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# An Optimization Problem of Boundary Type for Cooperative Hyperbolic Systems Involving Schrödinger Operator

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## Abstract

In this paper, we consider cooperative hyperbolic systems involving Schrödinger operator defined on $R^n$. First we prove the existence and uniqueness of the state for these systems. Then we find the necessary and sufficient conditions of optimal control for such systems of the boundary type. We also find the necessary and sufficient conditions of optimal control for same systems when the observation is on the boundary.

## Keywords

**Hyperbolic Systems, Schrödinger Operator, Boundary Control Problem, Boundary Observation, Cooperative**

## 1. Introduction

The optimal control problems of distributed systems involving Schrödinger operator have been widely discussed in many papers. One of the first studies was introduced by Serag [1], which discusses $2 \times 2$ cooperative systems of elliptic operator. Further research in this area developed the problem by studying different operator types (elliptic, parabolic, or hyperbolic) or higher system degree as in [2]-[6]. Many boundary control problems have been introduced in [7]-[10].

In [3], we discussed distributed control problem for $2 \times 2$ cooperative hyperbolic systems involving Schrödinger operator.

Here, using the theory of [11], we consider the following $2 \times 2$ cooperative hyperbolic systems involving Schrödinger operator:

$$\begin{cases} \dfrac{\partial^2 y_1(x)}{\partial t^2} + (-\Delta + q)\,y_1 = a y_1 + b y_2 + f_1(x,t), & \text{in } Q, \\[2mm] \dfrac{\partial^2 y_2(x)}{\partial t^2} + (-\Delta + q)\,y_2 = c y_1 + d y_2 + f_2(x,t), & \text{in } Q, \\[2mm] y_1, y_2 \to 0 \quad \text{as} \quad |x| \to \infty, \\[2mm] \left.\dfrac{\partial y_1}{\partial \nu}\right|_\Sigma = \left.\dfrac{\partial y_2}{\partial \nu}\right|_\Sigma = 0, \\[2mm] y_1(x,0) = y_{1,0}(x), \quad y_2(x,0) = y_{2,0}(x), & \text{in } R^n, \\[2mm] \dfrac{\partial y_1(x,0)}{\partial t} = y_{1,1}(x), \quad \dfrac{\partial y_2(x,0)}{\partial t} = y_{2,1}(x), & \text{in } R^n. \end{cases} \tag{1}$$

with $y_1, y_2 \in L^2\left(0,T;V_q\left(R^n\right)\right)$, $\dfrac{\partial y_1}{\partial t}, \dfrac{\partial y_2}{\partial t} \in L^2(Q)$.

where $a$, $b$, $c$ and $d$ are given numbers such that $b$, $c > 0$,

$$\textit{i.e. the system (1) is called cooperative} \tag{2}$$

$$q(x) \text{ is a positive function and tending to } \infty \text{ at infinity,} \tag{3}$$

and $Q = R^n \times ]0,T[$ with boundary $\Sigma = \Gamma \times ]0,T[$.

The model of the system (1) is given by:

$$B(t)\,y(x) = B(t)\big(y_1(x), y_2(x)\big) = \left( \dfrac{\partial^2 y_1}{\partial t^2} + (-\Delta + q)\,y_1 - a y_1 - b y_2, \dfrac{\partial^2 y_2}{\partial t^2} + (-\Delta + q)\,y_2 - c y_1 - d y_2 \right)$$

since $A(t)\,y(x) = \big((-\Delta + q)\,y_1 - a y_1 - b y_2, (-\Delta + q)\,y_2 - c y_1 - d y_2\big)$, $A(t)\,y \in \left(V_q'\left(R^n\right)\right)^2$.

We first prove the existence and uniqueness of the state for these systems, then we introduce the optimality conditions of boundary control, we also discuss them when the observation is on the boundary.

## 2. Some Concepts and Results

Here we shall consider some results about the following eigenvalue problem which introduced in [1] and [12]:

$$\begin{cases} (-\Delta + q)\phi = \lambda(q)\phi & \text{in } R^n \\ \phi(x) \to 0 & \text{as } |x| \to \infty, \quad \phi \succ 0 \end{cases} \tag{4}$$

The associated space is $V_q\left(R^n\right)$, with respect to the norm:

$$\|y\|_q = \left( \int_{R^n} \left[ |\nabla y|^2 + q|y|^2 \right] dx \right)^{1/2} \tag{5}$$

