

Regional Boundary Observability with Constraints of the Gradient

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ABSTRACT

The aim of this paper is to explore the concept of observability with constraints of the gradient for distributed parabolic system evolving in spatial domain Ω , and which the state gradient is to be observed only on a part of the boundary of the system evolution domain. It consists in the reconstruction of the initial state gradient which must be between two prescribed functions in a subregion Γ of $\partial\Omega$. Two necessary conditions are given. The first is formulated in terms of the subdifferential associated with a minimized functional, and the second uses the Lagrangian multiplier method. Numerical illustrations are given to show the efficiency of the second approach and lead to open questions.

Keywords: Distributed Systems; Parabolic Systems; Regional Observability with Constraints; Regional Reconstruction

1. Introduction

For a distributed parameter system evolving on a special domain Ω , the observability concept has been widely developed and survey of these developments can be found [1-3]. Later, the regional observability notion was introduced, and interesting results have been obtained [4,5], in particular, the possibility to observe a state only on a subregion ω interior to Ω . These results have been extended to the case where ω is a part of the boundary $\partial\Omega$ of Ω [6]. Then the concepts of regional gradient observability and regional observability with constraints were introduced and developed by [7-11] in the case where the subregion is interior to Ω and the case where the subregion is a part of $\partial\Omega$. Here we are interested to approach the initial state gradient and the reconstructed state between two prescribed functions given only on a boundary subregion Γ of system evolution domain. There are many reasons motivating this problem: first the mathematical model of system is obtained from measurements or from approximation techniques and is very often affected by perturbations. Consequently, the solution of such a system is approximately known, and second, in various real problems the target required to be between two bounds. This is the case, for example of a biological reactor “**Figure 1**” in which the concentration regulation of a substrate at the bottom of the reactor is observed between two levels.

The paper is organized as follows: first we provide results on regional observability for distributed parameter system of parabolic type and we give definitions related

to regional boundary observability with constraints of the gradient of parabolic systems. The next section is focused on the reconstruction of the initial state gradient by using an approach based on sub-differential tools. The same objective is achieved in Section 4 by applying the multiplier Lagrangian approach which gives a practice algorithm. The last section is devoted to compute the obtained algorithm with numerical example and simulations.

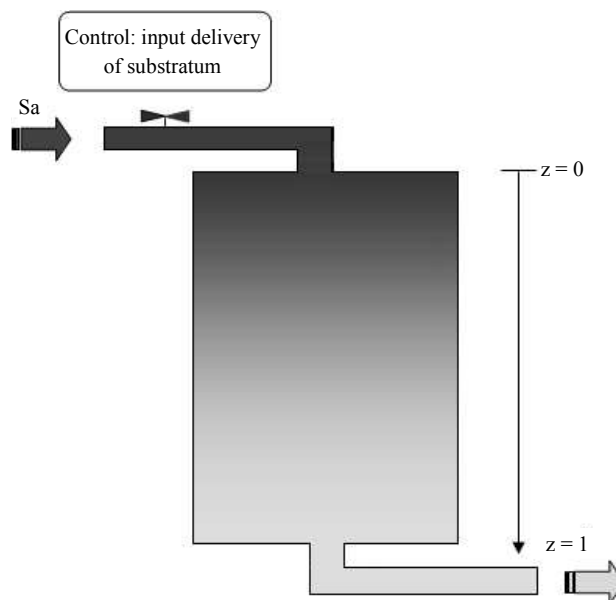


Figure 1. Regulation of the concentration flux of the substratum at a bottom of the reactor.

2. Problem Statement

Let Ω be an open bounded subset of IR^n ($n = 2, 3$) with regular boundary $\partial\Omega$ and a boundary subregion Γ of $\partial\Omega$. For a given time $T > 0$, let $Q = \Omega \times]0, T[$ and $\Sigma = \partial\Omega \times]0, T[$.

Consider a parabolic system defined by

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) = Ay(x, t) & \text{in } Q \\ y(x, 0) = y_0(x) & \text{in } \Omega \\ y(\xi, t) = 0 & \text{on } \Sigma \end{cases} \quad (1)$$

with the measurements given by the output function

$$z(t) = Cy(t) \quad (2)$$

where $C : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow IR^q(t)$ is linear and depends on the considered sensors structure.

The observation space is $O = L^2(0, T, IR^q)$.

A is a second order differential linear and elliptic operator which generates a strongly continuous semigroup

$$(S(t))_{t \geq 0}$$

in the Hilbert space $L^2(\Omega)$.

A denotes the adjoint operators of A .

The initial state y_0 and its gradient ∇y_0 are assumed to be unknown. The system (1) is autonomous and (2) allows writing

$$z(t) = CS(t)y_0, t \in]0, T[$$

We define the operator

$$\begin{aligned} K : H^2(\Omega) \cap H_0^1(\Omega) &\rightarrow O \\ z &\rightarrow CS(\cdot)z \end{aligned}$$

which is linear bounded with the adjoint K^* given by

$$\begin{aligned} K^* : O &\rightarrow H^2(\Omega) \cap H_0^1(\Omega) \\ z^* &\rightarrow \int_0^T S^*(t)C^*z^*(t)dt \end{aligned}$$

Consider the operator

$$\begin{aligned} \nabla : H^2(\Omega) \cap H_0^1(\Omega) &\rightarrow (L^2(\Omega))^n \\ y &\rightarrow \nabla y = \left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n} \right) \end{aligned}$$

While ∇^* denotes its adjoint given by

$$\begin{aligned} \nabla^* : (L^2(\Omega))^n &\rightarrow H^2(\Omega) \cap H_0^1(\Omega) \\ y &\rightarrow \nabla^* y = v \end{aligned}$$

where v is a solution of the Dirichlet's problem

$$\begin{cases} \Delta v = -\text{div}(y) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Let

$$\begin{aligned} \gamma : (L^2(\Omega))^n &\rightarrow (H^{1/2}(\Gamma))^n \\ z &\rightarrow \gamma z = (\gamma_0 z_1, \gamma_0 z_2 \dots \gamma_0 z_n) \end{aligned}$$

With $\gamma_0 : L^2(\Omega) \rightarrow (H^{1/2}(\Gamma))$ is the extension of the trace operator of order zero which is linear and surjective. γ^* , γ_0^* denotes the adjoint operators of γ and γ_0 .

