Intelligent Control and Automation, **2012, 3, 176-187**
http://dx.doi.org/10.4236/ica.2012.32020 Published Online May 2012 (http://www.SciRP.org/journal/ica)

Efficient Solutions of Coupled Matrix and Matrix Differential Equations

Zeyad Al-Zhour
Department of Basic Sciences and Humanities, College of Engineering, University of Dammam, Dammam, KSA
Email: zeyad1968@yahoo.com, zalzhour@ud.edu.sa

Received March 7, 2012; revised April 27, 2012; accepted May 4, 2012

ABSTRACT

In Kronecker products works, matrices are some times regarded as vectors and vectors are some times made in to matrices. To be precise about these reshaping we use the vector and diagonal extraction operators. In the present paper, the results are organized in the following ways. First, we formulate the coupled matrix linear least-squares problem and present the efficient solutions of this problem that arises in multistatic antenna array processing problem. Second, we extend the use of connection between the Hadamard (Kronecker) product and diagonal extraction (vector) operator in order to construct a computationally-efficient solution of non-homogeneous coupled matrix differential equations that useful in various applications. Finally, the analysis indicates that the Kronecker (Khatri-Rao) structure method can achieve good efficient while the Hadamard structure method achieve more efficient when the unknown matrices are diagonal.

Keywords: Matrix Products; Least-Squares Problem; Coupled Matrix and Matrix Differential Equations; Diagonal Extraction Operator

1. Introduction

Linear matrix and matrix differential equations show up in various fields including engineering, mathematics, physics, statistics, control, optimization, economic, linear system and linear differential system problems. For instance, the Lyapunov equations $A^*X + XA + Q = 0$ and $X - A^*XA = Q$ (where A^* is the conjugate transpose of *A*) are used to analyze of the stability of continuous-time and discrete-time systems, respectively [1]. The generalized Lyapunov equation:

$$AXB^T + CXD^T = Q. \tag{1}$$

(where B^T is the transpose of *B*) has been used to characterize structured covariance matrices [2]. Most of the existing results, however, are connected with particular systems of such matrix and matrix differential equations.

Coupled matrix and matrix differential equations have also been widely used in stability theory of differential equations, control theory, communication systems, perturbation analysis of linear and non-linear matrix equations and other fields of pure and applied mathematics and also recently in the context of the analysis and numerical simulation of descriptor systems. For instance, the canonical system

$$\begin{aligned} X'(t) &= AX(t) + BY(t), \\ Y'(t) &= CX(t) - A^TY(t). \end{aligned} \tag{2}$$

With the boundary conditions and $Y(b) = 0$ has been used to the solution of optimal control problem with the performance index [3]. In addition, many interesting problems lead to coupled Riccati matrix differential equations [4]:

$$\begin{aligned} X_1'(t) &= \{Q_1(t) + B_1(t)X_1(t) + X_1(t)A_1(t) + X_1(t)S_{11}(t)X_1(t) \\ &\quad + X_1(t)S_{22}(t)X_2(t) + X_2(t)S_{22}(t)X_1(t) + X_2(t)S_{12}(t)X_2(t)\}; \\ X_2'(t) &= \{Q_2(t) + B_2(t)X_2(t) + X_2(t)A_2(t) + X_2(t)S_{22}(t)X_2(t) \\ &\quad + X_2(t)S_{11}(t)X_1(t) + X_1(t)S_{11}(t)X_2(t) + X_1(t)S_{21}(t)X_1(t)\}. \\ X_1(t_f) &= X_{1f}, X_2(t_f) = X_{2f}, \end{aligned} \tag{3}$$

and the general class of non-homogeneous coupled matrix differential equations:

$$\begin{aligned} X_1'(t) &= A_{11}X_1(t)B_{11} + A_{12}X_2(t)B_{12} + \cdots + A_{1p}X_p(t)B_{1p} + U_1(t) \\ X_2'(t) &= A_{21}X_1(t)B_{21} + A_{22}X_2(t)B_{22} + \cdots + A_{2p}X_p(t)B_{2p} + U_2(t) \\ &\vdots \\ X_p'(t) &= A_{p1}X_1(t)B_{p1} + A_{p2}X_2(t)B_{p2} + \cdots + A_{pp}X_p(t)B_{pp} + U_p(t). \end{aligned} \quad (4)$$

where $A_{ij}, B_{ij} \in M_n$ are given scalar matrices, $U_i(t) \in M_n$ is a given matrix function, $X_i(t) \in M_n$ are the unknown diagonal matrix functions to be solved and $X_i(0) = C_i$; and where $X_i(t)$ denotes the derivative of matrix function $X_i(t)$. $(i, j = 1, 2, \cdots, p)$. (where $M_{m,n}$ is the set of all $m \times n$ matrices over the complex number field $\mathbb{C}$ and when $m = n$, we write M_m instead of $M_{m,n}$).

Examples of such situation are singular [5] and hybrid system control [6] and nonzero sum differential games [7]. Depending on the problem considered, different coupling terms may appear. However, in all the above mentioned cases the systems are difficult to solve.

Let us recall some concepts that will be used below. Given two matrices $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{p,q}$, then the Kronecker product of *A* and *B* is defined by (e.g. [8-12])

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}. \quad (5)$$

While if $A \in M_{m,n}$, $B \in M_{p,n}$, and let $\{a_i : 1 \le i \le n\}$ and $\{b_i : 1 \le i \le n\}$ be the columns of *A* and *B*, respectively, namely

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n], \; B = [b_1 \quad b_2 \quad \cdots \quad b_n].$$

The columns of the Kronecker product $A \otimes B$ are $\{a_i \otimes b_j\}$ for all *i*, *j* combinations in lexicographic order namely,

$$A \otimes B = [a_1 \otimes b_1 \cdots a_1 \otimes b_n \cdots a_n \otimes b_1 \cdots a_n \otimes b_n] \quad (6)$$

Thus, the Khatri-Rao product of *A* and *B* is defined by [13,14]:

$$A \Theta B = [a_1 \otimes b_1 \quad a_2 \otimes b_2 \quad \cdots \quad a_n \otimes b_n] \quad (7)$$

consists of a subset of the columns of $A \otimes B$. Notice that $A \otimes B$ is of order $mp \times n^2$ and $A \Theta B$ is of order $mp \times n$. This observation can be expressed in the following form [15]:

$$(A \otimes B) S_n = A \Theta B, \quad (8)$$

where the selection matrix S_n is of order $n^2 \times n$ and

$$S_n = [e_1 \quad e_{n+2} \quad e_{2n+3} \quad \cdots \quad e_{n^2}] \quad (9)$$

and e_k is an $n^2 \times 1$ column vector with a unity element in the *k*-th position and zeros elsewhere $(1 \le k \le n^2)$.

Additionally, if both matrices $A = [a_{ij}]$ and $B = [b_{ij}] \in M_{m,n}$ have the same size, then the *Hadamard product* of *A* and *B* is defined by [8-11,16]:

$$A \circ B = [a_{ij} b_{ij}] \in M_{m,n}. \quad (10)$$

This product is much simpler than Kronecker and Khatri-Rao products and it can be connected with isomorphic diagonal matrix representations that can have a certain interest in many fields of pure and applied mathematics, for example, Tauber [16] applied the Hadamard product to solving a partial differential equation coming from an air pollution problem. The Hadamard product is clearly commutative, associative, and distributive with respect to addition. It has been known that $A \circ B$ is a (principal) submatrix of $A \otimes B$ if *A* and *B* are (square) of the same size. This can be found in Visick [12] and even in Zhang's book [17]. Liv-Ari [13, Theorem 3.1, p. 128] gave the following new relations related to Kronecker, Khatri-Rao and Hadamard products:

$$S_n^T (A \Theta B) = A \circ B; \quad (11)$$

$$S_n^T (A \otimes B) S_n = A \circ B. \quad (12)$$

The Kronecker product and vector operator affirming their capability of solving some matrix and matrix differential equations. Such equations can be readily converted into the standard linear equation form by using the well-known identity (e.g. [17,18]):

$$Vec(AXB^T) = (B \otimes A) VecX, \quad (13)$$

Where $Vec(.)$ denotes a vectorization by columns of a matrix. The need to compute the e^A, $\cosh(A)$ and $\sinh(A)$ are due its appearance in the solutions of coupled matrix differential equations. Here

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \left(\frac{A^k}{k!} \right); \sinh(A) = \frac{e^A - e^{-A}}{2}; \\ \cosh(A) &= \frac{e^A + e^{-A}}{2}. \end{aligned} \quad (14)$$

For any matrix $A \in M_m$, the spectral representation of e^A and e^{At} assures that [9,18]:

$$e^A = \sum_{i=0}^{n} x_i y_i^T e^{\lambda_i}; \; e^{At} = \sum_{i=0}^{n} x_i y_i^T e^{\lambda_i t}, \quad (15)$$

where $\{\lambda_1, \cdots, \lambda_n\}$ and $\{x_1, \cdots, x_n\}$ are the eigenvalues and the corresponding eigenvectors of *A*, and $\{y_1, \cdots, y_n\}$

is the eigenvectors of matrix A^T.