Since the imbedding of $V_q\left(R^n\right)$ into $L^2\left(R^n\right)$ is compact, then the operator $(-\Delta + q)$ considered as an Operator in $L^2\left(R^n\right)$ is positive self-adjoint with compact inverse. Hence its spectrum consists of an infinite sequence of positive eigenvalues, tending to infinity; moreover the smallest one which is called the principal eigenvalue denoted by $\lambda(q)$ is simple and is associated with an eigenfunction which does not change sign in $R^n$. It is characterized by:

$$\lambda(q) \int_{R^n} |y|^2 \, dx \le \int_{R^n} \left[ |\nabla y|^2 + q|y|^2 \right] dx \quad \forall y \in V_q\left(R^n\right) \tag{6}$$

We have:

$$V_q\left(R^n\right) \times V_q\left(R^n\right) \subseteq L^2\left(R^n\right) \times L^2\left(R^n\right) \subseteq V_q'\left(R^n\right) \times V_q'\left(R^n\right)$$

which is continuous and compact.

Let us introduce the space $L^2\left(0,T;V_q\left(R^n\right)\right)$ of measurable function $t \to f(t)$ which is defined on open interval $(0,T)$ and the variable $t \in (0,T)$, $T < \infty$ denotes the time.

On $(0,T)$ with Lebesgue measure $\mathrm{d}t$ we have the norm:

$$\left\| f(t) \right\|_{L^2\left(0,T;V_q\left(R^n\right)\right)} = \left( \int_{(0,T)} \left\| f(t) \right\|^2_{V_q\left(R^n\right)} \mathrm{d}t \right)^{1/2} \prec \infty$$

and the scalar product

$$\left( f(t), g(t) \right)_{L^2\left(0,T;V_q\left(R^n\right)\right)} = \int_{(0,T)} \left( f(t), g(t) \right)_{V_q\left(R^n\right)} \mathrm{d}t ,$$

the space $L^2\left(0,T;V_q\left(R^n\right)\right)$ with the scalar product and the norm above is a Hilbert space.

Analogously, we can define the spaces $L^2\left(0,T;L^2\left(R^n\right)\right) = L^2(Q)$,

with the scalar product:

$$\left( f(t), g(t) \right)_{L^2(Q)} = \int_{(0,T)} \left( f(t), g(t) \right)_{L^2\left(R^n\right)} \mathrm{d}t = \int_Q f(t) \cdot g(t) \mathrm{d}x \mathrm{d}t$$

then we have:

$$L^2\left(0,T;V_q\left(R^n\right)\right) \times L^2\left(0,T;V_q\left(R^n\right)\right) \subseteq L^2(Q) \times L^2(Q) \subseteq L^2\left(0,T;V_q'\left(R^n\right)\right) \times L^2\left(0,T;V_q'\left(R^n\right)\right)$$

## 3. The Existence and Uniqueness for the State of the System (1)

We have the bilinear form:

$$\pi(t;y,\psi) = \frac{1}{b} \int_{R^n} \left[ \nabla y_1 \nabla \psi_1 + q y_1 \psi_1 \right] \mathrm{d}x + \frac{1}{c} \int_{R^n} \left[ \nabla y_2 \nabla \psi_2 + q y_2 \psi_2 \right] \mathrm{d}x - \int_{R^n} y_1 \psi_2 \mathrm{d}x - \frac{d}{c} \int_{R^n} y_2 \psi_2 \mathrm{d}x$$

$$- \frac{a}{b} \int_{R^n} y_1 \psi_1 \mathrm{d}x - \int_{R^n} y_2 \psi_1 \mathrm{d}x, \tag{7}$$

$$y = y_1, y_2, \quad \psi = (\psi_1, \psi_2) \in \left( V_q\left(R^n\right) \right)^2.$$

For all $y$, $\psi \in \left( V_q\left(R^n\right) \right)^2$ the function $t \to \pi(t;y,\psi)$ is measurable on $(0,T)$.

The coerciveness condition of the bilinear form (7) in $\left( V_q\left(R^n\right) \right)^2$ has been proved by Serag [1], by using the conditions for having the maximum principle for cooperative system (1) which have been obtained by Fleckinger [13], and take the form:

$$\begin{cases} a \prec \lambda(q), \quad d \prec \lambda(q), \\ \left( \lambda(q) - a \right)\left( \lambda(q) - d \right) \succ bc \end{cases} \tag{8}$$

that means:

$$\pi(t;y,y) \geq C\left( \left\| y_1 \right\|^2_{q,m} + \left\| y_2 \right\|^2_{q,m} \right), \quad C \succ 0 \tag{9}$$

*Theorem* (3.1):

Under the hypotheses (2) and (9), if $f_1, f_2 \in L^2\left(0,T;V_q'\left(R^n\right)\right)$, $y_{1,0}(x)$, $y_{2,0}(x) \in V_q\left(R^n\right)$ and $y_{1,1}(x)$, $y_{2,1}(x) \in L\left(R^n\right)$, then there exists a unique solution: $y = \{y_1, y_2\} \in \left( L^2\left(0,T;V_q\left(R^n\right)\right) \right)^2$ for system (1).