For $\Gamma \subset \partial\Omega$, we consider

$$\begin{aligned} \chi_\Gamma : (H^{1/2}(\partial\Omega))^n &\rightarrow (H^{1/2}(\Gamma))^n \\ y &\rightarrow \chi_\Gamma y = y|_\Gamma \end{aligned}$$

while χ_Γ^* denotes its adjoint.

We recall the following definitions

Definition 2.1

1. The system (1) together with the output (2) is said to be exactly (respectively weakly) gradient observable on Γ if

$$\text{Im}(\chi_\Gamma \mathcal{N} K^*) = (H^{1/2}(\Gamma))^n$$

(respectively $\ker(K \nabla^* \gamma^* \chi_\Gamma^*) = \{0\}$).

2. The sensor (D, f) (or a sequence of sensors) is said to be gradient strategic on Γ if the observed system is weakly gradient observable on Γ .

For more details, we refer the reader to [11].

Let $(\alpha_i(\cdot))_{i=1}^n$ and $(\beta_i(\cdot))_{i=1}^n$ be two functions defined in $(H^{1/2}(\Gamma))^n$ such that $\alpha_i(\cdot) \leq \beta_i(\cdot)$ a.e. on Γ for all $1 \leq i \leq n$. In the sequel we set

$$\begin{aligned} &[\alpha(\cdot), \beta(\cdot)] \\ &= \left\{ (y_1, y_2, \dots, y_n) \in (H^{1/2}(\Gamma))^n \mid \right. \\ &\quad \left. \alpha_i(\cdot) \leq y_i(\cdot) \leq \beta_i(\cdot) \text{ a.e. on } \Gamma \forall i \in \{1, 2, \dots, n\} \right\} \end{aligned}$$

Definition 2.2

- 1). The system (1) together with the output (2) is said to be exactly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ if

$$\text{Im}(\chi_\Gamma \mathcal{N} K^*) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset.$$

- 2). The system (1) together with the output (2) is said to be weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ if

$$\overline{\text{Im}(\chi_\Gamma \mathcal{N} K^*)} \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset.$$

- 3). A sensor (D, f) is said to be $[\alpha(\cdot), \beta(\cdot)]$ -gradient strategic on Γ if the observed system is weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ .

Remark 2.3

- 1). If the system (1) together with the output (2) is ex-

actly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ then it is weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ .

- 2). If the system (1) together with the output (2) is exactly gradient observable on Γ then it is exactly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ .
- 3). If the system (1) together with the output (2) is exactly (resp. weakly) $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ_1 then it is exactly (resp. weakly) $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on any $\Gamma_2 \subset \Gamma_1$.

There exist systems which are not weakly gradient observable on a subregion Γ but which are weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ .

Example 2.4

Consider the two-dimensional system described by the diffusion equation

$$\begin{cases} \frac{\partial y}{\partial t}(x_1, x_2, t) = \frac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \frac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) \text{ in } \mathcal{Q} \\ y(x_1, x_2, 0) = y_0(x_1, x_2) \text{ in } \Omega \\ y(\xi, \eta, t) = 0 \text{ on } \Sigma \end{cases} \quad (3)$$

where $\Omega =]0, 1[\times]0, 1[$, the time interval is $]0, T[$ and let Γ be the boundary subregion given by $\Gamma = [0, 1] \times \{0\}$. We consider the sensor (D, f) defined by $D = [0, 1] \times [0, 1]$ and

$$f(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2).$$

Thus, the output function is given by

$$z(t) = \int_D y(x_1, x_2, t) f(x_1, x_2) dx_1 dx_2 \quad (4)$$

The operator

$$A = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

generates a semigroup $(S(t))_{t \geq 0}$ in $L^2(\Omega)$ given by

$$S(t)y = \sum_{i,j=1}^n \exp(\lambda_{ij}t) \langle y, \varphi_{ij} \rangle_{H^1(\Omega)} \varphi_{ij} \quad (5)$$

where $\varphi_{ij}(x_1, x_2) = 2 \sin(i\pi x_1) \sin(j\pi x_2)$

and $\lambda_{ij} = -(i^2 + j^2)\pi^2$.

Then we have the result:

Proposition 2.5

The system (3) together with the output (4) is not weakly gradient observable on Γ but it is weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ .

Proof

Let g_1 be the function defined in Ω by

$$g_1(x_1, x_2) = (\cos(2\pi x_1) \sin(2\pi x_2), 0)$$

be the gradient to be observed on Γ and show that g_1 is

not weakly gradient observable on Γ .

$$\begin{aligned} & K\nabla^* \gamma^* \chi_\Gamma^* \chi_\Gamma \gamma(g_1) \\ &= \sum_{i,j=1}^\infty \exp(\lambda_{ij}t) \langle \chi_\Gamma \gamma(g_1) | \chi_\Gamma \gamma \mathcal{V}(\varphi_{ij}) \rangle_{(H^{1/2}(\Gamma))^n} \\ & \quad \times \int_D \varphi_{ij}(x_1, x_2) f(x_1, x_2) dx_1 dx_2 \\ &= 8 \sum_{ij=1}^\infty ij\pi^3 \exp(\lambda_{ij}t) \times \int_0^1 \sin(i\pi x) \sin(\pi x) dx_1 \\ & \quad \times \int_0^1 \sin(j\pi x_2) \sin(\pi x_2) dx_2 \times \int_0^1 \cos(i\pi x_1) \cos(2\pi x_1) dx_1 \\ &= 0 \end{aligned}$$

we have $K\nabla^* \gamma^* \chi_\Gamma^* \chi_\Gamma \gamma(g_1) = 0$. Consequently, the gradient g_1 is not weakly gradient observable on Γ . Then the system (3) together with the output (4) is not weakly gradient observable on Γ . but we can show that it is weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ , indeed, for