Finally, for any matrices $A, B, C, D \in M_n$, we shall make a frequent use the following properties of the Kronecker product (e.g. [9,18-20]) which are used to establish our results.

1) $(A\otimes B)(C\otimes D) = AC\otimes BD; (A\otimes B)^* = A^*\otimes B^*$ (16)

2) $e^{(A\otimes I_n + I_n + B)} = e^A\otimes e^B$; $e^{(A\otimes I_n)} = e^A\otimes I_n$;

$$e^{(I_n\otimes B)} = I_n\otimes e^A \quad (17)$$

3) $\sinh(A\otimes I_n) = (\sinh A)\otimes I_n$;

$$\sinh(I_n\otimes B) = I_n\otimes(\sinh B) \quad (18)$$

4) $\cosh(A\otimes I_n) = (\cosh A)\otimes I_n$;

$$\cosh(I_n\otimes B) = I_n\otimes(\cosh B). \quad (19)$$

In this paper, we present the efficient solution of coupled matrix linear least-squares problem and extend the use of diagonal extraction (vector) operator in order to construct a computationally-efficient solution of non-homogeneous coupled matrix linear differential equations.

2. Coupled Matrix Linear Least-Squares Problem

The multistatic antenna array processing problem can be written in matrix notation as [13]

$$Q = AXB^T;\ X = diag(\tau_i; 1\le i\le n). \quad (20)$$

where $A\in M_{m,n}$, $B\in M_{p,n}$ and $Q\in M_{m,p}$ are given (complex valued) matrices; and where the unknown matrix $X\in M_n$ is diagonal. We also assume that $n < mp$, so that we suggest using a least-squares approach, viz.,

$$\min_X \left\|Q - AXB^T\right\|_F^2, \quad (21)$$

where $\|A\|_F$ is called *Frobenius norm* of *A*. Using the identity in Equation (13) we can transform (21) into the vector LSP form:

$$\min_X \left\|VecQ - (B\otimes A)VecX\right\|_F^2. \quad (22)$$

which has the well-known solution:

$$VecX = \left((B\otimes A)^*(B\otimes A)\right)^{-1}(B\otimes A)^* VecQ, \quad (23)$$

provided $(B\otimes A)^*(B\otimes A)$ is invertible.

Applying the direct vector transformation in Equation (13) to $Q - AXB^T$ results in a highly inefficient least-square problem, because *VecX* is very sparse. Liv-Ari [13] described an alternative approach based on:

$$Vec(AXB^T) = (B\Theta A)Vecd(X),\ X \text{ is diagonal} \quad (24)$$

which involves the so-called Khatri-Rao product Θ, as well as the *diagonal extraction operator* $vecd(X)$:

$$Vecd(X) = \begin{pmatrix} x_{11} & x_{22} & \cdots & x_{nn}\end{pmatrix}^T \quad (25)$$

which forms a column vector consisting of the diagonal elements of the $n\times n$ square matrix *X*, instead of the much longer column vector *VecX*. In addition, if *Y* is any matrix of order $m\times p$, then

$$Vecd(A^T YB) = (B\Theta A)^T VecY. \quad (26)$$

As we have observed earlier, when the unknown matrix *X* is diagonal, solving for *VecX* is highly inefficient, since most of the elements of *X* vanish. Instead Liv-Ari [13] used the more compact vectorization identity to rewrite matrix LSP (21) in the vector form:

$$\min_X \left\|VecQ - (B\Theta A)Vecd(X)\right\|_F^2. \quad (27)$$

Notice that $Vecd(X)$ consists of only the nontrivial (*i.e.*, diagonal) elements of the matrix *X*. The explicit solution of (27) is

$$Vecd(X) = \left((B\Theta A)^*(B\Theta A)\right)^{-1}(B\Theta A)^* VecQ. \quad (28)$$

provided $(B\Theta A)^*(B\Theta A)$ is invertible.

It turns out that this expression can also be implemented using Hadamard product, resulting in a significant reduction in computational cost, as implied the following result [13]:

$$(A\Theta B)^*(A\Theta B) = (A^*A)\circ(B^*B), \quad (29)$$

where $A\in M_{m,n}$ and $B\in M_{p,n}$.

When $n < \min\{m,p\}$, we observe that the left-hand side expression in Equation (29) requires $mpn + mpn(n+1)/2$ multiplications, while forming the equivalent right-hand side expression requires only $(m+p+1)n(n+1)/2$ multiplications. Thus the latter offers significant computational savings, especially when $mp \gg m+p+1$.

Now, using (26) we can rewrite (28) in the more compact form:

$$Vecd(X) = \left((B^*B)\circ(A^*A)\right)^{-1} Vecd\left\{A^*Qconj(B)\right\}. \quad (30)$$

This expression which requires $O(n^3) + O([m+p]n^2)$ (multiply and add) operations is much more efficient than (28), which requires $O(n^3) + O(mpn^2)$ operations. It means that the computational advantage of using the Hadamard product expression is particularly evident when $n < \min\{m,p\}$, which implies that $mp \gg m+p \gg n$. In order to be able to use (30) we must ascertain that the matrix $(B^*B)\circ(A^*A)$ is invertible. This will hold, for instance, when both *A* and *B* have full column rank.

As for the *diagonal extraction operator* $vecd(.)$, we observe that for any square $n\times n$ matrix $Y=\left[y_{ij}\right]$,

$$Vecd(Y)=S_n^T VecY . \quad (31)$$

If Y is diagonal, then we also have

$$VecY=S_n Vecd(Y),\ Y \text{ is diagonal.} \quad (32)$$

Moreover, the columns of the $n^2\times n$ selection matrix S_n are mutually orthonormal, viz.,

$$S_n^T S_n=I_n . \quad (33)$$

Using (32) and (11), we get the fundamental relation between the Hadamard product and *diagonal extraction operator* $vecd(.)$ which is given by

$$Vecd\left(AXB^T\right)=(B\circ A)Vecd(X),\ X \text{ is diagonal} \quad (34)$$

where A, B and X is $n\times n$ diagonal matrix.

Now we will discuss the efficient and more efficient least-squares solutions of coupled matrix linear equations:

$$AXC^T+BYC^T=E,\quad BXC^T+AYC^T=F \quad (35)$$

where $A, B\in M_{m,n}$ $C\in M_{p,n}$, $E, F\in M_{m,p}$ are given scalar matrices and $X, Y\in M_n$ are unknown matrices to be solved. We also assume that $n<mp$, so that the coupled matrix linear Equations (35) is over-determined, which suggests using a least squares approach. We consider the coupled matrix linear least-squares problem (CLSP):

$$\min_{X,Y}\left\|\begin{bmatrix}E\\F\end{bmatrix}-\begin{bmatrix}AXC^T+BYC^T\\BXC^T+AYC^T\end{bmatrix}\right\|_F^2 . \quad (36)$$

The solution procedure presented here may be considered as a continuation of the method proposed to solve least-squares problem in (21).