*Proof*:

Let $\psi \to L(\psi)$ be a continuous linear form defined on $\left(V_q\left(R^n\right)\right)^2$ by:

$$L(\psi) = \frac{1}{b}\int_Q f_1(x,t)\psi_1(x)\mathrm{d}x\mathrm{d}t + \frac{1}{c}\int_Q f_2(x,t)\psi_2(x)\mathrm{d}x\mathrm{d}t$$

$$+\frac{1}{b}\int_{R^n} y_{1,1}(x)\frac{\partial\psi_1(x,0)}{\partial t}\mathrm{d}x + \frac{1}{c}\int_{R^n} y_{2,1}(x)\frac{\partial\psi_2(x,0)}{\partial t}\mathrm{d}x, \quad \forall(\psi_1,\psi_2)\in\left(V_q\left(R^n\right)\right)^2. \tag{10}$$

then by Lax-Milgram lemma, there exists a unique element $y = (y_1, y_2)\in\left(V_q\left(R^n\right)\right)^2$ such that:

$$\pi(t; y,\psi) = L(\psi), \quad \forall \psi = (\psi_1,\psi_2)\in\left(V_q\left(R^n\right)\right)^2 \tag{11}$$

Now, let us multiply both sides of first equation of system (1) by $\frac{1}{b}\psi_1(x)$, and the second equation by:

$\frac{1}{c}\psi_2(x)$ then integration over $Q$, we have:

$$\frac{1}{b}\int_Q\left[\frac{\partial^2 y_1(x)}{\partial t^2}+(-\Delta+q)\,y_1-ay_1-by_2\right]\psi_1\mathrm{d}x\mathrm{d}t = \frac{1}{b}\int_Q f_1(x,t)\psi_1\mathrm{d}x\mathrm{d}t$$

$$\frac{1}{c}\int_Q\left[\frac{\partial^2 y_2(x)}{\partial t^2}+(-\Delta+q)\,y_2-cy_1-dy_2\right]\psi_2\mathrm{d}x\mathrm{d}t = \frac{1}{c}\int_Q f_2(x,t)\psi_2\mathrm{d}x\mathrm{d}t$$

By applying Green's formula:

$$\frac{1}{b}\int_{R^n}\frac{\partial y_1(x,0)}{\partial t}\frac{\partial\psi_1(x,0)}{\partial t}\mathrm{d}x+\frac{1}{b}\int_\Sigma\psi_1\frac{\partial y_1}{\partial t}v\mathrm{d}\Sigma+\frac{1}{b}\int_Q\nabla y_1\nabla\psi_1\mathrm{d}x\mathrm{d}t-\frac{1}{b}\int_\Sigma\psi_1\frac{\partial y_1}{\partial v_A}\mathrm{d}\Sigma+\int_Q\left(\frac{q}{b}y_1-\frac{a}{b}y_1-y_2\right)\psi_1\mathrm{d}x\mathrm{d}t$$

$$=\frac{1}{b}\int_Q f_1(x,t)\psi_1\mathrm{d}x\mathrm{d}t,$$

$$\frac{1}{c}\int_{R^n}\frac{\partial y_2(x,0)}{\partial t}\frac{\partial\psi_2(x,0)}{\partial t}\mathrm{d}x+\frac{1}{c}\int_\Sigma\psi_2\frac{\partial y_2}{\partial t}v\mathrm{d}\Sigma+\frac{1}{c}\int_Q\nabla y_2\nabla\psi_2\mathrm{d}x\mathrm{d}t-\frac{1}{c}\int_\Sigma\psi_2\frac{\partial y_2}{\partial v_A}\mathrm{d}\Sigma+\int_Q\left(\frac{q}{c}y_2-y_1-\frac{d}{c}y_2\right)\psi_1\mathrm{d}x\mathrm{d}t$$

$$=\frac{1}{c}\int_Q f_2(x,t)\psi_c\mathrm{d}x\mathrm{d}t,$$

By sum the two equations we get:

$$\frac{1}{b}\int_{R^n}\frac{\partial y_1(x,0)}{\partial t}\frac{\partial\psi_1(x,0)}{\partial t}\mathrm{d}x+\frac{1}{b}\int_\Sigma\psi_1\frac{\partial y_1}{\partial t}v\mathrm{d}\Sigma-\frac{1}{b}\int_\Sigma\psi_1\frac{\partial y_1}{\partial v_A}\mathrm{d}\Sigma+\frac{1}{c}\int_{R^n}\frac{\partial y_2(x,0)}{\partial t}\frac{\partial\psi_2(x,0)}{\partial t}\mathrm{d}x+\frac{1}{c}\int_\Sigma\psi_2\frac{\partial y_2}{\partial t}v\mathrm{d}\Sigma$$

$$-\frac{1}{c}\int_\Sigma\psi_2\frac{\partial y_2}{\partial v_A}\mathrm{d}\Sigma = \frac{1}{b}\int_{R^n} y_{1,1}(x)\frac{\partial\psi_1(x,0)}{\partial t}\mathrm{d}x+\frac{1}{c}\int_{R^n} y_{2,1}(x)\frac{\partial\psi_2(x,0)}{\partial t}\mathrm{d}x,$$

by comparing the previous equation with (7), (10) and (11) we deduce that:

$$\left.\frac{\partial y_1}{\partial v}\right|_\Sigma = \left.\frac{\partial y_2}{\partial v}\right|_\Sigma = 0$$

$$\frac{\partial y_1(x,0)}{\partial t} = y_{1,1}(x), \quad \frac{\partial y_2(x,0)}{\partial t} = y_{2,1}(x) \quad \text{in } R^n$$

then the proof is complete.

## 4. Formulation of the Control Problem

The space $L^2(\Sigma)\times L^2(\Sigma)$ is the space of controls. For a control $u = (u_1, u_2)\in\left(L_2(\Sigma)\right)^2$, the state

$y(u) = (y_1(u), y_2(u)) \in \left(L^2\left(0,T; V_q\left(R^n\right)\right)\right)^2$  of the system is given by the solution of

$$\begin{cases} \dfrac{\partial^2 y_1(u)}{\partial t^2} + (-\Delta + q) y_1(u) = a y_1(u) + b y_2(u) + f_1 & \text{in } Q, \\[2mm] \dfrac{\partial^2 y_2(u)}{\partial t^2} + (-\Delta + q) y_2(u) = c y_1(u) + d y_2(u) + f_2 & \text{in } Q, \\[2mm] y_1, y_2 \to 0 \quad \text{as} \quad |x| \to \infty, \\[2mm] \dfrac{\partial y_1(u)}{\partial \nu}\bigg|_{\Sigma} = u_1, \quad \dfrac{\partial y_2(u)}{\partial \nu}\bigg|_{\Sigma} = u_2, \\[2mm] y_1(x,0,u) = y_{1,0}(x), \quad y_2(x,0,u) = y_{2,0}(x) & \text{in } R^n, \\[2mm] \dfrac{\partial y_1(x,0,u)}{\partial t} = y_{1,1}(x), \quad \dfrac{\partial y_2(x,0,u)}{\partial t} = y_{2,1}(x) & \text{in } R^n. \end{cases} \qquad (12)$$

with $y_1(u), y_2(u) \in L^2\left(0,T; V_q\left(R^n\right)\right)$, $\dfrac{\partial y_1(u)}{\partial t}, \dfrac{\partial y_2(u)}{\partial t} \in L^2(Q)$.

The observation equation is given by $z(u) = (z_1(u), z_2(u)) = y(u) = (y_1(u), y_2(u))$.

For a given $z_d = (z_{d1}, z_{d2}) \in \left(L^2(Q)\right)^2$, the cost function is given by:

$$J(v) = \|y_1(v) - z_{d1}\|^2_{L^2(Q)} + \|y_2(v) - z_{d2}\|^2_{L^2(Q)} + (Nv, v)_{\left(L^2(\Sigma)\right)^2}. \qquad (13)$$

where $N \in L\left(\left(L^2(\Sigma)\right)^2, \left(L^2(\Sigma)\right)^2\right)$ is hermitian positive definite operator:

$$(Nu, u)_{\left(L^2(\Sigma)\right)^2} \geq \gamma \|u\|^2_{\left(L^2(\Sigma)\right)^2}, \quad \gamma \succ 0 \qquad (14)$$

The control problem then is to find $u = \{u_1, u_2\} \in U_{ad}$ such that $J(u) \leq J(v)$, where $U_{ad}$ is a closed convex subset of $\left(L^2(\Sigma)\right)^2$.