$$g_2(x_1, x_2) = (\cos(\pi x_1) \sin(\pi x_2), 0)$$

we have

$$\begin{aligned} & K\nabla^* \gamma^* \chi_\Gamma^* \chi_\Gamma \gamma(g_2) \\ &= \sum_{i,j=1}^\infty \exp(\lambda_{ij}t) \langle \chi_\Gamma \gamma(g_2) | \chi_\Gamma \gamma \mathcal{V}(\varphi_{ij}) \rangle_{(H^{1/2}(\Gamma))^n} \\ & \quad \times \int_D \varphi_{ij}(x_1, x_2) f(x_1, x_2) dx_1 dx_2 \\ &= 4 \sum_{ij=1}^\infty ij\pi^3 \exp(\lambda_{ij}t) \times \int_0^1 \sin(i\pi x) \sin(\pi x) dx_1 \\ & \quad \times \int_0^1 \sin(j\pi x_2) \sin(\pi x_2) dx_2 \times \int_0^1 \cos(i\pi x_1) \cos(2\pi x_1) dx_1 \\ &= \frac{1}{2} \pi^3 \exp(\lambda_{ij}t) \\ & \neq 0 \end{aligned}$$

which show that the gradient g_2 is weakly gradient observable on Γ .

For $\alpha(x) = (-5 + x_1 + x_2, -1)$ and $\beta(x) = (1 + x_1, 2)$, we have that $g_2 \in [\alpha(\cdot), \beta(\cdot)]$, then the system (3) together with the output (4) is weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ .

Proposition 2.6

The system (1) together with the output (2) is exactly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ if and only if

$$\ker \chi_\Gamma + \text{Im}(\gamma \mathcal{V} K^*) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

Proof

- If $\ker \chi_\Gamma + \text{Im}(\gamma \mathcal{V} K^*) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$

then, we can find $z \in [\alpha(\cdot), \beta(\cdot)]$ such that

$$z \in \ker \chi_\Gamma + \text{Im}(\gamma \mathcal{V} K^*),$$

then $z = z_1 + z_2$ where $\chi_\Gamma z_1 = 0$ and $z_2 = \gamma \nabla K^* \theta$ with $\theta \in \mathcal{O}$, then

$$\begin{aligned} \chi_\Gamma z &= \chi_\Gamma z_1 + \chi_\Gamma z_2 \\ &= \chi_\Gamma z_2 \\ &= \chi_\Gamma \gamma \nabla K^* \theta \end{aligned}$$

and $\chi_\Gamma z \in \text{Im}(\chi_\Gamma \gamma \nabla K^*)$ thus

$$\text{Im}(\chi_\Gamma \gamma \nabla K^*) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

which shows that the system (1) together with the output (2) is exactly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ .

- Assume that the system (1) together with the output (2) is exactly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ , which is equivalent to

$$\text{Im}(\chi_\Gamma \gamma \nabla K^*) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

then there exists $z \in [\alpha(\cdot), \beta(\cdot)]$ and $\theta \in \mathcal{O}$ such that $\chi_\Gamma z = \chi_\Gamma \gamma \nabla K^* \theta$ which gives

$$\chi_\Gamma (z - \gamma \nabla K^* \theta) = 0.$$

Let $y_1 = z - \gamma \nabla K^* \theta$ and $y_2 = \gamma \nabla K^* \theta$, then $z = y_1 + y_2$ with $y_1 \in \ker \chi_\Gamma$ and $y_2 \in \text{Im}(\gamma \nabla K^*)$ which shows that

$$z \in \ker \chi_\Gamma + \text{Im}(\gamma \nabla K^*)$$

and therefore

$$(\ker \chi_\Gamma + \text{Im}(\gamma \nabla K^*)) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

Proposition 2.7

The system (1) together with the output (2) is weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ if and only if

$$(\ker \chi_\Gamma + \overline{\text{Im}(\gamma \nabla K^*)}) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

Proof

- Suppose that

$$(\ker \chi_\Gamma + \overline{\text{Im}(\gamma \nabla K^*)}) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

then, there exists $z \in [\alpha(\cdot), \beta(\cdot)]$ such that

$$z \in \ker \chi_\Gamma + \overline{\text{Im}(\gamma \nabla K^*)}$$

so $z = z_1 + z_2$, where $\chi_\Gamma z_1 = 0$ and

$$z_2 = \lim_{n \rightarrow +\infty} \gamma \nabla K^* \theta_n$$

with $\theta_n \in \mathcal{O}$, $\forall n \in \mathbb{N}$, then

$$\begin{aligned} \chi_\Gamma z &= \chi_\Gamma z_2 \\ &= \chi_\Gamma \left(\lim_{n \rightarrow +\infty} \gamma \nabla K^* \theta_n \right) \\ &= \lim_{n \rightarrow +\infty} \chi_\Gamma \gamma \nabla K^* \theta_n \end{aligned}$$

and

$$\chi_\Gamma z \in \overline{\text{Im}(\chi_\Gamma \gamma \nabla K^*)}$$

therefore

$$\overline{\text{Im}(\chi_\Gamma \gamma \nabla K^*)} \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

which implies that the system (1) together with the output (2) is weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ .

- Suppose that the system (1) together with the output (2) is weakly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ , which is equivalent to

$$\overline{\text{Im}(\chi_\Gamma \gamma \nabla K^*)} \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

then there exists $z \in [\alpha(\cdot), \beta(\cdot)]$ and θ_n a sequence of elements of \mathcal{O} such that

$$\chi_\Gamma z = \lim_{n \rightarrow +\infty} \chi_\Gamma \gamma \nabla K^* \theta_n$$

which gives

$$\chi_\Gamma \left(z - \lim_{n \rightarrow +\infty} \gamma \nabla K^* \theta_n \right) = 0.$$

Let

$$y_1 = z - \lim_{n \rightarrow +\infty} \gamma \nabla K^* \theta_n$$

and

$$y_2 = \lim_{n \rightarrow +\infty} \gamma \nabla K^* \theta_n,$$

then $z = y_1 + y_2$ with $y_1 \in \ker \chi_\Gamma$ and

$$y_2 \in \overline{\text{Im}(\gamma \nabla K^*)}$$

which shows that

$$z \in \ker \chi_\Gamma + \overline{\text{Im}(\gamma \nabla K^*)}$$

and therefore

$$(\ker \chi_\Gamma + \overline{\text{Im}(\gamma \nabla K^*)}) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset$$

3. Subdifferential Approach

This section is focused on the characterization of the initial state of the system (1) together with the output (2) in the nonempty subregion Γ with constraints on the gradient by using an approach based on subdifferential tools [12]. So we consider the optimization problem

$$\begin{cases} \min \|Ky - z\|_0^2 \\ y \in Y \end{cases} \quad (6)$$

where

$$Y = \{y \in H^2(\Omega) \cap H_0^1(\Omega) \mid \chi_\Gamma \gamma \nabla y \in [\alpha(\cdot), \beta(\cdot)]\}$$

Let us denote by

- $\Gamma_0(H^2(\Omega) \cap H_0^1(\Omega))$ the set of functions

$$f : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow \widetilde{IR} =]-\infty, +\infty]$$

proper, lower semi-continuous (l.s.c.) and convex.