Using the identity (13) we can transform (36) into the vector CLSP form [10]:

$$\min_{X,Y}\left\|\begin{bmatrix}VecE\\VecF\end{bmatrix}-\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}\begin{bmatrix}VecX\\VecY\end{bmatrix}\right\|_F^2 \quad (37)$$

which has the following solution

$$\begin{bmatrix}VecX\\VecY\end{bmatrix}=\left(\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}^*\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}\right)^{-1}\times\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}^*\begin{bmatrix}VecE\\VecF\end{bmatrix}. \quad (38)$$

One can easily show that

$$\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}=U\begin{bmatrix}C\otimes(A+B) & 0\\0 & C\otimes(A-B)\end{bmatrix}U^T \quad (39)$$

where $U=\frac{1}{\sqrt{2}}\begin{bmatrix}I & -I\\I & I\end{bmatrix}$ is a unitary matrix. So

$$\begin{aligned}\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}^*\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}&=U\begin{bmatrix}(C\otimes(A+B))^* & 0\\0 & (C\otimes(A-B))^*\end{bmatrix}U^T\times U\begin{bmatrix}C\otimes(A+B) & 0\\0 & C\otimes(A-B)\end{bmatrix}U^T\\&=U\begin{bmatrix}(C\otimes(A+B))^*(C\otimes(A+B)) & 0\\0 & (C\otimes(A-B))^*(C\otimes(A-B))\end{bmatrix}U^T\end{aligned} \quad (40)$$

Suppose that $H=C\otimes(A+B)$ and $W=C\otimes(A-B)$, we then have

$$\begin{aligned}\left(\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}^*\begin{bmatrix}C\otimes A & C\otimes B\\C\otimes B & C\otimes A\end{bmatrix}\right)^{-1}&=U\begin{bmatrix}\left((C\otimes(A+B))^*(C\otimes(A+B))\right)^{-1} & 0\\0 & \left((C\otimes(A-B))^*(C\otimes(A-B))\right)^{-1}\end{bmatrix}U^T\\&=\frac{1}{2}\begin{bmatrix}I & -I\\I & I\end{bmatrix}\begin{bmatrix}(H^*H)^{-1} & 0\\0 & (W^*W)^{-1}\end{bmatrix}\begin{bmatrix}I & I\\-I & I\end{bmatrix}=\frac{1}{2}\begin{bmatrix}(H^*H)^{-1}+(W^*W)^{-1} & (H^*H)^{-1}-(W^*W)^{-1}\\(H^*H)^{-1}-(W^*W)^{-1} & (H^*H)^{-1}+(W^*W)^{-1}\end{bmatrix}.\end{aligned} \quad (41)$$

Now the least—squares solutions (38) can be rewrite into the form:

$$\begin{bmatrix}VecX\\VecY\end{bmatrix}=\frac{1}{2}\begin{bmatrix}(H^*H)^{-1}+(W^*W)^{-1} & (H^*H)^{-1}-(W^*W)^{-1}\\(H^*H)^{-1}-(W^*W)^{-1} & (H^*H)^{-1}+(W^*W)^{-1}\end{bmatrix}\times\begin{bmatrix}(C\otimes A)^* & (C\otimes B)^*\\(C\otimes B)^* & (C\otimes A)^*\end{bmatrix}\begin{bmatrix}VecE\\VecF\end{bmatrix}. \quad (42)$$

This gives

$$\begin{aligned} VecX &= \frac{1}{2}\times\left\{\left(\left(H^*H\right)^{-1}+\left(W^*W\right)^{-1}\right)\left(C\otimes A\right)^*+\left(\left(H^*H\right)^{-1}-\left(W^*W\right)^{-1}\right)\left(C\otimes B\right)^*\right\}\times VecE \\ &+\frac{1}{2}\times\left\{\left(\left(H^*H\right)^{-1}+\left(W^*W\right)^{-1}\right)\left(C\otimes B\right)^*+\left(\left(H^*H\right)^{-1}-\left(W^*W\right)^{-1}\right)\left(C\otimes A\right)^*\right\}\times VecF \\ VecY &= \frac{1}{2}\left\{\left(\left(H^*H\right)^{-1}-\left(W^*W\right)^{-1}\right)\left(C\otimes A\right)^*+\left(\left(H^*H\right)^{-1}+\left(W^*W\right)^{-1}\right)\left(C\otimes B\right)^*\right\}\times VecE \\ &+\frac{1}{2}\left\{\left(\left(H^*H\right)^{-1}-\left(W^*W\right)^{-1}\right)\left(C\otimes B\right)^*+\left(\left(H^*H\right)^{-1}+\left(W^*W\right)^{-1}\right)\left(C\otimes A\right)^*\right\}\times VecF \end{aligned} \tag{43}$$

where $H = C\otimes(A+B)$ and $W = C\otimes(A-B)$.

In order to be able to use (38) and (43) we must ascertain that the matrix:

$$\begin{bmatrix} C\otimes A & C\otimes B \\ C\otimes B & C\otimes A \end{bmatrix}^* \begin{bmatrix} C\otimes A & C\otimes B \\ C\otimes B & C\otimes A \end{bmatrix}$$

is invertible if and only one

$$H^*H = \left(C\otimes(A+B)\right)^*\left(C\otimes(A+B)\right)$$

and

$$W^*W = \left(C\otimes(A-B)\right)^*\left(C\otimes(A-B)\right)$$

are invertible matrices.

As we observed, when the unknown matrices X and $Y \in M_n$ are diagonal, solving for $VecX$ and $VecY$ are highly inefficient, since most of the elements of X and Y vanish. Instead we can use the more compact vectorization identity (24) to rewrite the coupled matrix linear least-squares problem (37) in the *reduced-order vector* form:

$$\min_{X,Y}\left\|\begin{bmatrix} VecE \\ VecF \end{bmatrix}-\begin{bmatrix} C\Theta A & C\Theta B \\ C\Theta B & C\Theta A \end{bmatrix}\begin{bmatrix} Vecd\{X\} \\ Vecd\{Y\} \end{bmatrix}\right\|_F^2. \tag{44}$$

Notice that $Vecd\{X\}$ and $Vecd\{Y\}$ consists of only the nontrivial (*i.e.*, diagonal) elements of matrices X and Y. The explicit efficient solution of (44) is

$$\begin{aligned} Vecd\{X\} &= \frac{1}{2}\left\{\left(\left(R^*R\right)^{-1}+\left(S^*S\right)^{-1}\right)\left(C\Theta A\right)^*+\left(\left(R^*R\right)^{-1}-\left(S^*S\right)^{-1}\right)\left(C\Theta B\right)^*\right\}\times VecE \\ &+\frac{1}{2}\left\{\left(\left(R^*R\right)^{-1}+\left(S^*S\right)^{-1}\right)\left(C\Theta B\right)^*+\left(\left(R^*R\right)^{-1}-\left(S^*S\right)^{-1}\right)\left(C\Theta A\right)^*\right\}\times VecF \\ Vecd\{Y\} &= \frac{1}{2}\left\{\left(\left(R^*R\right)^{-1}-\left(S^*S\right)^{-1}\right)\left(C\Theta A\right)^*+\left(\left(R^*R\right)^{-1}+\left(S^*S\right)^{-1}\right)\left(C\Theta B\right)^*\right\}\times VecE \\ &+\frac{1}{2}\left\{\left(\left(R^*R\right)^{-1}-\left(S^*S\right)^{-1}\right)\left(C\Theta B\right)^*+\left(\left(R^*R\right)^{-1}+\left(S^*S\right)^{-1}\right)\left(C\Theta A\right)^*\right\}\times VecF \end{aligned} \tag{45}$$

where $R = C\Theta(A+B)$ and $S = C\Theta(A-B)$.

In order to be able to use (45), we must ascertain that the matrix

$$R^*R = \left(C\Theta(A+B)\right)^*\left(C\Theta(A+B)\right)$$

and

$$S^*S = \left(C\Theta(A-B)\right)^*\left(C\Theta(A-B)\right)$$

are invertible matrices.

