Intelligent Control and Automation, 2012, 3, 78-89
http://dx.doi.org/10.4236/ica.2012.31010 Published Online February 2012 (http://www.SciRP.org/journal/ica)

Sensors and Regional Gradient Observability of Hyperbolic Systems

Samir Benhadid[1], Soraya Rekkab[1], El Hassane Zerrik[2]
[1]Mathematics Department, Faculty of Exact Sciences, University Mentouri, Constantine, Algeria
[2]MACS Team, Department of Mathematics, Faculty of Sciences, University Moulay Ismail, Meknes, Morocco
Email: {ihebmath, rekkabsoraya, zerrik3}@yahoo.fr

Received August 2, 2011; revised November 3, 2011; accepted November 10, 2011

ABSTRACT

This paper presents a method to deal with an extension of regional gradient observability developed for parabolic system [1,2] to hyperbolic one. This concerns the reconstruction of the state gradient only on a subregion of the system domain. Then necessary conditions for sensors structure are established in order to obtain regional gradient observability. An approach is developed which allows the reconstruction of the system state gradient on a given subregion. The obtained results are illustrated by numerical examples and simulations.

Keywords: Distributed Systems; Hyperbolic Systems; Observability; Regional Gradient Observability; Sensors; Gradient Reconstruction

1. Introduction

For a distributed parameter system evolving on a spatial domain $\Omega \subset IR^n$, the notion of regional observability concerns the reconstruction of the initial state on a subregion ω of Ω. Characterization results and approaches for the reconstruction of regional state are given in [3,4]. Similar results were developed for the state gradient of parabolic systems in [2]. This led to the so-called regional gradient observability and concerns the possibility to reconstruct the gradient on a subregion ω without the knowledge of the system state. The study of gradient observability is motivated by real applications, the case of insulation problems, also there exist systems for which the state is not observable but the state gradient is observable, example is given in [1].

In this paper we present an extension of the above results on regional gradient observability to hyperbolic systems evolving on a spatial domain Ω. That is to say one may be concerned with the observability of the state gradient only in a critical subregion ω of Ω. More precisely let (S) be a linear hyperbolic system with suitable state space and suppose that the initial state $\bar{y}^0$ and its gradient $\bar{\nabla}\bar{y}^0$ are unknown and that measurements are given by means of output functions (depending on the number and structure of the sensors). The problem concerns the reconstruction of the state gradient on the subregion ω of the system domain Ω without taking into account the residuel part on $\Omega \setminus \omega$.

Here, we consider the problem of regional gradient observability of hyperbolic systems and we establish condition that allows the reconstruction of the initial gradient on such a subregion. And the paper is organized as follows.

The second section is devoted to definitions and characterizations of this notion for hyperbolic systems. In the third section we establish a relation between regional gradient observability and sensors structure. The fourth section is focused on regional reconstruction of the initial gradient. In the last section we give a numerical approach, extending the Hilbert Uniqueness Method developed by J.L. Lions [5], and illustrations with efficient simulations.

2. Regional Gradient Observability

Let Ω be an open bounded subset of IR^n with a regular boundary $\partial\Omega$. Fix $T > 0$ and let denote by $Q = \Omega \times]0,T[$ and $\Sigma = \partial\Omega \times]0,T[$.

Consider the system described by the hyperbolic equation

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = A\, y(x,t) & Q \\ y(x,0) = y^0, \dfrac{\partial y(x,0)}{\partial t} = y^1 & \Omega \\ y(\zeta,t) = 0 & \Sigma \end{cases} \quad (1)$$

where A is the second order elliptic linear operator with regular coefficients.

Equation (1) has a unique solution

$$y\in C\left(0,T;H_0^1(\Omega)\right)\cap C^1\left(0,T;L^2(\Omega)\right) \text{ [6].}$$

Suppose that measurements on system (1) are given by an output function:

$$z(t)=Cy(t). \tag{2}$$

where $C:H_0^1(\Omega)\to IR^q$ is a linear operator depending on the structure of q sensors.

Let us recall that a sensor is defined by a couple (D,f), where $D\subset\Omega$ is the location of the sensor and $f\in L^2(D)$ is the spatial distribution of measurements on D. In the case of a pointwise sensor, $D=b\in\Omega$ and $f=\delta_b$ is the Dirac mass concentrated in b see [7].