- For $f \in \Gamma_0(H^2(\Omega) \cap H_0^1(\Omega))$ the polar function f^* of f be given by

$$f^*(y^*) = \sup \{ \langle y^*, y \rangle - f(y) \},$$

$$\forall y^* \in H^2(\Omega) \cap H_0^1(\Omega), y \in \text{dom}(f)$$

where

$$\text{dom}(f) = \{ y \in H^2(\Omega) \cap H_0^1(\Omega) \mid f(y) < +\infty \}.$$

- For $y^0 \in \text{dom}(f)$ the set

$$\begin{aligned} & \partial f(y^0) \\ &= \{ y^* \in H^2(\Omega) \cap H_0^1(\Omega) \mid f(y) \\ & \geq f(y^0) + \langle y^*, y - y^0 \rangle, \forall y \in H^2(\Omega) \cap H_0^1(\Omega) \} \end{aligned}$$

denotes the subdifferential of f at y^0 , then we have $y_1 \in \partial f(y^*)$ if and only if

$$f(y^*) + f^*(y_1) = \langle y^*, y_1 \rangle.$$

- For D a nonempty subset of $H^2(\Omega) \cap H_0^1(\Omega)$

$$\Psi_D(y) = \begin{cases} 0 & \text{if } y \in D \\ +\infty & \text{otherwise} \end{cases}$$

denotes the indicator functional of D .

With these notations the problem (6) is equivalent to the problem:

$$\begin{cases} \inf (\|Ky - z\|_0^2 + \Psi_Y(y)) \\ y \in H^2(\Omega) \cap H_0^1(\Omega) \end{cases} \quad (7)$$

The solution of this problem may be characterized by the following result.

Proposition 3.1

If the system (1) together with the output (2) is exactly $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ , then y^* is a solution of (7) if and only if $y^* \in Y$ and

$$\Psi_Y(-2K^*(Ky^* - z)) = -2\|Ky^*\|_0^2 + 2\langle K^*z, y^* \rangle$$

Proof

We have that y^* is a solution of (7) if and only if

$$0 \in \partial(F + \Psi_Y)(y^*),$$

with

$$F(y^*) = \|Ky^* - z\|_0^2,$$

since

$$F \in \Gamma_0(H^2(\Omega) \cap H_0^1(\Omega)),$$

Y is closed, convex and nonempty, then

$$\Psi_Y \in \Gamma_0(H^2(\Omega) \cap H_0^1(\Omega))$$

Also, according to the hypothesis of the $[\alpha(\cdot), \beta(\cdot)]$ -gradient observability on Γ , we have

$$\text{Dom}(F) \cap \text{Dom}(\Psi_Y) \neq \emptyset.$$

Now F is continuous, then

$$\partial(F + \Psi_Y)(y^*) = \partial F(y^*) + \partial \Psi_Y(y^*)$$

it follows that y^* is a solution of (7) if and only if

$$0 \in \partial F(y^*) + \partial \Psi_Y(y^*).$$

On the other hand F is Frechet-differentiable, then

$$\partial f(y^*) = \{ \nabla F(y^*) \} = \{ 2K^*(Ky^* - z) \}$$

and y^* is a solution of (7) if and only if

$$-2K^*(Ky^* - z) \in \partial \Psi_Y(y^*)$$

which is equivalent to $y^* \in Y$ and

$$\Psi_Y(y^*) + \Psi_Y(-2K^*(Ky^* - z)) = \langle y^*, -2K^*Ky^* + 2K^*z \rangle$$

and consequently $y^* \in Y$ and

$$\Psi_Y(-2K^*(Ky^* - z)) = -2\|Ky^*\|_0^2 + 2\langle K^*z, y^* \rangle$$

4. Lagrangian Multiplier Approach

In this section we propose to solve the problem (6) using the Lagrangian multiplier method [13]. Also we describe a numerical algorithm which allows the computation of the initial state gradient on the boundary subregion Γ and finally we illustrate the obtained results by numerical simulation which tests the efficiency of the numerical scheme.

From the definition of the exact $[\alpha(\cdot), \beta(\cdot)]$ -gradient observability on Γ all state we will consider are of the form $K^*\theta$ such that $\theta \in L^2(0, T; IR^q)$. So the last minimization problem is equivalent to

$$\begin{cases} \min \|KK^*\theta - z\|_0^2 \\ \theta \in G \end{cases} \quad (8)$$

with

$$G = \{ \hat{\theta} \in L^2(0, T; IR^q) \mid \chi_\Gamma \mathcal{N}K^*\hat{\theta} \in [\alpha(\cdot), \beta(\cdot)] \}.$$

Then we have the following result:

Proposition 4.1

If the system (1) together with the output (2) is exactly observable in Ω , exactly $[\alpha_1(\cdot), \beta_1(\cdot)]$ -boundary gradient observable on Γ with $\alpha_1 = \alpha + \delta$, $\beta_1 = \beta - \delta$,