It turns out that the expression (45) can also be implemented using Hadamard product by the same technique in the expression (30). Note that the least squares solutions in term of Hadamard product is more efficient than (45) and (43).

3. Non-Homogeneous Matrix Differential Equations

The solution procedure presented here may be considered as a continuation of the method proposed to solve the homogenous coupled matrix differential equations in [18]. We will use our knowledge of the solution of the of simplest homogeneous matrix differential equation:

$$X'(t) = AX(t),\quad X(0) = C \tag{46}$$

where $A\in M_m$, $C\in M_{m,n}$ are given scalar matrices, and $X(t)\in M_{m,n}$ is the unknown matrix function to be solved. In fact the unique solution of (46) is given by:

$$X(t) = \mathrm{e}^{At}C. \tag{47}$$

Theorem 3.1 Let $A\in M_m$, $C\in M_{m,n}$ are given scalar matrices, $U(t)\in M_{m,n}$ is a given matrix function and $X(t)\in M_{m,n}$ is the unknown matrix. Then the general solution of the non-homogeneous matrix differential equation:

$$X'(t) = AX(t)+U(t),\quad X(0) = C \tag{48}$$

is given by

$$X(t)=\mathrm{e}^{At}C+\mathrm{e}^{At}*U(t). \tag{49}$$

Where $\mathrm{e}^{At}*U(t)=\int_0^t \mathrm{e}^{A(t-s)}U(s)\mathrm{d}s$ is well-defined, which involves the *convolution product* of two matrices e^{At} and $U(t)$.

Proof: Suppose that $X_p(t)=\mathrm{e}^{At}G(t)$ is the particular solution of (48). The product rule of differentiation gives

$$X_p(t)=\mathrm{e}^{At}G'(t)+A\mathrm{e}^{At}G(t).$$

Substituting these in (48) we obtain

$$\mathrm{e}^{At}G'(t)+A\mathrm{e}^{At}G(t)=A\mathrm{e}^{At}G(t)+U(t)$$

Thus

$$\mathrm{e}^{At}G'(t)=U(t). \tag{50}$$

Multiplying both sides of (50) by $\mathrm{e}^{-At}=\left(\mathrm{e}^{At}\right)^{-1}$ gives

$$G'(t)=\mathrm{e}^{-At}U(t) \tag{51}$$

Integrating both sides of (51) between 0 and *t* gives

$$G(t)=\int_0^t \mathrm{e}^{-As}U(s)\cdot\mathrm{d}s \tag{52}$$

Hence, by assumption, we conclude that the particular solution of equation (48) is

$$X_p(t)=\mathrm{e}^{At}G(t)=\int_0^t \mathrm{e}^{A(t-s)}U(s)\mathrm{d}s=\mathrm{e}^{At}*U(t). \tag{53}$$

Now from (47) and (53) we get (49).

Theorem 3.2 Let $A=\left[a_{ij}\right]$, $B=\left[b_{ij}\right]$, $C\in M_n$ are given scalar matrices, $U(t)\in M_n$ is a given matrix function and $X(t)\in M_n$ is unknown diagonal matrix function. Then the general solution of non-homogeneous matrix differential equation

$$X'(t)=AX(t)+X(t)B+U(t),\ X(0)=C \tag{54}$$

is given by

$$Vecd\left\{X(t)\right\}=diag\left(\mathrm{e}^{(a_{11}+b_{11})t},\cdots,\mathrm{e}^{(a_{nn}+b_{nn})t}\right)Vecd\left\{C\right\}+diag\left(\mathrm{e}^{(a_{11}+b_{11})t},\cdots,\mathrm{e}^{(a_{nn}+b_{nn})t}\right)*Vecd\left\{U(t)\right\}. \tag{55}$$

Proof: Using the identity (34) we can transform (54) into the vector form:

$$\begin{aligned}Vecd\left\{X'(t)\right\}&=\left(I_n\circ A+B^T\circ I_n\right)Vecd\left\{X(t)\right\}+Vecd\left\{U(t)\right\}=\left\{\left(A+B^T\right)\circ I_n\right\}Vecd\left\{X(t)\right\}+Vecd\left\{U(t)\right\}\\&=diag\left(a_{11}+b_{11},\cdots,a_{nn}+b_{nn}\right)Vecd\left\{X(t)\right\}+Vecd\left\{U(t)\right\}.\end{aligned} \tag{56}$$

Now, applying (49), then the unique solution of (56) is

$$\begin{aligned}Vecd\left\{X(t)\right\}&=\mathrm{e}^{diag(a_{11}+b_{11},\cdots,a_{nn}+b_{nn})t}Vecd\left\{C\right\}+\mathrm{e}^{diag(a_{11}+b_{11},\cdots,a_{nn}+b_{nn})t}*Vecd\left\{U(t)\right\}\\&=diag\left(\mathrm{e}^{(a_{11}+b_{11})t},\cdots,\mathrm{e}^{(a_{nn}+b_{nn})t}\right)Vecd\left\{C\right\}+diag\left(\mathrm{e}^{(a_{11}+b_{11})t},\cdots,\mathrm{e}^{(a_{nn}+b_{nn})t}\right)*Vecd\left\{U(t)\right\}.\end{aligned}$$

If we put $U(t)=0$ in Theorem 3.2 we obtain the following result.

Corollary 3.3 Let *A*, *B*, $C\in M_n$ are given scalar matrices. Then general solution of the homogeneous matrix differential equation:

$$\begin{aligned}&X'(t)=AX(t)+X(t)B,\ X(0)=C,\\&X(t)\ \text{is diagonal.}\end{aligned} \tag{57}$$

is given by

$$Vecd\left\{X(t)\right\}=diag\left(\mathrm{e}^{(a_{11}+b_{11})t},\cdots,\mathrm{e}^{(a_{nn}+b_{nn})t}\right)Vecd\left\{C\right\} \tag{58}$$

Now we will discuss the general class of non-homogeneous coupled matrix differential equations which defined in (4): By using the $Vecd(.)$-notation of (4), we have

$$\begin{bmatrix}Vecd\left\{X_1'(t)\right\}\\Vecd\left\{X_2'(t)\right\}\\\vdots\\Vecd\left\{X_p'(t)\right\}\end{bmatrix}=\begin{bmatrix}B_{11}^T\circ A_{11} & B_{12}^T\circ A_{12} & \cdots & B_{1p}^T\circ A_{1p}\\B_{21}^T\circ A_{21} & B_{22}^T\circ A_{22} & \cdots & B_{2p}^T\circ A_{2p}\\\vdots & \vdots & \vdots & \vdots\\B_{p1}^T\circ A_{p1} & B_{p2}^T\circ A_{p2} & \cdots & B_{pp}^T\circ A_{pp}\end{bmatrix}\times\begin{bmatrix}Vecd\left\{X_1(t)\right\}\\Vecd\left\{X_2(t)\right\}\\\vdots\\Vecd\left\{X_p(t)\right\}\end{bmatrix}+\begin{bmatrix}Vecd\left\{U_1(t)\right\}\\Vecd\left\{U_2(t)\right\}\\\vdots\\Vecd\left\{U_p(t)\right\}\end{bmatrix}. \tag{59}$$

Let

$$x'(t)=\begin{bmatrix} Vecd\{X_1'(t)\} \\ Vecd\{X_2'(t)\} \\ \vdots \\ Vecd\{X_p'(t)\} \end{bmatrix}, x(t)=\begin{bmatrix} Vecd\{X_1(t)\} \\ Vecd\{X_2(t)\} \\ \vdots \\ Vecd\{X_p(t)\} \end{bmatrix}, c=\begin{bmatrix} Vecd\{C_1\} \\ Vecd\{C_2\} \\ \vdots \\ Vecd\{C_p\} \end{bmatrix}, u(t)=\begin{bmatrix} Vecd\{U_1(t)\} \\ Vecd\{U_2(t)\} \\ \vdots \\ Vecd\{U_p(t)\} \end{bmatrix}$$
$$H=\begin{bmatrix} B_{11}^T\circ A_{11} & B_{12}^T\circ A_{12} & \cdots & B_{1p}^T\circ A_{1p} \\ B_{21}^T\circ A_{21} & B_{22}^T\circ A_{22} & \cdots & B_{2p}^T\circ A_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ B_{p1}^T\circ A_{p1} & B_{p2}^T\circ A_{p2} & \cdots & B_{pp}^T\circ A_{pp} \end{bmatrix}. \quad (60)$$

Now (59) can be written as

$$x'(t)=Hx(t)+u(t),\ x(0)=c$$

and the general solution is given by:

$$x(t)=e^{Ht}c+e^{Ht}*u(t). \quad (61)$$

Note that there is many special cases can be considered from the above general class coupled matrix differential equations; now we will discuss some important special cases in the next results.

