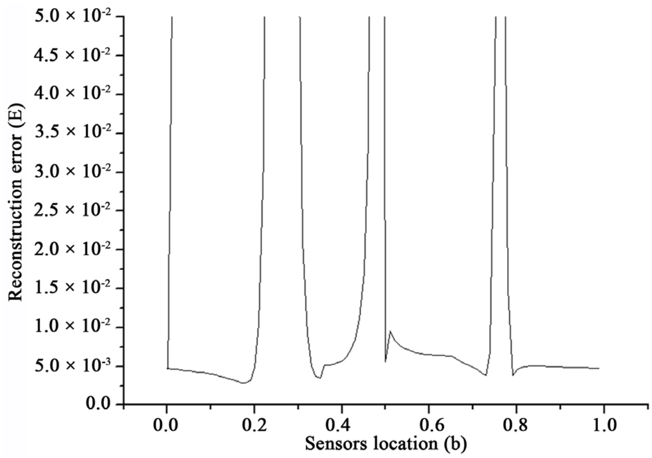

Figure 3. Evolution of the estimated gradient error with respect to the sensor location *b*.

Figure 3 shows how the worst locations of the sensor correspond to a great error, which corresponds to the non-strategic sensor location.

6. Conclusion

In this work, we have considered the problem of regional enlarged observability of the gradient for parabolic linear systems. We explored an approach that leads to the reconstruction of the initial gradient state between two prescribed functions. The obtained results were applied to the head equation in a one-dimen- sional case and illustrated by numerical example and simulations. Future works aim to extend this notion in Γ a part of the boundary $\partial\Omega$ of the system evolution domain Ω.

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