Let $\bar{y}=(y,\partial y/\partial t)$ and $\bar{C}\bar{y}=(Cy,0)$ then the system (1) may be written in the form

$$\begin{cases}\dfrac{\partial\bar{y}}{\partial t}=\bar{A} & \text{in } \Omega\\ \bar{y}^0=\left(y^0,y^1\right) & \text{in } \Omega\end{cases} \tag{3}$$

with $\bar{A}=\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}$.

$\bar{A}$ has a compact resolvent and generates a strongly continuous semi-group $\left(S(t)\right)_{t\ge 0}$ on a subspace of the Hilbert state space $L^2(\Omega)\times L^2(\Omega)$ given by

$$S(t)\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}\displaystyle\sum_{m\ge 1}\sum_{j=1}^{r_m}\left(\langle y_1,w_{mj}\rangle\cos\sqrt{-\lambda_m}\,t+\frac{1}{\sqrt{-\lambda_m}}\langle y_2,w_{mj}\rangle\sin\sqrt{-\lambda_m}\,t\right)w_{mj}\\ \displaystyle\sum_{m\ge 1}\sum_{j=1}^{r_m}\left(-\sqrt{-\lambda_m}\langle y_1,w_{mj}\rangle\sin\sqrt{-\lambda_m}\,t+\langle y_2,w_{mj}\rangle\cos\sqrt{-\lambda_m}\,t\right)w_{mj}\end{pmatrix}$$

$\left(w_{mj}\right)$ is a basis in $H_0^1(\Omega)$ of eigenfunctions of A, orthonormal in $L^2(\Omega)$ and $\lambda_m<0$ the associated eigenvalues with multiplicity r_m. Then (3) admits a unique solution $\bar{y}=S(t)\bar{y}^0$.

Let us define the observability operator

$$\begin{aligned}K:H_0^1(\Omega)\times H_0^1(\Omega)&\to L^2\left(0,T;IR^q\right)\\ h&\to\bar{C}S(.)h\end{aligned}$$

which is linear and bounded with its adjoint denoted by K^* and let $\bar{\nabla}$ be the operator

$$\begin{aligned}\bar{\nabla}:H_0^1(\Omega)\times H_0^1(\Omega)&\to\left(L^2(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n\\ (y_1,y_2)&\to\bar{\nabla}(y_1,y_2)=(\nabla y_1,\nabla y_2)\end{aligned}$$

where

$$\begin{aligned}\nabla:H_0^1(\Omega)&\to\left(L^2(\Omega)\right)^n\\ y&\to\nabla y=\left(\frac{\partial y}{\partial x_1},\frac{\partial y}{\partial x_2},\dots,\frac{\partial y}{\partial x_n}\right)\end{aligned}$$

while their adjoints are denoted by $\bar{\nabla}^*$ and ∇^* respectively.

2.1. Definition 2.1

The system (1) together with the output (2) is said to be exactly (resp. approximately) gradient observable if

$$\operatorname{Im}\left(\bar{\nabla}K^*\right)=\left(L^2(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n,$$

$$\left(\text{resp. } Ker\left(K\bar{\nabla}^*\right)=\{0\}\right)$$

Such a system will be said exactly (resp. approximately) G-observable.

For a positive Lebesgue measure subset ω of Ω, we also consider the operators

$$\begin{aligned}\bar{\chi}_\omega:\left(L^2(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n&\to\left(L^2(\omega)\right)^n\times\left(L^2(\omega)\right)^n\\ (y_1,y_2)&\to(\chi_\omega y_1,\chi_\omega y_2)\end{aligned}$$

where

$$\begin{aligned}\chi_\omega:\left(L^2(\Omega)\right)^n&\to\left(L^2(\omega)\right)^n\\ y&\to y|_\omega\end{aligned}$$

and

$$\begin{aligned}\tilde{\chi}_\omega:L^2(\Omega)&\to L^2(\omega)\\ y&\to y|_\omega\end{aligned}$$

while their adjoints, denoted by $\bar{\chi}_\omega^*$, χ_ω^* and $\tilde{\chi}_\omega^*$ respectively and given by

$$\begin{aligned}\bar{\chi}_\omega^*:\left(L^2(\omega)\right)^n\times\left(L^2(\omega)\right)^n&\to\left(L^2(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n\\ (y_1,y_2)&\to(\chi_\omega^* y_1,\chi_\omega^* y_2)\end{aligned}$$