$\delta = (\varepsilon_1 > 0, \dots, \varepsilon_n > 0)$ then the solution of (8) is given by

$$\begin{aligned} \theta^* &= (KK^*KK^*)^{-1} KK^*z \\ &\quad - \frac{1}{2}(KK^*KK^*)^{-1} K\nabla^* \gamma^* \chi_\Gamma^* \lambda^* \end{aligned} \tag{9}$$

and the gradient in Γ of the solution of the problem (6) is given by

$$y^* = R_\Gamma K^* z - \frac{1}{2} R_\Gamma \nabla^* \gamma^* \chi_\Gamma^* \lambda^* \tag{10}$$

where λ^* is the solution of

$$\begin{cases} \frac{1}{2} R_\Gamma \nabla^* \gamma^* \chi_\Gamma^* \lambda^* = -y^* + R_\Gamma K^* z \\ y^* = P_{[\alpha(\cdot), \beta(\cdot)]}(\rho \lambda^* + y^*) \end{cases} \tag{11}$$

while

$$P_{[\alpha(\cdot), \beta(\cdot)]} : (H^{1/2}(\Gamma))^n \rightarrow [\alpha(\cdot), \beta(\cdot)],$$

denotes the projection operator, $\rho > 0$ and

$$R_\Gamma = \chi_\Gamma \gamma \nabla K^* (KK^*KK^*)^{-1} K.$$

Proof

The system (1) together with the output (2) is $[\alpha(\cdot), \beta(\cdot)]$ -gradient observable on Γ then $G \neq \emptyset$ and the problem (8) has a solution. The problem (8) is equivalent to the saddle point problem

$$\begin{cases} \min \|KK^* \theta - z\|_0^2 \\ (\theta, y) \in W \end{cases} \tag{12}$$

where

$$W = \{(\theta, y) \in O \times [\alpha(\cdot), \beta(\cdot)] \mid \chi_\Gamma \gamma \nabla K^* \theta - y = 0\}$$

We associate with problem (12) the Lagrangian functional L defined by the formula

$$L(\theta, y, \lambda) = \|KK^* \theta - z\|_0^2 + \langle \lambda, \chi_\Gamma \gamma \nabla K^* \theta - y \rangle_{(H^{1/2}(\Gamma))^n}$$

for all

$$(\theta, y, \lambda) \in O \times [\alpha(\cdot), \beta(\cdot)] \times (H^{1/2}(\Gamma))^n.$$

Let us recall that $(\theta^*, y^*, \lambda^*)$ is a saddle point of L if

$$\max L(\theta^*, y^*, \lambda) = L(\theta^*, y^*, \lambda^*) = \min L(\theta, y, \lambda^*)$$

$$\begin{aligned} \lambda \in (H^{1/2}(\Gamma))^n & \quad \theta \in O, \\ y \in [\alpha(\cdot), \beta(\cdot)] \end{aligned}$$

- The system (1) together with the output (2) is $[\alpha_1(\cdot), \beta_1(\cdot)]$ -gradient observable on Γ and, there-

fore, there exist $\delta > 0$ and $\theta \in O$ such that

$$\chi_\Gamma \gamma \nabla K^* \theta \in [\alpha(\cdot) + \delta, \beta(\cdot) - \delta],$$

which implies

$$\inf \{L(\theta, y, \lambda), (\theta, y) \in O \times [\alpha(\cdot), \beta(\cdot)]\} \rightarrow -\infty,$$

as

$$\|\lambda\|_{(H^{1/2}(\Gamma))^n} \rightarrow +\infty$$

moreover, there exists $\lambda_0 \in (H^{1/2}(\Gamma))^n$ such that

$$\lim_{\|(\theta, y)\| \rightarrow +\infty} L(\theta, y, \lambda_0) = +\infty$$

then L admits a saddle point.

- Let $(\theta^*, y^*, \lambda^*)$ be a saddle point of L and prove that $y^* = \chi_\Gamma \gamma \nabla K^* \theta^*$ is the restriction gradient on Γ of the solution of (6).

We have

$$L(\theta^*, y^*, \lambda) \leq L(\theta^*, y^*, \lambda^*) \leq L(\theta, y, \lambda^*)^*$$

for all

$$(\theta, y, \lambda) \in O \times [\alpha(\cdot), \beta(\cdot)] \times (H^{1/2}(\Gamma))^n.$$

The first inequality above gives

$$\begin{aligned} &\langle \lambda, \chi_\Gamma \gamma \nabla K^* \theta^* - y^* \rangle_{(H^{1/2}(\Gamma))^n} \\ &\leq \langle \lambda^*, \chi_\Gamma \gamma \nabla K^* \theta^* - y^* \rangle_{(H^{1/2}(\Gamma))^n} \\ &\quad \forall \lambda \in (H^{1/2}(\Gamma))^n. \end{aligned}$$

which implies that $\chi_\Gamma \gamma \nabla K^* \theta^* = y^*$ and hence

$$\chi_\Gamma \gamma \nabla K^* \theta^* \in [\alpha(\cdot), \beta(\cdot)].$$

The second inequality means that

$$\forall (\theta, y) \in O \times [\alpha(\cdot), \beta(\cdot)]$$

we have

$$\begin{aligned} &\|KK^* \theta^* - z\|_0^2 + \langle \lambda^*, \chi_\Gamma \gamma \nabla K^* \theta^* - y^* \rangle \\ &\leq \|KK^* \theta - z\|_0^2 + \langle \lambda^*, \chi_\Gamma \gamma \nabla K^* \theta - y \rangle \end{aligned}$$

Since $y^* = \chi_\Gamma \gamma \nabla K^* \theta^*$ we have

$$\|KK^* \theta^* - z\|_0^2 \leq \|KK^* \theta - z\|_0^2 + \langle \lambda^*, \chi_\Gamma \gamma \nabla K^* \theta - y \rangle$$

for all

$$(\theta, y) \in O \times [\alpha(\cdot), \beta(\cdot)].$$

Taking

$$y = \chi_\Gamma \gamma \nabla K^* \theta \in [\alpha(\cdot), \beta(\cdot)],$$

we obtain

$$\|KK^*\theta^* - z\|_0^2 \leq \|KK^*\theta - z\|_0^2$$

which implies that θ^* minimize $\|KK^*\theta - z\|_0^2$, and so $y_0^* = K^*\theta^*$ whose the restriction of the gradient on Γ is $y^* = \chi_\Gamma \gamma \nabla K^* \theta^*$ minimize the function $\|Ky - z\|_0^2$ for all the states which are of the form $K^*\theta$ with $\theta \in O$.