Since the cost function (14) can be written as (see [11]):

$$J(v) = a(v,v) - 2L(v) + \|y(0) - z_d\|^2_{\left(L^2(Q)\right)^2}$$

where $a(v,v)$ is a continuous coercive bilinear form and $L(v)$ is a continuous linear form on $\left(L^2(\Sigma)\right)^2$. Then there exists a unique optimal control $u \in U_{ad}$ such that $J(u) = \inf J(v)$ for all $v \in U_{ad}$ by using the general theory of Lions [11]. Moreover, we have the following theorem which gives the necessary and sufficient conditions of optimality:

*Theorem* (4.1):

Assume that (9) and (14) hold. If the cost function is given by (13), the optimal control $u = (u_1, u_2) \in \left(L_2(\Sigma)\right)^2$ is then characterized by the following equations and inequalities:

$$\begin{cases} \dfrac{\partial^2 p_1(u)}{\partial t^2} + (-\Delta + q) p_1(u) - a p_1(u) - c p_2(u) = y_1(u) - z_{d1} & \text{in } Q, \\[2mm] \dfrac{\partial^2 p_2(u)}{\partial t^2} + (-\Delta + q) p_2(u) - b p_1(u) - d p_2(u) = y_2(u) - z_{d2} & \text{in } Q, \\[2mm] p_1, p_2 \to 0 \quad \text{as} \quad |x| \to \infty, \\[2mm] \dfrac{\partial p_1(u)}{\partial \nu}\bigg|_{\Sigma} = 0, \quad \dfrac{\partial p_2(u)}{\partial \nu}\bigg|_{\Sigma} = 0, \\[2mm] p_1(x,T,u) = p_2(x,T,u) = 0 & \text{in } R^n, \\[2mm] \dfrac{\partial p_1(x,T,u)}{\partial t} = \dfrac{\partial p_2(x,T,u)}{\partial t} = 0 & \text{in } R^n. \end{cases} \qquad (15)$$

with $\quad p_1(u),\, p_2(u) \in L^2\left(0,T;V_q\left(R^n\right)\right),\ \dfrac{\partial p_1(u)}{\partial t},\ \dfrac{\partial p_2(u)}{\partial t} \in L^2(Q)$

$$\left(p(u)+Nu, v-u\right)_{\left(L^2(\Sigma)\right)^2} \geq 0, \quad \forall v = (v_1, v_2) \in U_{ad} \tag{16}$$

together with (12), where $\quad p(u) = \left(p_1(u), p_2(u)\right)$ is the adjoint state.

_Proof_:

The optimal control $\quad u = (u_1, u_2) \in \left(L_2(\Sigma)\right)^2$ is characterized by [11]

$$J'(u)(v-u) \geq 0, \quad \forall v \in U_{ad},$$

Which is equivalent to:

$$\left(y(u)-z_d,\, y(v)-y(u)\right)_{\left(L^2(Q)\right)^2} + \left(Nu, v-u\right)_{\left(L^2(\Sigma)\right)^2} \geq 0$$

*i.e.*

$$\left(y_1(u)-z_{d1},\, y_1(v)-y_1(u)\right)_{L^2(Q)} + \left(y_2(u)-z_{d2},\, y_2(v)-y_2(u)\right)_{L^2(Q)} + \left(Nu, v-u\right)_{\left(L^2(\Sigma)\right)^2} \geq 0 \tag{17}$$

this inequality can be written as:

$$\int_0^T \left[ \left(y_1(u)-z_{d1},\, y_1(v)-y_1(u)\right)_{L^2\left(R^n\right)} + \left(y_2(u)-z_{d2},\, y_2(v)-y_2(u)\right)_{L^2\left(R^n\right)} \right] dt + \left(Nu, v-u\right)_{\left(L^2(\Sigma)\right)^2} \geq 0 \tag{18}$$