where

$$\begin{aligned}\chi_\omega^*:\left(L^2(\omega)\right)^n&\to\left(L^2(\Omega)\right)^n\\ y&\to\chi_\omega^* y=\begin{cases}y & \text{on } \omega\\ 0 & \text{on } \Omega\backslash\omega\end{cases}\end{aligned}$$

and

$$\begin{aligned}\tilde{\chi}_\omega^*:L^2(\omega)&\to L^2(\Omega)\\ y&\to\tilde{\chi}_\omega^* y=\begin{cases}y & \text{on } \omega\\ 0 & \text{on } \Omega\backslash\omega\end{cases}\end{aligned}$$

We finally introduce the operator

$$\mathrm{H}=\bar{\chi}_\omega\bar{\nabla}K^*:L^2\left(0,T;IR^q\right)\to\left(L^2(\omega)\right)^n\times\left(L^2(\omega)\right)^n.$$

2.2. Definition 2.2

1) The system (1) together with the output Equation (2) is said to be exactly regionally gradient observable or exactly G-observable on ω if

$$\mathrm{Im}(\mathrm{H}) = \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n$$

2) The system (1) together with the output equation (2) is said to be approximately regionally gradient observable or approximately G-observable on ω if $Ker\left(\mathrm{H}^*\right) = \{0\}$.

The notion of regional G-observability on ω may be characterized by the following results.

2.3. Proposition 2.3

1) The system (1) together with the output Equation (2) is exactly G-observable on ω if and only if one of the following propositions is holds.

a) For all $z^* \in \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n$, there exists $c > 0$, such that

$$\left\| z^* \right\|_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n} \le c \left\| K \nabla^* \overline{\chi}_\omega^* z^* \right\|_{L^2\left(0,T;IR^q\right)}$$

b) $Ker\ \overline{\chi}_\omega + \mathrm{Im}\left(\overline{\nabla} K^*\right) = \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n$

2) The system (1) together with the output Equation (2) is approximately G-observable on ω if and only if the operator $N_\omega = \mathrm{H}\,\mathrm{H}^*$ is positive definite.

2.4. Proof

1) a) let us consider the operator $h = Id_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n}$ and $g = \overline{\chi}_\omega \overline{\nabla} K$.

Since the system is exactly G-observable on ω, we have $\mathrm{Im}\, h \subset \mathrm{Im}\, g$, and by the general result given in [8], this is equivalent to $\exists\ c > 0$ such that

$$\forall z^* \in \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n;$$
$$\left\| h^* z^* \right\|_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n} \le c \left\| g^* z^* \right\|_{L^2\left(0,T;IR^q\right)}.$$

b) Let $y \in \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n$, then

$$\overline{\chi}_\omega y \in \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n,$$

since the system (1) is exactly G-observable on ω, there exists $z \in L^2\left(0,T;IR^q\right)$ such that $\overline{\chi}_\omega\left(y - \overline{\nabla} K^* z\right) = 0$. Let put $y = y_1 + y_2$ where $y_1 = y - \overline{\nabla} K^* z$ and $y_2 = \overline{\nabla} K^* z$, then $y_1 \in Ker\ \overline{\chi}_\omega$ and $y_2 \in \mathrm{Im}\left(\overline{\nabla} K^*\right)$.

Conversely, let $y \in \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n$, then $\overline{\chi}_\omega^* y \in \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n$, there exist $y_1 \in Ker\,\overline{\chi}_\omega$ and $y_2 \in \mathrm{Im}\left(\overline{\nabla} K^*\right)$ such that $\overline{\chi}_\omega^* y = y_1 + y_2$ and $\overline{\chi}_\omega \overline{\chi}_\omega^* y = \overline{\chi}_\omega y_1 + \overline{\chi}_\omega y_2 = \overline{\chi}_\omega y_2$.

Since $y_2 \in \mathrm{Im}\left(\overline{\nabla} K^*\right)$, there exists $z \in L^2\left(0,T;IR^q\right)$ such that $y_2 = \overline{\nabla} K^* z$. Thus $\overline{\chi}_\omega \overline{\chi}_\omega^* y = \overline{\chi}_\omega \overline{\nabla} K^* z$, which gives $y = \overline{\chi}_\omega \overline{\nabla} K^* z \in \mathrm{Im}\,\mathrm{H}$.