- Now if $(\theta^*, y^*, \lambda^*)$ is a saddle point of L , then the following assumptions hold

$$2\langle KK^*\theta^* - z, KK^*(\theta - \theta^*) \rangle + \langle \lambda^*, \chi_\Gamma \gamma \nabla K^*(\theta - \theta^*) \rangle = 0, \quad \forall \theta \in O \tag{13}$$

$$-\langle \lambda^*, y - y^* \rangle \geq 0, \quad \forall y \in [\alpha(\cdot), \beta(\cdot)] \tag{14}$$

$$\langle \lambda - \lambda^*, \chi_\Gamma \gamma \nabla K^* \theta^* - y^* \rangle = 0, \quad \forall \lambda \in (H^{1/2}(\Gamma))^n \tag{15}$$

From (13) we have

$$\begin{aligned} & \left\langle 2(KK^*)^*(KK^*\theta^* - z), (\theta - \theta^*) \right\rangle \\ & + \left\langle (\chi_\Gamma \gamma \nabla K^*)^* \lambda^*, (\theta - \theta^*) \right\rangle = 0, \quad \forall \theta \in O \end{aligned}$$

Then

$$-2(KK^*)^* KK^* \theta^* + 2(KK^*)^* z = (\chi_\Gamma \gamma \nabla K^*)^* \lambda^*$$

we assume that the system is observable in Ω , then KK^*KK^* is invertible, and

$$\begin{aligned} \theta^* &= (KK^*KK^*)^{-1} KK^* z \\ & - \frac{1}{2}(KK^*KK^*)^{-1} K \nabla^* \gamma^* \chi_\Gamma^* \lambda^* \end{aligned}$$

so y^* is given by

$$\begin{aligned} y^* &= \chi_\Gamma \gamma \nabla K^* (KK^*KK^*)^{-1} KK^* z \\ & - \frac{1}{2} \chi_\Gamma \gamma \nabla K^* (KK^*KK^*)^{-1} K \nabla^* \gamma^* \chi_\Gamma^* \lambda^* \end{aligned}$$

Then

$$y^* = R_\Gamma K^* z - \frac{1}{2} R_\Gamma \nabla^* \gamma^* \chi_\Gamma^* \lambda^*$$

with

$$R_\Gamma = \chi_\Gamma \gamma \nabla K^* (KK^*KK^*)^{-1} K.$$

By (14) we have

$$\begin{aligned} & \langle (\rho \lambda^* + y^*) - y^*, y - y^* \rangle \leq 0, \\ & \forall y \in [\alpha(\cdot), \beta(\cdot)], \forall \rho > 0 \end{aligned}$$

Then

$$y^* = P_{[\alpha(\cdot), \beta(\cdot)]}(\rho \lambda^* + y^*)$$

Corollary 4.2

If the system (1) together with the output (2) is observable in Ω , exactly gradient observable on Γ and the function

$$\begin{aligned} L_\Gamma &= \left[(K \nabla^* \gamma^* \chi_\Gamma^*)^* K \nabla^* \gamma^* \chi_\Gamma^* \right]^{-1} (K \nabla^* \gamma^* \chi_\Gamma^*)^* \\ & \times KK^* KK^* \left[(\chi_\Gamma \gamma \nabla K^*)^* \chi_\Gamma \gamma \nabla K^* \right]^{-1} [\chi_\Gamma \gamma \nabla K^*]^* \end{aligned}$$

is coercive, then for ρ conveniently chosen, the system (11) has a unique solution (λ^*, y^*) .

Proof

We have

$$\begin{aligned} y^* &= \chi_\Gamma \gamma \nabla K^* (KK^*KK^*)^{-1} KK^* z \\ & - \frac{1}{2} \chi_\Gamma \gamma \nabla K^* (KK^*KK^*)^{-1} K \nabla^* \gamma^* \chi_\Gamma^* \lambda^* \end{aligned}$$

then

$$\begin{aligned} \lambda^* &= -2L_\Gamma y^* \\ & + 2 \left[(\chi_\Gamma \gamma \nabla K^*) (K \nabla^* \gamma^* \chi_\Gamma^*) \right]^{-1} (K \nabla^* \gamma^* \chi_\Gamma^*)^* KK^* z \end{aligned}$$

So if $(\theta^*, y^*, \lambda^*)$ is a saddle point of L then the system (11) is equivalent to

$$\begin{cases} \lambda^* = 2 \left[(\chi_\Gamma \gamma \nabla K^*) (K \nabla^* \gamma^* \chi_\Gamma^*) \right]^{-1} (K \nabla^* \gamma^* \chi_\Gamma^*)^* KK^* z \\ \quad - 2L_\Gamma y^* \\ y^* = P_{[\alpha(\cdot), \beta(\cdot)]} \left(2\rho \left[(\chi_\Gamma \gamma \nabla K^*) (K \nabla^* \gamma^* \chi_\Gamma^*) \right]^{-1} \right. \\ \quad \left. \times (K \nabla^* \gamma^* \chi_\Gamma^*)^* KK^* z - 2\rho L_\Gamma y^* + y^* \right) \end{cases}$$

It follows that y^* is a fixed point of the function

$$\begin{cases} F_\rho : [\alpha(\cdot), \beta(\cdot)] \rightarrow [\alpha(\cdot), \beta(\cdot)] \\ y \mapsto P_{[\alpha(\cdot), \beta(\cdot)]} \left(+2\rho \left[(\chi_\Gamma \gamma \nabla K^*) (K \nabla^* \gamma^* \chi_\Gamma^*) \right]^{-1} \right. \\ \quad \left. \times (K \nabla^* \gamma^* \chi_\Gamma^*)^* KK^* z - 2\rho L_\Gamma y + y \right) \end{cases}$$

Now the operator L_Γ is coercive, so $\exists m > 0$ such that

$$\langle L_\Gamma y, y \rangle \geq m \|y\|^2 \quad \forall y \in (H^{1/2}(\Gamma))^n$$

It follows that

$$\forall t_1, t_2 \in [\alpha_1(\cdot), \beta_1(\cdot)]$$

$$\begin{aligned} \|F_\rho(t_1) - F_\rho(t_2)\|^2 &\leq \|-2\rho L_\Gamma(t_1 - t_2) + (t_1 - t_2)\|^2 \\ &\leq 4\rho^2 \|L_\Gamma\|^2 \|(t_1 - t_2)\|^2 + \|(t_1 - t_2)\|^2 \\ &\quad - 4\rho \langle L_\Gamma(t_1 - t_2), (t_1 - t_2) \rangle \\ &\leq (4\rho^2 \|L_\Gamma\|^2 + 1 - 4\rho m) \|(t_1 - t_2)\|^2 \end{aligned}$$

and we deduce that if

$$\rho < \frac{m}{\|L_\Gamma\|^2},$$

then F_ρ is a contractor map, which implies the uniqueness of y^* and λ^* .