Theorem 3.4 Let *A*, *B*, *C*, *D*, *E*, $F\in M_n$ are given scalar matrices such that

$$\left(D^T\circ C\right)\left(B^T\circ A\right)=\left(B^T\circ A\right)=\left(D^T\circ C\right);$$
$$U(t),V(t)\in M_n$$

are given matrix functions and $X(t)$, $Y(t)\in M_n$ are the unknown diagonal matrices. Then the general solution of non-homogeneous coupled matrix differential equations:

$$X'(t)=AX(t)B+CY(t)D+U(t),$$
$$Y'(t)=CX(t)D+AY(t)B+V(t), \quad (62)$$
$$X(0)=E, Y(0)=F$$

is given by

$$Vecd\{X(t)\}=e^{\left(B^T\circ A\right)t}\left\{\left[\cosh\left(D^T\circ C\right)t\right]Vecd\{E\}+\left[\sinh\left(D^T\circ C\right)t\right]Vecd\{F\}\right\}$$
$$+e^{\left(B^T\circ A\right)t}\times\left\{\left[\cosh\left(D^T\circ C\right)t\right]*Vecd\{U(t)\}+\left[\sinh\left(D^T\circ C\right)t\right]*Vecd\{V(t)\}\right\}Vecd\{Y(t)\}$$
$$Vecd\{Y(t)\}=e^{\left(B^T\circ A\right)t}\left\{\left[\sinh\left(D^T\circ C\right)t\right]Vecd\{E\}+\left[\cosh\left(D^T\circ C\right)t\right]Vecd\{F\}\right\}$$
$$+e^{\left(B^T\circ A\right)t}\left\{\left[\sinh\left(D^T\circ C\right)t\right]*Vecd\{U(t)\}+\left[\cosh\left(D^T\circ C\right)t\right]*Vecd\{V(t)\}\right\} \quad (63)$$

Proof: Using the identity (34) we can transform (62) into the vector form:

$$\begin{bmatrix} Vecd\{X'(t)\} \\ Vecd\{Y'(t)\} \end{bmatrix}=\begin{bmatrix} B^T\circ A & D^T\circ C \\ D^T\circ C & B^T\circ A \end{bmatrix}\begin{bmatrix} Vecd\{X(t)\} \\ Vecd\{Y(t)\} \end{bmatrix}+\begin{bmatrix} Vecd\{U(t)\} \\ Vecd\{V(t)\} \end{bmatrix} \quad (64)$$

From (61), this system has the following solution:

$$\begin{bmatrix} Vecd\{X(t)\} \\ Vecd\{Y(t)\} \end{bmatrix}=e^{\begin{bmatrix} B^T\circ A & D^T\circ C \\ D^T\circ C & B^T\circ A \end{bmatrix}t}\times\begin{bmatrix} Vecd\{E\} \\ Vec\{F\} \end{bmatrix}+e^{\begin{bmatrix} B^T\circ A & D^T\circ C \\ D^T\circ C & B^T\circ A \end{bmatrix}t}*\begin{bmatrix} Vecd\{U(t)\} \\ Vecd\{V(t)\} \end{bmatrix}. \quad (65)$$

Now we will deal with

$$e^{\begin{bmatrix} B^T\circ A & D^T\circ C \\ D^T\circ C & B^T\circ A \end{bmatrix}}. \quad (66)$$

Since $\left(D^T\circ C\right)\left(B^T\circ A\right)=\left(B^T\circ A\right)=\left(D^T\circ C\right)$, then we have

$$\begin{bmatrix} B^T\circ A & 0 \\ 0 & B^T\circ A \end{bmatrix}\begin{bmatrix} 0 & D^T\circ C \\ D^T\circ C & 0 \end{bmatrix}$$
$$=\begin{bmatrix} 0 & D^T\circ C \\ D^T\circ C & 0 \end{bmatrix}\begin{bmatrix} B^T\circ A & 0 \\ 0 & B^T\circ A \end{bmatrix}.$$

Then

$$e^{\begin{bmatrix} B^T\circ A & D^T\circ C \\ D^T\circ C & B^T\circ A\end{bmatrix}} = e^{\begin{bmatrix} B^T\circ A & 0 \\ 0 & B^T\circ A\end{bmatrix}+\begin{bmatrix} 0 & D^T\circ C \\ D^T\circ C & 0\end{bmatrix}} = e^{\begin{bmatrix} B^T\circ A & 0 \\ 0 & B^T\circ A\end{bmatrix}}\cdot e^{\begin{bmatrix} 0 & D^T\circ C \\ D^T\circ C & 0\end{bmatrix}}.$$

But

$$e^{\begin{bmatrix} B^T\circ A & 0 \\ 0 & B^T\circ A\end{bmatrix}} = \begin{bmatrix} e^{(B^T\circ A)} & 0 \\ 0 & e^{(B^T\circ A)}\end{bmatrix};$$

$$e^{\begin{bmatrix} 0 & D^T\circ C \\ D^T\circ C & 0\end{bmatrix}} = \begin{bmatrix} \frac{e^{(D^T\circ C)}+e^{-(D^T\circ C)}}{2} & \frac{e^{(D^T\circ C)}-e^{-(D^T\circ C)}}{2} \\ \frac{e^{(D^T\circ C)}-e^{-(D^T\circ C)}}{2} & \frac{e^{(D^T\circ C)}+e^{-(D^T\circ C)}}{2}\end{bmatrix}.$$

So

$$e^{\begin{bmatrix} B^T\circ A & D^T\circ C \\ D^T\circ C & B^T\circ A\end{bmatrix}} = \begin{bmatrix} e^{(B^T\circ A)} & 0 \\ 0 & e^{(B^T\circ A)}\end{bmatrix} \times \begin{bmatrix} \frac{e^{(D^T\circ C)}+e^{-(D^T\circ C)}}{2} & \frac{e^{(D^T\circ C)}-e^{-(D^T\circ C)}}{2} \\ \frac{e^{(D^T\circ C)}-e^{-(D^T\circ C)}}{2} & \frac{e^{(D^T\circ C)}+e^{-(D^T\circ C)}}{2}\end{bmatrix} = \begin{bmatrix} e^{(B^T\circ A)}\cosh(D^T\circ C) & e^{(B^T\circ A)}\sinh(D^T\circ C) \\ e^{(B^T\circ A)}\sinh(D^T\circ C) & e^{(B^T\circ A)}\cosh(D^T\circ C)\end{bmatrix} \tag{67}$$

Due to (67) we have

$$e^{\begin{bmatrix} B^T\circ A & D^T\circ C \\ D^T\circ C & B^T\circ A\end{bmatrix}t} \times \begin{bmatrix} Vecd\{E\} \\ Vecd\{F\}\end{bmatrix} = \begin{bmatrix} e^{(B^T\circ A)t}\cosh(D^T\circ C)t & e^{(B^T\circ A)t}\sinh(D^T\circ C)t \\ e^{(B^T\otimes A)t}\sinh(D^T\circ C)t & e^{(B^T\circ A)t}\cosh(D^T\circ C)t\end{bmatrix} \times \begin{bmatrix} Vecd\{E\} \\ Vecd\{F\}\end{bmatrix}; \tag{68}$$

$$e^{\begin{bmatrix} B^T\circ A & D^T\circ C \\ D^T\circ C & B^T\circ A\end{bmatrix}t} * \begin{bmatrix} Vecd\{U(t)\} \\ Vecd\{V(t)\}\end{bmatrix} = \begin{bmatrix} e^{(B^T\circ A)t}\cosh(D^T\circ C)t & e^{(B^T\circ A)t}\sinh(D^T\circ C)t \\ e^{(B^T\circ A)t}\sinh(D^T\circ C)t & e^{(B^T\circ A)t}\cosh(D^T\circ C)t\end{bmatrix} * \begin{bmatrix} Vecd\{U(t)\} \\ Vecd\{V(t)\}\end{bmatrix}. \tag{69}$$

Now substitute (68) and (69) in (65), we get (63).