2) Let $z^* \in \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n$ such that $\left\langle N_\omega z^*, z^* \right\rangle_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n} = 0$.

So $\left\langle \mathrm{H}^* z^*, \mathrm{H}^* z^* \right\rangle_{L^2\left(0,T;IR^q\right)} = 0$ which means that $z^* \in Ker\ \mathrm{H}^*$ and since (1) is approximately G-observable then $z^* = 0$, that is N_ω is positive definite.

Conversely, let $z^* \in \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n$ such that $\mathrm{H}^* z^* = 0$, then $\left\langle N_\omega z^*, z^* \right\rangle_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n} = 0$, there for $z^* = 0$, that is the system is approximately G-observable on ω.

2.5. Remark 2.4

1) If a system is exactly (resp. approximately) G-observable on ω_2, it is exactly (resp. approximately) G-observable on $\omega_1 \subseteq \omega_2$.

2) There exist systems which are not G-observable on the whole domain but may be G-observable on some subregion.

2.6. Example 2.5

Let $\Omega =]0,1[\times]0,1[$, we consider the two-dimensional system described by the hyperbolic system

$$\begin{cases} \dfrac{\partial^2 y(x_1,x_2,t)}{\partial t^2} = \Delta y(x_1,x_2,t) & Q \\ y(x_1,x_2,0) = y^0(x_1,x_2) & \Omega \\ \dfrac{\partial y(x_1,x_2,0)}{\partial t} = y^1(x_1,x_2) & \Omega \\ y(\zeta,\eta,t) = 0 & \Sigma \end{cases}$$

The operator $A = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, which the eigenvalues are $\lambda_{ij} = -\left(i^2 + j^2\right)\pi^2$ associated to the eigenfunctions $w_{ij}(x_1,x_2) = 2\sin(i\pi x_1)\sin(j\pi x_2)$.

Measurements are given by the output function

$$z(t) = \int_D y(x_1,x_2,t)\ f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

where $D = \left\{\dfrac{1}{2}\right\} \times \left]0,1\right[$ is the sensor support and $f(x_1,x_2) = \sin(\pi x_2)$ is the function measure.

Let the subregion $\omega =]0,1[\times \left]\dfrac{1}{6},\dfrac{1}{3}\right[$ and we consider the initial state

$$y^0(x_1,x_2)=\sin(\pi x_1)\sin(2\pi x_2),$$

$$y^1(x_1,x_2)=\cos(\pi x_1)\cos(3\pi x_2)$$

Then the initial state gradient to be observed is

$$g^0(x_1,x_2)=\begin{pmatrix}\pi\cos(\pi x_1)\sin(2\pi x_2)\\ 2\pi\sin(\pi x_1)\cos(2\pi x_2)\end{pmatrix}^t$$

$$g^1(x_1,x_2)=\begin{pmatrix}-\pi\sin(\pi x_1)\cos(3\pi x_2)\\ -3\pi\cos(\pi x_1)\sin(3\pi x_2)\end{pmatrix}^t$$

We have the result.

2.7. Proposition 2.6

The gradient g is not approximately G-observable on the whole domain Ω, however it is approximately G-observable on the subregion ω.

2.8. Proof

To prove that g is not approximately G-observable on Ω, we must show that $g\in Ker\left(K\overline{\nabla}^*\right)$. We have

$$K\overline{\nabla}^*\left(g^0,g^1\right)=\sum_{ij=1}^{+\infty}\left[\left\langle\nabla^* g^0,w_{ij}\right\rangle\cos\sqrt{-\lambda_{ij}}t+\frac{1}{\sqrt{-\lambda_{ij}}}\left\langle\nabla^* g^1,w_{ij}\right\rangle\sin\sqrt{-\lambda_{ij}}t\right]\left\langle w_{ij},f\right\rangle$$