Now, since:

$$\left(p, By\right)_{\left(L^2(Q)\right)^2} = \int_0^T \left( p_1(u),\, \dfrac{\partial^2 y_1(u)}{\partial t^2} + (-\Delta+q)\, y_1(u) - ay_1(u) - by_2(u) \right)_{L^2\left(R^n\right)} dt$$

$$+ \int_0^T \left( p_2(u),\, \dfrac{\partial^2 y_2(u)}{\partial t^2} + (-\Delta+q)\, y_2(u) - cy_1(u) - dy_2(u) \right)_{L^2\left(R^n\right)} dt.$$

where

$By(u) = B\left(y_1(u), y_2(u)\right)$

$$= \left( \dfrac{\partial^2 y_1(u)}{\partial t^2} + (-\Delta+q)\, y_1(u) - ay_1(u) - by_2(u),\, \dfrac{\partial^2 y_2(u)}{\partial t^2} + (-\Delta+q)\, y_2(u) - cy_1(u) - dy_2(u) \right).$$

by using Green formula and (12), we have:

$$\left(p, By\right)_{\left(L^2(Q)\right)^2} = \int_0^T \left( \dfrac{\partial^2 p_1(u)}{\partial t^2} + (-\Delta+q)\, p_1(u) - ap_1(u) - cp_2(u),\, y_1(u) \right)_{L^2\left(R^n\right)} dt$$

$$+ \int_0^T \left( \dfrac{\partial^2 p_2(u)}{\partial t^2} + (-\Delta+q)\, p_2(u) - bp(u) - dp_2(u),\, y_2(u) \right)_{L^2\left(R^n\right)} dt$$

$$= \left(B^* p, y\right)_{\left(L^2(Q)\right)^2}.$$

then

$B^* p(u) = B^* \left(p_1(u), p_2(u)\right)$

$$= \left\{ \dfrac{\partial^2 p_1(u)}{\partial t^2} + (-\Delta+q)\, p_1(u) - ap_1(u) - cp_2(u),\, \dfrac{\partial^2 p_2(u)}{\partial t^2} + (-\Delta+q)\, p_2(u) - bp_1(u) - dp_2(u) \right\}.$$

and $\quad A^* p(u) = \left(p_1(u), p_2(u)\right) = \left((-\Delta+q)\, p_1(u) - ap_1(u) - cp_2(u),\, (-\Delta+q)\, p_2(u) - bp_1(u) - dp_2(u)\right)$

since the adjoint equation takes the form [11]: $\dfrac{\partial^2 p(u)}{\partial t^2} + A^* p(u) = y(u) - z_d$

and from theorem (3.1), we have a unique solution $p(u) \in \left(L^2\left(0,T;V_q\left(R^n\right)\right)\right)^2$ which satisfies $p_1(u)$,

$p_2(u) \in L^2\left(0,T;V_q\left(R^n\right)\right)$, $\dfrac{\partial p_1(u)}{\partial t}$, $\dfrac{\partial p_2(u)}{\partial t} \in L^2(Q)$.

This proves system (15).

Now, we transform (18) by using (15) as follows:

$$\int_0^T \left(\frac{\partial^2 p_1(u)}{\partial t^2} + (-\Delta + q)\, p_1(u) - ap_1(u) - cp_2(u), y_1(v) - y_1(u)\right)_{L^2\left(R^n\right)} dt$$

$$+ \int_0^T \left(\frac{\partial^2 p_2(u)}{\partial t^2} + (-\Delta + q)\, p_2(u) - bp_1(u) - dp_2(u), y_2(v) - y_2(u)\right)_{L^2\left(R^n\right)} dt$$

$$+ (Nu, v-u)_{\left(L^2(\Sigma)\right)^2} \geq 0.$$

Using Green formula, we obtain:

$$\int_0^T \left(p_1(u), \left(\frac{\partial^2}{\partial t^2} + (-\Delta + q)\right) y_1(v) - y_1(u)\right)_{L^2\left(R^n\right)} dt + \int_0^T -a\left(p_1(u), y_1(v) - y_1(u)\right)_{L^2\left(R^n\right)} dt$$

$$+ \int_0^T -c\left(p_2(u), y_1(v) - y_1(u)\right)_{L^2\left(R^n\right)} dt + \int_0^T \left(p_2(u), \left(\frac{\partial^2}{\partial t^2} + (-\Delta + q)\right) y_2(v) - y_2(u)\right)_{L^2\left(R^n\right)} dt$$

$$+ \int_0^T -b\left(p_1(u), y_2(v) - y_2(u)\right)_{L^2\left(R^n\right)} dt + \int_0^T -d\left(p_2(u), y_2(v) - y_2(u)\right)_{L^2\left(R^n\right)} dt + (Nu, v-u)_{\left(L^2(\Sigma)\right)^2} \geq 0.$$

Using (12), we have:

$$\left(p(u) + Nu, v-u\right)_{\left(L^2(\Sigma)\right)^2} \geq 0.$$

Thus the proof is complete.