4.1. Numerical Approach

In this section we describe a numerical scheme which allows the calculation of the initial state gradient between $\alpha(\cdot)$ and $\beta(\cdot)$ on the subregion Γ .

We have seen in the previous section that in order to reconstruct the initial state between $\alpha(\cdot)$ and $\beta(\cdot)$, it is sufficient to solve the Equations (9)-(11), which can be done by the following algorithm of Uzawa type. Let $T = KK^*KK^*$, if we choose two functions

$$(y_0^*, \lambda_1^*) \in [\alpha_1(\cdot), \beta_1(\cdot)] \times (H^{1/2}(\Gamma))^n$$

and

$$\theta_n^* = T^{-1}KK^*z - \frac{1}{2}T^{-1}K\nabla^* \gamma^* \chi_\Gamma^* \lambda_n^*$$

$$y_n^* = P_{[\alpha(\cdot), \beta(\cdot)]}(\rho \lambda_n^* + y_{n-1}^*)$$

$$\lambda_{n+1}^* = \lambda_n^* + (\chi_\Gamma \gamma \nabla K^* \theta_n^* - y_n^*)$$

then we obtain the following algorithm (Table 1).

4.2. Simulation Results

In this section we give a numerical example which

Table 1. Algorithm.

-
- 1) Given the subregion Γ , the location of the sensor b , the function of measure distribution f , the functions $\alpha(\cdot)$ and $\beta(\cdot)$, initiate two functions $y_0^* \in [\alpha(\cdot), \beta(\cdot)]$ and $\lambda_1^* \in (H^{1/2}(\Gamma))^n$ and $\varepsilon > 0$ the threshold.
 - 2) Solve $T(\theta_n^*) = KK^*z - \frac{1}{2}K\nabla^* \gamma^* \chi_\Gamma^* \lambda_n^*, n \geq 1$.
 - 3) Calculate $y_n^* = P_{[\alpha(\cdot), \beta(\cdot)]}(\rho \lambda_n^* + y_{n-1}^*), n \geq 1$.
 - 4) Calculate $\lambda_{n+1}^* = \lambda_n^* + (\chi_\Gamma \gamma \nabla K^* \theta_n^* - y_n^*), n \geq 1$.
 - 5) If $\|y_n^* - y_{n-1}^*\|_{(H^{1/2}(\Gamma))^n} \leq \varepsilon$, stop, else.
 - 6) $\lambda_{n+1}^* = \lambda_n^* + (\chi_\Gamma \gamma_0 \nabla K^* \theta_n^* - y_n^*), n \geq 1$ and return to step 2.
-

illustrates the efficiency of the previous approach. The results are related to the choice of the subregion, the initial conditions and the sensor location. Let us consider a two-dimensional system defined in $\Omega =]0, 1[\times]0, 1[$ and described by the following parabolic equation

$$\begin{cases} \frac{\partial y}{\partial t}(x_1, x_2, t) = 0.01 \left[\frac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \frac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) \right] \text{ in } Q \\ y(x_1, x_2, 0) = y_0(x_1, x_2) \text{ in } \Omega \\ y(\xi, \eta, t) = 0 \text{ on } \Sigma \end{cases}$$

The measurements are given by a pointwise sensor $z(t) = y(b, t)$ with b is the location of the sensor and $T = 2$. Let $\Gamma = \{0\} \times]0, 1[$ and

$$g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2))$$

the initial gradient to be observed on Γ with g_1 and g_2 are given by

$$g_1(x_1, x_2) = 4x_1x_2^3 - 8x_1x_2^2 + 4x_1x_2 - 2x_2^3 + 4x_2^2 - 2x_2$$

and

$$g_2(x_1, x_2) = 6x_1^2x_2^2 - 8x_1^2x_2 + 2x_1^2 - 6x_1x_2^2 + 8x_1x_2 - 2x_1$$

For

$$\alpha(x_1, x_2) = (\alpha_1(x_1, x_2), \alpha_2(x_1, x_2))$$

and

$$\beta(x_1, x_2) = (\beta_1(x_1, x_2), \beta_2(x_1, x_2))$$

with

$$\begin{cases} \alpha_1(x_1, x_2) = 4x_1x_2^3 - 8x_1x_2^2 + 4x_1x_2 - 2x_2^3 + 4x_2^2 - 2x_2 - 0.5 \\ \alpha_2(x_1, x_2) = 4x_1x_2^3 - 8x_1^2x_2^2 + 4x_1x_2 + 2x_2^2 - 2x_2^3 - 2x_2 - 0.4 \\ \beta_1(x_1, x_2) = 8x_1x_2^3 - 8x_1x_2^2 + 4x_1x_2 - 2x_2^3 + 4x_2^2 - 2x_2 + 0.5 \\ \beta_2(x_1, x_2) = 4x_1^4x_2^2 + 8x_1x_2^2 + 2x_1x_2 - 4x_2^6 + 4x_2^3 + 2x_2 + 0.4 \end{cases}$$

Applying the previous algorithm for $b = (0.34, 0.78)$, we obtain

“**Figures 2 and 3**” show that the estimated initial gradient is between $\alpha(\cdot)$ and $\beta(\cdot)$ on the subregion Γ , and show that the sensor located in $b = (0.34, 0.78)$ is $[\alpha(\cdot), \beta(\cdot)]$ -gradient strategic on Γ . The estimated initial gradient is obtained with reconstruction error $\varepsilon = 5.54 \times 10^{-5}$.