Since

$$\left\langle w_{ij},f\right\rangle_{L^2(D)}=\begin{cases}0 & \text{for } j>1\\ \sin\left(i\frac{\pi}{2}\right) & \text{for } j=1\end{cases}$$

we have $\forall i\in IN^*$

$$\left\langle\nabla^* g^0,w_{i1}\right\rangle_{L^2(\Omega)}=\int_0^1 2i\pi^2\cos(\pi x_1)\cos(i\pi x_1)\,dx_1\int_0^1\sin(2\pi x_2)\sin(\pi x_2)\,dx_2+4\pi^2\int_0^1\sin(\pi x_1)\sin(i\pi x_1)\,dx_1\int_0^1\cos(2\pi x_2)\cos(\pi x_2)\,dx_2=0$$

and

$$\left\langle\nabla^* g^1,w_{i1}\right\rangle_{L^2(\Omega)}=\int_0^1-2i\pi^2\sin(\pi x_1)\cos(i\pi x_1)\,dx_1\int_0^1\cos(3\pi x_2)\sin(\pi x_2)\,dx_2-6\pi^2\int_0^1\cos(\pi x_1)\sin(i\pi x_1)\,dx_1\int_0^1\sin(3\pi x_2)\cos(\pi x_2)\,dx_2=0$$

This gives $K\overline{\nabla}^*\left(g^0,g^1\right)=0$, and then the system is not approximately G-observable once Ω.

On the other hand g may be approximately G-observable on ω.

Indeed, suppose that $K\overline{\nabla}^*\overline{\chi}_\omega^*\overline{\chi}_\omega\left(g^0,g^1\right)=0$, then

$$\sum_{ij=1}^{\infty}\left[\left\langle\nabla^* g^0,w_{ij}\right\rangle\cos\sqrt{-\lambda_{ij}}t+\frac{1}{\sqrt{-\lambda_{ij}}}\left\langle\nabla^* g^1,w_{ij}\right\rangle\sin\sqrt{-\lambda_{ij}}t\right]\left\langle w_{ij},f\right\rangle=0$$

Since for T large enough, the set $\left\{\sin\sqrt{-\lambda_{ij}}t,\cos\sqrt{-\lambda_{ij}}t\right\}_{ij\ge1}$ forms a complete orthonormal set of $L^2(0,T)$, we have

$$\left\langle\nabla^* g^0,w_{ij}\right\rangle_{L^2(\omega)}\left\langle w_{ij},f\right\rangle_{L^2(D)}=\left\langle\nabla^* g^1,w_{ij}\right\rangle_{L^2(\omega)}\left\langle w_{ij},f\right\rangle_{L^2(D)}=0;\forall i,j\ge1$$

but for $j=1$ and $i\in 2IN+1$, we have

$$\left\langle w_{i1},f\right\rangle_{L^2(D)}=\sin\left(i\frac{\pi}{2}\right)\neq0.$$

witch gives, $\forall i\in 2IN+1$

$$\left\langle\nabla^* g^0,w_{i1}\right\rangle_{L^2(\omega)}=\left\langle\nabla^* g^1,w_{i1}\right\rangle_{L^2(\omega)}=0.$$

But for $i=1$, we have

$$\left\langle\nabla^* g^0,w_{11}\right\rangle_{L^2(\omega)}==\frac{\pi}{4}\left(3\sqrt{3}-\frac{11}{3}\right)\neq0.$$

Thus $K\overline{\nabla}^*\overline{\chi}_\omega^*\overline{\chi}_\omega\left(g^0,g^1\right)\neq0$.

3. Gradient Strategic Sensors

The purpose of this section is to establish a link between regional gradient observability and the sensors structure.

Let us consider the system (1) observed by q sensors $(D_i,f_i)_{1\le i\le q}$ which may be pointwise or zone.

3.1. Definition 3.1

A sensor (D,f) (or a sequence of sensors) is said to be gradient strategic on ω if the observed system is G-observable on ω, such a sensor will be said G-strategic on ω.

We assume that the operator A is of constant coefficients and has a complete set of eigenfunctions in $H_0^1(\Omega)$ denoted by (w_i) orthonormal in $L^2(\Omega)$

associated to the eigenvalues λ_i of multiplicity r_i. Assume also that $r = \sup r_i$ is finite, then we have the following result.