## 5. Formulation of the Problem When the Observation Is on the Boundary

The observation equation is given by:

$$z(u) = \left(z_1(u), z_2(u)\right) = M\left(y(u)\big|_\Sigma\right) = M\left(\left(y_1(u)\big|_\Sigma\right), \left(y_2(u)\big|_\Sigma\right)\right)$$

$$M \in L\left(\left(L^2(\Sigma)\right)^2, \left(L^2(\Sigma)\right)^2\right).$$

This is interpreted as follows [11]: we take the trace of $y(u)$ on $\Sigma$, which is particular in $\left(L^2(\Sigma)\right)^2$. Let this be denoted by $y(u)\big|_\Sigma$.

For a given $z_d = (z_{d1}, z_{d2}) \in \left(L^2(\Sigma)\right)^2$, the cost function is given by:

$$J(v) = \left\| y_1(u)\big|_\Sigma - z_{d1} \right\|_{L^2(\Sigma)}^2 + \left\| y_2(u)\big|_\Sigma - z_{d2} \right\|_{L^2(\Sigma)}^2 + (Nv, v)_{\left(L^2(\Sigma)\right)^2}. \tag{19}$$

where $N \in L\left(\left(L^2(\Sigma)\right)^2, \left(L^2(\Sigma)\right)^2\right)$ is defined as in (14).

The control problem then is to find $u = (u_1, u_2) \in U_{ad}$ such that $J(u) \leq J(v)$, where $U_{ad}$ is a closed con-

vex subset of $\left(L^2\left(\Sigma\right)\right)^2$.

Since the cost function (19) can be written as [11]:

$$J\left(v\right) = a\left(v,v\right) - 2L\left(v\right) + \left\|y\left(0\right) - z_d\right\|^2_{\left(L^2\left(\Sigma\right)\right)^2},$$

where $a\left(v,v\right)$ is a continuous coercive bilinear form and $L\left(v\right)$ is a continuous linear form on $\left(L^2\left(\Sigma\right)\right)^2$. Then using the general theory of Lions [11], there exists a unique optimal control $u \in U_{ad}$ such that $J\left(u\right) = \inf J\left(v\right)$ for all $v \in U_{ad}$. Moreover, we have the following theorem which gives the necessary and sufficient conditions of optimality:

_Theorem_ (5.1):

Assume that (9) and (14) hold. If the cost function is given by (19), the optimal control $u = \left(u_1, u_2\right) \in \left(L_2\left(\Sigma\right)\right)^2$ is then characterized by the following equations and inequalities:

$$\begin{cases} \dfrac{\partial^2 p_1\left(u\right)}{\partial t^2} + \left(-\Delta + q\right)p_1\left(u\right) - ap_1\left(u\right) - cp_2\left(u\right) = 0 & \text{in } Q, \\[2mm] \dfrac{\partial^2 p_2\left(u\right)}{\partial t^2} + \left(-\Delta + q\right)p_2\left(u\right) - bp_1\left(u\right) - dp_2\left(u\right) = 0 & \text{in } Q, \\[2mm] p_1, p_2 \to 0 \quad \text{as} \quad \left|x\right| \to \infty, \\[2mm] \dfrac{\partial p_1\left(u\right)}{\partial \nu}\bigg|_{\Sigma} = y_1\left(u\right)\big|_{\Sigma} - z_{d1}, \quad \dfrac{\partial p_2\left(u\right)}{\partial \nu}\bigg|_{\Sigma} = y_2\left(u\right)\big|_{\Sigma} - z_{d2}, \\[2mm] p_1\left(x,T,u\right) = p_2\left(x,T,u\right) = 0 & \text{in } R^n, \\[2mm] \dfrac{\partial p_1\left(x,T,u\right)}{\partial t} = \dfrac{\partial p_2\left(x,T,u\right)}{\partial t} = 0 & \text{in } R^n. \end{cases} \quad (20)$$

with $p_1\left(u\right), p_2\left(u\right) \in L^2\left(0,T;V_q\left(R^n\right)\right)$, $\dfrac{\partial p_1\left(u\right)}{\partial t}, \dfrac{\partial p_2\left(u\right)}{\partial t} \in L^2\left(Q\right)$ together with (16) and (12).