If we put $U(t)=V(t)=0$ in **Theorem 3.4** we obtain the following result.

Corollary 3.5 Let $A, B, C, D, E, F\in M_n$ are given scalar matrices such that

$$(D^T\circ C)(B^T\circ A)=(B^T\circ A)=(D^T\circ C),$$

and $X(t), Y(t)\in M_n$ are the unknown diagonal matrices. Then the general solution of homogeneous coupled matrix differential equations:

$$\begin{aligned} X'(t) &= AX(t)B+CY(t)D, \\ Y'(t) &= CX(t)D+AY(t)B, \\ X(0) &= E, Y(0)=F \end{aligned} \tag{70}$$

is given by

$$\begin{aligned} Vecd\{X(t)\} &= e^{(B^T\circ A)t}\left\{\left[\cosh(D^T\circ C)t\right]Vecd\{E\}+\left[\sinh(D^T\circ C)t\right]Vecd\{F\}\right\}; \\ Vecd\{Y(t)\} &= e^{(B^T\circ A)t}\left\{\left[\sinh(D^T\circ C)t\right]Vecd\{E\}+\left[\cosh(D^T\circ C)t\right]Vecd\{F\}\right\}. \end{aligned} \tag{71}$$

Corollary 3.6 Let $B=[b_{ij}]$, $D=[d_{ij}]$, E, $F\in M_n$ are given scalar matrices and $X(t)$, $Y(t)\in M_n$ are the unknown diagonal matrices. Then the general solution of homogeneous coupled matrix differential equations:

$$\begin{aligned} X'(t) &= X(t)B+Y(t)D, \\ Y'(t) &= X(t)D+Y(t)B, \\ X(0) &= E, Y(0)=F \end{aligned} \tag{72}$$

is given by

$$\begin{aligned} Vecd\{X(t)\} &= \left\{diag\left(e^{b_{11}t}\cosh d_{11}t,\cdots,e^{b_{nn}t}\cosh d_{nn}t\right)\right\}Vecd\{E\}+\left\{diag(e^{b_{11}t}\sinh d_{11}t,..,e^{b_{nn}t}\sinh d_{nn}t)\right\}Vecd\{F\}; \\ Vecd\{Y(t)\} &= \left\{diag\left(e^{b_{11}t}\sinh d_{11}t,\cdots,e^{b_{nn}t}\sinh d_{nn}t\right)\right\}Vecd\{E\}+\left\{diag(e^{b_{11}t}\cosh d_{11}t,..,e^{b_{nn}t}\cosh d_{nn}t)\right\}Vecd\{F\}. \end{aligned} \tag{73}$$

Proof: For any matrix $A=[a_{ij}]\in M_n$, it is easily to show that

$$\cosh(A^T\circ I)t = diag\left(\cosh(a_{11}t),\cosh(a_{22}t),\cdots,\cosh(a_{nn}t)\right); \tag{74}$$

$$\sinh\left(A^T \circ I\right)t = diag\left(\sinh\left(a_{11}t\right), \sinh\left(a_{22}t\right), \cdots, \sinh\left(a_{nn}t\right)\right). \tag{75}$$

Now put $A = C = I_n$ in **Corollary 3.5** we have

$$\begin{aligned} Vecd\left\{X(t)\right\} &= e^{\left(B^T \otimes I_n\right)t}\left\{\left[\cosh\left(D^T \circ I_n\right)t\right]Vecd\left\{E\right\} + \left[\sinh\left(D^T \circ I_n\right)t\right]Vecd\left\{F\right\}\right\} \\ &= \left\{diag\left(e^{b_{11}t}, \cdots, e^{b_{nn}t}\right) \cdot diag\left(\cosh d_{11}t, \cdots, \cosh d_{nn}t\right)\right\}Vecd\left\{E\right\} \\ &\quad + \left\{diag\left(e^{b_{11}t}, \cdots, e^{b_{nn}t}\right) \cdot diag\left(\sinh d_{11}t, \cdots, \sinh d_{nn}t\right)\right\}Vecd\left\{F\right\} \\ &= \left\{diag\left(e^{b_{11}t}\cosh d_{11}t, \cdots, e^{b_{nn}t}\cosh d_{nn}t\right)\right\}Vecd\left\{E\right\} + \left\{diag\left(e^{b_{11}t}\sinh d_{11}t, ..., e^{b_{nn}t}\sinh d_{nn}t\right)\right\}Vecd\left\{F\right\} \end{aligned}$$

Similarly,

$$Vecd\left\{Y(t)\right\} = \left\{diag\left(e^{b_{11}t}\sinh d_{11}t, \cdots, e^{b_{nn}t}\sinh d_{nn}t\right)\right\}Vecd\left\{E\right\} + \left\{diag\left(e^{b_{11}t}\cosh d_{11}t, \cdots, e^{b_{nn}t}\cosh d_{nn}t\right)\right\}Vecd\left\{F\right\}.$$

While if we applying the fundamental relation between $Vec(.)$ and Kronecker product defined in (13) and using the same technique in the proof of Theorem 3.4 we obtain (for any matrix $X \in M_n$) the following result.

Theorem 3.7 Let A, B, C, D, E, $F \in M_n$ are given scalar matrices such that $AC = CA$, $BD = DB$, $U(t)$, $V(t) \in M_n$ are given matrix functions and $X(t)$, $Y(t) \in M_n$ are the unknown matrices. Then the general solution of non-homogeneous coupled matrix differential equations:

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t) \\ Y'(t) &= CX(t)D + AY(t)B + V(t) \\ X(0) &= E, Y(0) = F \end{aligned} \tag{76}$$

is given by

$$\begin{aligned} VecX(t) &= e^{\left(B^T \otimes A\right)t}\left\{\left[\cosh\left(D^T \otimes C\right)t\right]VecE + \left[\sinh\left(D^T \otimes C\right)t\right]VecF\right\} \\ &\quad + e^{\left(B^T \otimes A\right)t}\left\{\left[\cosh\left(D^T \otimes C\right)t\right] * VecU(t) + \left[\sinh\left(D^T \otimes C\right)t\right] * VecV(t)\right\} \\ VecY(t) &= e^{\left(B^T \otimes A\right)t}\left\{\left[\sinh\left(D^T \otimes C\right)t\right]VecE + \left[\cosh\left(D^T \otimes C\right)t\right]VecF\right\} \\ &\quad + e^{\left(B^T \otimes A\right)t}\left\{\left[\sinh\left(D^T \otimes C\right)t\right] * VecU(t) + \left[\cosh\left(D^T \otimes C\right)t\right] * VecV(t)\right\} \end{aligned} \tag{77}$$

If we put $U(t) = V(t) = 0$ and $A = C = I_n$ in **Theorem 3.7** and using properties (16)-(19) we obtain the following results.