3.2. Proposition 3.2

If the sequence of sensors $(D_i, f_i)_{1 \le i \le q}$ is G-strategic on ω, then $q \ge r$ and $rank\left(M_m \gamma_\omega^m\right) = r_m$, $\forall m \ge 1$, where $1 \le i \le q$ and $1 \le j \le r_m$

$$\left(M_m\right)_{i,j} = \begin{cases} \sum_{k=1}^{n} \dfrac{\partial w_{m_j}(b_i)}{\partial x_k} & \text{point twise case} \\ \sum_{k=1}^{n} \left\langle \dfrac{\partial w_{m_j}}{\partial x_k}, f_i \right\rangle_{L^2(D_i)} & \text{zone case.} \end{cases} \tag{4}$$

and $\gamma_\omega^m = \begin{pmatrix} \gamma_\omega^{m_1} \\ \gamma_\omega^{m_2} \\ \vdots \\ \gamma_\omega^{m_{r_m}} \end{pmatrix}$

is the row vector the elements of which are $\gamma_\omega^{m_i} = \left(\gamma_\omega^{m_i j_k}\right)$, with $\gamma_\omega^{m_i j_k} = \int_\omega w_{m_i} w_{j_k} \mathrm{d}x$; for $j = 1, \cdots \infty; k = 1, \cdots r_j$.

3.3. Proof

The proof is developed in the case zone sensors.

The sequence of sensors (D_i, f_i) is G-strategic on ω if and only if

$$\left\{ z \in \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n \Big/ \langle \mathrm{H}u, z \rangle_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n} = 0, \forall u \in L^2\left(0, T; IR^q\right) \right\} = \{0\}$$

Suppose that the sequence of sensors (D_i, f_i) is G-strategic on ω and there exists $m_0 \ge 1$, with $rank\left(M_{m_0} \gamma_\omega^{m_0}\right) \ne r_{m_0}$ then there exists

$Z_{m_0} = \left(0, \cdots, z_{m_0}, \cdots, 0\right)^T$ such that

$$z_{m_0} = \begin{pmatrix} z_{m_0}^1 \\ \vdots \\ z_{m_0}^{r_{m_0}} \end{pmatrix} \ne 0 \quad \text{and} \quad M_{m_0} \gamma_\omega^{m_0} z_{m_0} = 0. \tag{5}$$

Let $z_1^* = \begin{pmatrix} z_{11}^* \\ \vdots \\ z_{1n}^* \end{pmatrix} \in \left(L^2(\omega)\right)^n$ verifying

$$\begin{cases} \left\langle z_{1k}^*, w_{m_0 j} \right\rangle_{L^2(\omega)} = z_{m_0}^j, \forall j = 1, \cdots, r_{m_0}, \forall k = 1, \cdots, n \\ \left\langle z_{1k}^*, w_{m_j} \right\rangle_{L^2(\omega)} = 0, \forall m \ne m_0, \forall j = 1, \cdots, r_m, \forall k = 1, \cdots, n \end{cases} \tag{6}$$

Let $z_2^* = \begin{pmatrix} z_{21}^* \\ \vdots \\ z_{2n}^* \end{pmatrix} \in \left(L^2(\omega)\right)^n$ verifying

$$\begin{cases} \left\langle z_{2k}^*, w_{m_0 j} \right\rangle_{L^2(\omega)} = z_{m_0}^j, \forall j = 1, \cdots, r_{m_0}, \forall k = 1, \cdots, n \\ \left\langle z_{2k}^*, w_{m_j} \right\rangle_{L^2(\omega)} = 0, \forall m \ne m_0, \forall j = 1, \cdots, r_m, \\ \qquad \forall k = 1, \cdots, n \end{cases} \tag{7}$$

and let $z^* = \left(z_1^*, z_2^*\right) \in \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n$

and $z^* \ne 0$ then

$$\left\langle \mathrm{H}u, z^* \right\rangle_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n} = \left\langle \overline{\chi}_\omega \overline{\nabla} K^* u, z^* \right\rangle_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n}$$

assume that

$$K^* u = \left(v_1, v_2\right) \in H_0^1(\Omega) \times H_0^1(\Omega)$$

then

$$\left\langle \mathrm{H}u, z^* \right\rangle_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n} = \left\langle \nabla v_1, \chi_\omega^* z_1^* \right\rangle_{\left(L^2(\Omega)\right)^n} + \left\langle \nabla v_2, \chi_\omega^* z_2^* \right\rangle_{\left(L^2(\Omega)\right)^n}$$