_Proof_:

The optimal control $u = \left(u_1, u_2\right) \in \left(L_2\left(\Sigma\right)\right)^2$ is characterized by [11]:

$$J'\left(u\right)\left(v - u\right) \geq 0, \quad \forall v \in U_{ad}$$

Which is equivalent to:

$$\left(y\left(u\right) - z_d, y\left(c\right) - y\left(u\right)\right)_{\left(L^2\left(\Sigma\right)\right)^2} + \left(Nu, v - u\right)_{\left(L^2\left(\Sigma\right)\right)^2} \geq 0$$

i.e.

$$\left(y_1\left(u\right) - z_{d1}, y_1\left(v\right) - y_1\left(u\right)\right)_{L^2\left(\Gamma\right)} + \left(y_2\left(v\right) - z_{d2}, y_2\left(v\right) - y_2\left(u\right)\right)_{L^2\left(\Gamma\right)} + \left(Nu, v - u\right)_{\left(L^2\left(\Sigma\right)\right)^2} \geq 0 \quad (21)$$

this inequality can be written as:

$$\int_0^T \left[\left(y_1\left(u\right) - z_{d1}, y_1\left(v\right) - y_1\left(u\right)\right)_{L^2\left(\Gamma\right)} + \left(y_2\left(v\right) - z_{d2}, y_2\left(v\right) - y_2\left(u\right)\right)_{L^2\left(\Gamma\right)}\right] dt + \left(Nu, v - u\right)_{\left(L^2\left(\Sigma\right)\right)^2} \geq 0 \quad (22)$$

since the adjoint system takes the form [11]:

$$\begin{cases} \dfrac{\partial^2 p\left(u\right)}{\partial t^2} + A^* p\left(u\right) = 0 & \text{in } Q \\[2mm] \dfrac{\partial p\left(u\right)}{\partial \nu}\bigg|_{\Sigma} = y\left(u\right) - z_d & \text{on } \Sigma \end{cases}$$

and from theorem (3.1), we get a unique solution $p(u) \in \left( L^2\left(0,T;V_q\left(R^n\right)\right) \right)^2$ which satisfies:

$$p_1(u),\ p_2(u) \in L^2\left(0,T;V_q\left(R^n\right)\right),\ \frac{\partial p_1(u)}{\partial t},\ \frac{\partial p_2(u)}{\partial t} \in L^2(Q).$$

This proves system (20).

Now, we transform (22) by using (20) as follows:

$$\int_0^T \left( \frac{\partial p_1(u)}{\partial \nu}, y_1(v) - y_1(u) \right)_{L^2(\Gamma)} dt + \int_0^T \left( \frac{\partial p_2(u)}{\partial \nu}, y_2(v) - y_2(u) \right)_{L^2(\Gamma)} dt + \left( Nu, v-u \right)_{\left(L^2(\Sigma)\right)^2} \geq 0$$

Using Green formula, we obtain:

$$\int_0^T \left( p_1(u), \frac{\partial y_1(v)}{\partial \nu} - \frac{\partial y_1(u)}{\partial \nu} \right)_{L^2(\Gamma)} dt + \int_0^T \left( p_2(u), \frac{\partial y_2(v)}{\partial \nu} - \frac{\partial y_2(u)}{\partial \nu} \right)_{L^2(\Gamma)} dt + \left( Nu, v-u \right)_{\left(L^2(\Sigma)\right)^2} \geq 0$$

Using (12), we have:

$$\int_0^T \left( p_1(u), v_1 - u_1 \right)_{L^2(\Gamma)} dt + \int_0^T \left( p_2(u), v_2 - u_2 \right)_{L^2(\Gamma)} dt + \left( Nu, v-u \right)_{\left(L^2(\Sigma)\right)^2} \geq 0,$$

which is equivalent to:

$$\left( p(u) + Nu, v-u \right)_{\left(L^2(\Sigma)\right)^2} \geq 0.$$

Thus the proof is complete.

## 6. Conclusions

In this paper, we have some important results. First of all we proved the existence and uniqueness of the state for system (1), which is (2 × 2) cooperative hyperbolic system involving Schrödinger operator defined on $R^n$ (*Theorem* 3.1). Then we found the necessary and sufficient conditions of optimality for system (1), that give the characterization of optimal control (*Theorem* 4.1). Finally, we also find the necessary and sufficient conditions of optimal control when the observation is on the boundary (*Theorem* 5.1).

Also it is evident that by modifying:

-the nature of the control (distributed, boundary),
-the nature of the observation (distributed, boundary),
-the initial differential system,
-the type of equation (elliptic, parabolic and hyperbolic),
-the type of system (non-cooperative, cooperative),
-the order of equation,

many of variations on the above problem are possible to study with the help of Lions formalism.

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