Corollary 3.8 Let B, D, E, $F \in M_n$ are given scalar matrices such that $BD = DB$ and $X(t)$, $Y(t) \in M_n$ are the unknown matrices. Then the general solution of homogeneous coupled matrix differential equations:

$$\begin{aligned} X'(t) &= X(t)B + Y(t)D, \\ Y'(t) &= X(t)D + Y(t)B, \\ X(0) &= E, Y(0) = F \end{aligned} \tag{78}$$

is given by

$$\begin{aligned} X(t) &= \left\{E\cosh(Dt) + F\sinh(Dt)\right\}e^{Bt}, \\ Y(t) &= \left\{E\sinh(Dt) + F\cosh(Dt)\right\}e^{Bt}. \end{aligned} \tag{79}$$

Corollary 3.9 Let A, C, E, $F \in M_n$ are given scalar matrices such that $AC = CA$ and $X(t)$, $Y(t) \in M_n$ are the unknown matrices .Then the general solution of homogeneous coupled matrix differential equations:

$$\begin{aligned} X'(t) &= AX(t) + CY(t), \\ Y'(t) &= CX(t) + AY(t), \\ X(0) &= E, Y(0) = F \end{aligned} \tag{80}$$

is given by

$$\begin{aligned} X(t) &= e^{At}\left\{\cosh(Ct) \cdot E + \sinh(Ct) \cdot F\right\}, \\ Y(t) &= e^{At}\left\{\sinh(Ct) \cdot E + \cosh(Ct) \cdot F\right\}. \end{aligned} \tag{81}$$

4. Concluding Remarks

We have studied an explicit characterization of the mappings

$$A \otimes B \Rightarrow A\Theta B \Rightarrow A \circ B$$

in terms of the selection matrix S_n as in (11) and (12).

We have also observed that the same matrix relates the two operators $Vec(.)$ and $Vecd(.)$ as in (31) and (32). We used the fundamental relation between the Hadamard (Kronecker) product and diagonal extraction (vector) operator in (34) and (13) to derive our main results in Section 2 and 3 and, subsequently, to construct a computationally-efficient solution of coupled matrix least-squares problem and non-homogeneous coupled matrix differential equations. In fact, the Kronecker (Hadamard) product and operator $Vec(.)$ ($Vecd(.)$) affirming their capability of solving matrix and matrix differential equations fast (more fast when the unknown matrices are diagonal). To demonstrate the usefulness of applying some properties of the Kronecker products, suppose we have to solve, for example, the following system:

$$BXA^T = C , \tag{82}$$

where $A,\ B \in M_n$ are given scalar matrices and $X \in M_n$ is unknown matrix to be solved. Then it is not hard by using the $Vec(.)$-notation to establish the following equivalence:

$$(A \otimes B)VecX = VecC , \tag{83}$$

and thus also by using the $Vecd(.)$-notation product to establish the following equivalence:

$$(A \circ B)Vecd(X) = Vecd(C),\ X \text{ is diagonal.} \tag{84}$$

If we ignore the Kronecker (Hadamard) product structure, then we need to solve the following both matrix equations:

- $BY = C$ (85)

Here, Y can be obtained in $O(n^3)$ arithmetic operations (flops) by using LU factorization of matrix B (*Forward Substitution*).

- $XA^T = Y$ (86)

Here X can be obtained also in $O(n^3)$ operations (flops) by using LU factorization of matrix A (*Back Substitution*).

Now without exploiting the Kronecker product structure, an $n^2 \times n^2$ system defined in (82) would normally (by Gaussian elimination) require $O(n^6)$ operations to solve. But when we use Kronecker product structure: $(A \otimes B)VecX = VecC$, the calculations shows that $VecX$ can be obtained only in $O(n^3)$ operations by using LU factorization of matrices A and B [20, pp. 87]. We can say that the system of the form: $(A \otimes B)VecX = VecC$ can be solved fast and the Kronecker structure also a voids the formation of $n^2 \times n^2$ matrices, only the smaller lower and upper triangular matrices L_A, L_B, U_A, U_B are needed. While if X is $n \times n$ diagonal matrix and use the Hadamard product structure: $(A \circ B)Vecd(X) = Vecd(C)$, the calculations shows that $Vecd(X)$ can be obtained only in $O(n)$ operations by using LU factorization of $A \circ B$.

We can say that the system of the form: $(A \circ B)Vecd(X) = Vecd(C)$ can be solved more fast than Kronecker structure, only the very smaller lower and upper triangular matrices $L_{A\circ B}$ and $U_{A\circ B}$ are needed. For example, consider A, B are 3 × 3 matrices and C is 9 × 1 vector. To demonstrate the usefulness of applying Kronecker product and $Vec(.)$-notation, we return to the system problem $(A \otimes B)VecX = VecC$. If $A \otimes B$ is non-singular and regarding with LU factorizetions of $A = L_A U_A$ and $B = L_B U_B$, then a solution of system exists and can be written as:

$$(U_A \otimes U_B)VecX = z, (L_A \otimes L_B)z = VecC . \tag{87}$$

First, the lower triangular system $(L_A \otimes L_B)z = VecC$ can be solved by forward substitution as the following:

$$\left(\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}\right) \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_9 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_9 \end{bmatrix}$$

i.e.,

$$L_A \otimes L_B = \begin{bmatrix} a_{11}b_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11}b_{21} & a_{11}b_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21}b_{11} & 0 & 0 & a_{22}b_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21}b_{21} & a_{21}b_{22} & 0 & a_{22}b_{21} & a_{22}b_{22} & 0 & 0 & 0 & 0 \\ a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33} & 0 & 0 & 0 \\ a_{31}b_{11} & 0 & 0 & a_{32}b_{11} & 0 & 0 & a_{33}b_{11} & 0 & 0 \\ a_{31}b_{21} & a_{31}b_{22} & 0 & a_{32}b_{21} & a_{32}b_{22} & 0 & a_{33}b_{21} & a_{33}b_{22} & 0 \\ a_{31}b_{31} & a_{31}b_{32} & a_{31}b_{33} & a_{32}b_{31} & a_{32}b_{32} & a_{32}b_{33} & a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{bmatrix}$$

which can be solved in $O\left(n^2\right)=O(9)$ operations. The first three equations are:

- $a_{11}b_{11}z_1=c_1 \Rightarrow z_1=\dfrac{c_1}{a_{11}b_{11}}$. (88)
- $a_{11}b_{21}z_1+a_{11}b_{22}z_2=c_2 \Rightarrow z_2=\dfrac{b_{11}c_2-b_{21}c_1}{a_{11}b_{11}b_{22}}$. (89)
- $a_{11}b_{31}z_1+a_{11}b_{32}z_2+a_{11}b_{33}z_3=c_3$

$$\Rightarrow z_3=\frac{b_{11}b_{22}c_3-b_{11}b_{32}c_2-b_{22}b_{31}c_1+b_{32}b_{21}c_1}{a_{11}b_{11}b_{22}b_{33}} \quad (90)$$

Now the next three equations are:

- $\boldsymbol{a_{21}b_{11}z_1}+a_{22}b_{11}z_4=c_4$. (91)
- $\boldsymbol{a_{21}b_{21}z_1}+\boldsymbol{a_{21}b_{12}z_2}+a_{22}b_{21}z_4+a_{22}b_{22}z_5=c_5$. (92)
- $\boldsymbol{a_{21}b_{31}z_1}+\boldsymbol{a_{21}b_{32}z_2}+\boldsymbol{a_{21}b_{33}z_3}$

$$+a_{22}b_{31}z_4+a_{22}b_{32}z_5+a_{22}b_{33}z_6=c_6 . \quad (93)$$

The first boldface expression $\boldsymbol{a_{21}b_{11}z_1}$ in (91) can be computed as $\dfrac{a_{21}c_1}{a_{11}}$. The second boldface expression $\boldsymbol{a_{21}b_{21}z_1}+\boldsymbol{a_{21}b_{12}z_2}$ in (92) can be also computed as $\dfrac{a_{21}c_2}{a_{11}}$. While the third boldface expression $\boldsymbol{a_{21}b_{31}z_1}+\boldsymbol{a_{21}b_{32}z_2}+\boldsymbol{a_{21}b_{33}z_3}$ in (93) can be also computed as $\dfrac{a_{21}c_3}{a_{11}}$.