If we take $b = (0.15, 0.52)$, we obtain “**Figure 4**”

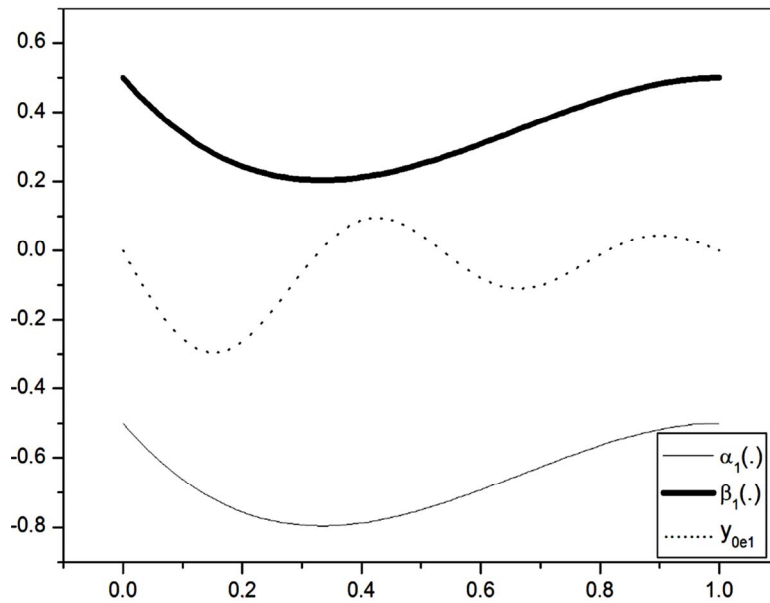


Figure 2. The first component of the estimated initial gradient, $\alpha_1(\cdot)$ and $\beta_1(\cdot)$ on Γ .

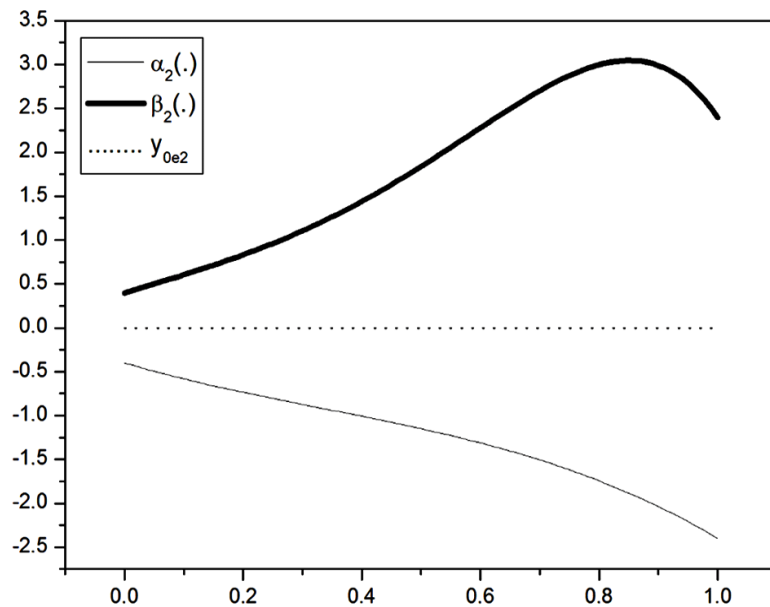


Figure 3. The second component of the estimated initial gradient, $\alpha_2(\cdot)$ and $\beta_2(\cdot)$ on Γ .

shows that the estimated initial gradient is not between $\alpha(\cdot)$ and $\beta(\cdot)$ on the subregion Γ , which implies that the sensor located in $b = (0.15, 0.52)$ is not $[\alpha(\cdot), \beta(\cdot)]$ -gradient strategic on Γ .

Remark 4.2

The above results are obtained with pointwise measurement, and one can obtain similar results with zone (internal or boundary) measurement.

5. Conclusions

The problem of $[\alpha(\cdot), \beta(\cdot)]$ -boundary gradient ob-

servability on Γ of parabolic system is considered. The initial state gradient is characterized by two approaches based on regional observability tools in connection with Lagrangian and subdifferential techniques.

Moreover, we have explored a useful numerical algorithm which allows the computation of initial state gradient and which is illustrated by numerical example and simulations. Various questions are still open. The characterization of $[\alpha(\cdot), \beta(\cdot)]$ -boundary gradient observability by a rank condition as stated for usual gradient observability or regional gradient observability of distributed parameter systems is of great interest. This

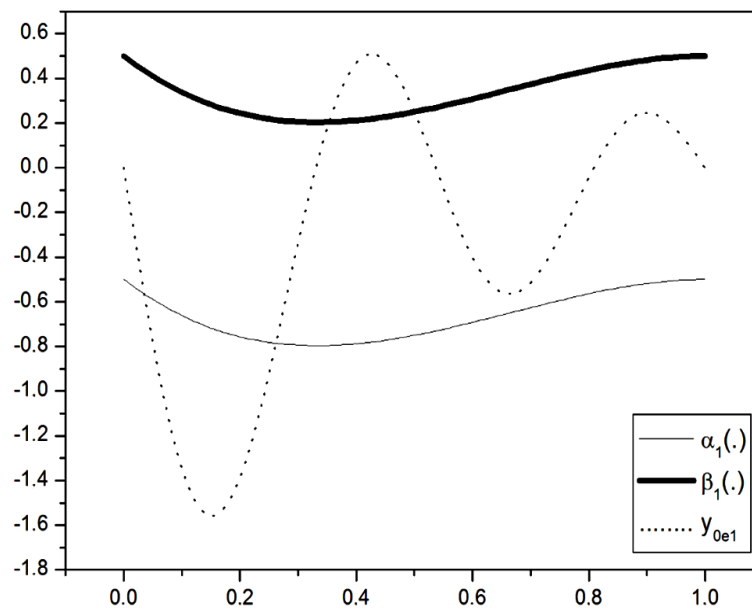


Figure 4. The first component of the estimated initial gradient, $\alpha_1(\cdot)$ and $\beta_1(\cdot)$ on Γ .

question is under consideration and will be the subject of the future paper.

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