We use the previous expressions for obtaining z_1, z_2 and z_3 in the first set of equations to simplify the second set of three equations. The simplified second set of equations becomes

$$a_{22}b_{11}z_4=c_4-\frac{a_{21}c_1}{a_{11}} . \quad (94)$$

$$a_{22}b_{21}z_4+a_{22}b_{22}z_5=c_5-\frac{a_{21}c_2}{a_{11}} . \quad (95)$$

$$a_{22}b_{31}z_4+a_{22}b_{32}z_5+a_{22}b_{33}z_6=c_6-\frac{a_{21}c_3}{a_{11}} . \quad (96)$$

Solving the second set of equations takes $O(n)$ operations and the forward solve step takes $O\left(n^2\right)$ operations, so obtaining z_4, z_5 and z_6 takes $O\left(n^2\right)$ time. This simplification and using the work from the previous solution step continuous so that solving each of n-sets of n-equations takes $O\left(n^2\right)$ time, resulting in an overall solution time of $O\left(n^2\right)$. Exploiting the Kronecker structure reduce the usual, expected $O\left(n^4\right)$ time to solve $\left(L_A\otimes L_B\right)z=VecC$ to $O\left(n^2\right)$.

One final note regarding the exploitation of the Kronecker structure of the system remains. Suppose the matrices A and B are different sizes. Then, the time required to solve the system $(A\otimes B)VecX=VecC$ is $O\left(n_A n_B^2\right)$, where n_A is the size of A and n_B is the size of B. In our work, the modeler has some choice for the size of the A and B matrices. Thus, a wise choice would make n_B small, reducing the effect of the n_B^2 term in the $O\left(n_A n_B^2\right)$ computation time.

While when X is $n\times n$ diagonal matrix and applying $Vecd(.)$-notation, we return to the system problem: $(A\circ B)Vecd(X)=Vecd(C)$. If $A\circ B$ is non-singular matrix and regarding with LU factorizations of $A\circ B=L_{A\circ B}U_{A\circ B}$, then a solution of system exists and can be written as:

$$U_{A\circ B}Vecd(X)=y,\; L_{A\circ B}y=Vecd(C) . \quad (97)$$

First, the lower triangular system $L_{A\circ B}y=Vecd(C)$ can be solved by forward substitution as the following:

$$\begin{bmatrix} a_{11}b_{11} & 0 & 0 \\ a_{21}b_{21} & a_{22}b_{22} & 0 \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \end{bmatrix},$$

which can be solved in $O(n)=O(3)$ operations as follows:

- $a_{11}b_{11}y_1=c_{11} \Rightarrow y_1=\dfrac{c_{11}}{a_{11}b_{11}}$. (98)
- $a_{21}b_{21}y_1+a_{22}b_{22}y_2=c_{22} \Rightarrow y_2=\dfrac{a_{11}b_{11}c_{22}-a_{21}b_{21}c_{11}}{a_{11}b_{11}a_{22}b_{22}}$. (99)
- $a_{31}b_{31}y_1+a_{32}b_{32}y_2+a_{33}b_{33}y_3=c_{33} \Rightarrow y_3=\dfrac{a_{11}b_{11}a_{22}b_{22}c_{33}-\left[a_{31}b_{31}a_{22}b_{22}-a_{32}b_{32}a_{21}b_{21}\right]c_{11}-a_{32}b_{32}a_{11}b_{11}c_{22}}{a_{11}b_{11}a_{22}b_{22}a_{33}b_{33}}$. (100)

5. Conclusion

The solution of coupled matrix linear least-squares problems and coupled matrix differential equations is studied and some important special cases are discussed. The analysis indicates that solving for $Vec(.)$ is efficient and solving for $Vecd(.)$ is more efficient when the unknown matrices are diagonal. Although the algorithms are presented for non-homogeneous coupled matrix and matrix linear differential equations, the idea adopted can be easily extended to study coupled matrix nonlinear differential equations, e.g., the coupled matrix Riccati differential equations.

6. Acknowledgements

The author expresses his sincere thanks to referee (s) for careful reading of the manuscript and several helpful suggestions. The author also gratefully acknowledges that this research was partially supported by Deanship of Scientific Research/University of Dammam/Kingdom of Saudi Arabia/under the Grant No. 2012152.

REFERENCES

[1] T. Kailath, "Linear Systems," Prentice-Hall, Englewood Cliffs, 1980.

[2] T. Kailath and A. H. Sayed, "Displacement Structure: Theory and Applications," *SIAM Review*, Vol. 37, No. 3, 1995, pp. 297-386. doi:10.1137/1037082

[3] G. Mouroutsos and P. D. Sparis, "Taylor Series Approach to System Identification, Analysis and Optimal Control," *Journal of the Franklin Institute*, Vol. 319, No. 3, pp. 359-371. doi:10.1016/0016-0032(85)90056-0

[4] L. Jódar and H. Abou-Kandil, "Kronecker Products and Coupled Matrix Riccati Differential Systems," *Linear Algebra and its Applications*, Vol. 121, 1989, pp. 39-51. doi:10.1016/0024-3795(89)90690-3

[5] S. Campbell, "Singular Systems of Differential Equations II," Pitman, London, 1982.

[6] M. Mariton, "Les Systèmes Linéaires á Sauts Markoviens," Thèse d'Etat, Université Paris-Sud, 1986.

[7] J. B. Cruz and C. I. Chen, "Series Nash Solution of Two-Person Nonzero Sum Linear Differential Games," *Journal of Optimization Theory and Applications*, Vol. 7, No. 4, 1971, pp. 240-257. doi:10.1007/BF00928706

[8] Z. Al-Zhour and A. Kilicman, "Matrix Equalities and Inequalities Involving Khatri-Rao and Tracy-Singh Sums," *Journal of Inequalities in Pure & Applied Mathematics*, Vol. 7, No. 1, 2006, pp. 496-513.

[9] A. Graham, "Kronecker Products and Matrix Calculus with Applications," Ellis Horwood Ltd., New York, 1981.

[10] A. Kilicman and Z. Al-Zhour, "Vector Least-Squares Solutions of Coupled Singular Matrix Equations," *Journal of Computational and Applied Mathematics*, Vol. 206, No. 2, 2007, pp. 1051-1069. doi:10.1016/j.cam.2006.09.009

[11] W.-H. Steeb, "Matrix Calculus and Kronecker Product with Applications and C_{++} Programs," World Scientific Publishing Co. Pte. Ltd., Singapore, 1997.

[12] G. Visick, "A Quantitative Version of the Observation that the Hadamard Product Is A Principle Submatrix of the Kronecker Product," *Linear Algebra and its Applications*, Vol. 304, No. 1-3, 2000, pp. 45-68. doi:10.1016/S0024-3795(99)00187-1

[13] H. Lev-Ari, "Efficient Solution of Linear Matrix Equations with Application to Multistatic Antenna Array Processing," *Communications in Information & Systems*, Vol. 5, No. 1, 2005, pp. 123-130.

[14] C. R. Rao and M. B. Rao, "Matrix Algebra and Its Applications to Statistics and Econometrics," World Scientific Publishing Co. Pte. Ltd., Singapore, 1998.

[15] F. Ding and T. Chen, "Iterative Least-Squares Solutions of Coupled Sylvester Matrix Equations," *Systems & Control Letters*, Vol. 54, No. 2, 2005, pp. 95-107. doi:10.1016/j.sysconle.2004.06.008

[16] S. Tauber, "An Applications of the Hadamard Product to Air Pollution," *Applied Mathematics and Computation*, Vol. 4, No. 2, 1978, pp. 167-176. doi:10.1016/0096-3003(78)90020-6

[17] F. Zhang, Matrix Theory: Basic Results and Techniques, Springer-Verlag, New York, 1999.

[18] A. Kilicman and Z. Al-Zhour, "The General Common Exact Solutions of Coupled Linear Matrix and Matrix Differential Equations," *Journal of Computational Analysis and Applications*, Vol. 1, No. 1, 2005, pp. 15-29.

[19] J. R. Magnus and H. Neudecker, "Matrix Differential Calculus with Applications in Statistics and Econometrics," John Wiley and Sons Ltd., New York, 1999.

[20] G. F. Van Loan, "The Ubiquitous Kronecker Product," *Journal of Computational and Applied Mathematics*, Vol. 123, No. 1-2, 2000, pp. 85-100. doi:10.1016/S0377-0427(00)00393-9