Integrating on Ω we obtain

$$\left\langle \nabla v_1, \chi_\omega^* z_1^* \right\rangle_{\left(L^2(\Omega)\right)^n} = -\left\langle v_1, \sum_{k=1}^{n} \frac{\partial\left(\tilde{\chi}_\omega^* z_{1k}^*\right)}{\partial x_k} \right\rangle_{L^2(\Omega)}$$

and

$$\left\langle \nabla v_2, \chi_\omega^* z_2^* \right\rangle_{\left(L^2(\Omega)\right)^n} = -\left\langle v_2, \sum_{k=1}^{n} \frac{\partial\left(\tilde{\chi}_\omega^* z_{2k}^*\right)}{\partial x_k} \right\rangle_{L^2(\Omega)}$$

then

$$\left\langle \mathrm{H}u, z^* \right\rangle = -\left\langle u, K \begin{pmatrix} \sum_{k=1}^{n} \dfrac{\partial\left(\tilde{\chi}_\omega^* z_{1k}^*\right)}{\partial x_k} \\ \sum_{k=1}^{n} \dfrac{\partial\left(\tilde{\chi}_\omega^* z_{2k}^*\right)}{\partial x_k} \end{pmatrix} \right\rangle$$

but we have

$$\tilde{\chi}_\omega^* z_{1k}^* = \sum_{m=1}^{\infty} \sum_{j=1}^{r_m} \left\langle \tilde{\chi}_\omega^* z_{1k}^*, w_{mj} \right\rangle_{L^2(\Omega)} w_{mj}$$

and

$$\tilde{\chi}_\omega^* z_{2k}^* = \sum_{m=1}^{\infty} \sum_{j=1}^{r_m} \left\langle \tilde{\chi}_\omega^* z_{2k}^*, w_{mj} \right\rangle_{L^2(\Omega)} w_{mj}$$

Using the fact that

$$\left\langle \tilde{\chi}_\omega^* z_{1k}^*, w_{mj} \right\rangle_{L^2(\Omega)} = \sum_{l=1}^{\infty} \sum_{p=1}^{r_l} \gamma_\omega^{m j l_p} \left\langle z_{1k}^*, w_{l_p} \right\rangle_{L^2(\omega)}$$

and

$$\left\langle \tilde{\chi}_{\omega}^{*} z_{2k}^{*}, w_{m_j} \right\rangle_{L^2(\Omega)} = \sum_{l=1}^{\infty}\sum_{p=1}^{r_l} \gamma_{\omega}^{m_j l_p} \left\langle z_{2k}^{*}, w_{l_p} \right\rangle_{L^2(\omega)}$$

then we obtain $\forall i = 1,\cdots,q$

$$\sum_{k=1}^{n} \frac{\partial\left(\tilde{\chi}_{\omega}^{*} z_{1k}^{*}\right)}{\partial x_k} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{r_m}\sum_{p=1}^{r_l}\sum_{k=1}^{n} \gamma_{\omega}^{m_j l_p} \left\langle z_{1k}^{*}, w_{l_p} \right\rangle \frac{\partial w_{mj}}{\partial x_k}$$

and

$$\sum_{k=1}^{n} \frac{\partial \tilde{\chi}_{\omega}^{*} z_{2k}^{*}}{\partial x_k} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{r_m}\sum_{p=1}^{r_l}\sum_{k=1}^{n} \gamma_{\omega}^{m_j l_p} \left\langle z_{2k}^{*}, w_{l_p} \right\rangle \frac{\partial w_{mj}}{\partial x_k}$$

from (5), (6) and (7) we obtain

$$\sum_{m=1}^{\infty}\sum_{j=1}^{r_m}\sum_{k=1}^{n}\left[\sum_{l=1}^{\infty}\sum_{p=1}^{r_l} \gamma_{\omega}^{m_j l_p} \left\langle z_{1k}^{*}, w_{l_p} \right\rangle \cos\sqrt{-\lambda_m}t + \frac{1}{\sqrt{-\lambda_m}} \right.$$
$$\left. \sum_{l=1}^{\infty}\sum_{p=1}^{r_l} \gamma_{\omega}^{m_j l_p} \left\langle z_{2k}^{*}, w_{l_p} \right\rangle \sin\sqrt{-\lambda_m}t \right]\left\langle \frac{\partial w_{mj}}{\partial x_k}, f_i \right\rangle = 0$$

Thus

$$K\begin{pmatrix} \sum_{k=1}^{n} \frac{\partial\left(\tilde{\chi}_{\omega}^{*} z_{1k}^{*}\right)}{\partial x_k} \\ \sum_{k=1}^{n} \frac{\partial\left(\tilde{\chi}_{\omega}^{*} z_{2k}^{*}\right)}{\partial x_k} \end{pmatrix} = 0$$

this gives $\left\langle \mathrm{H}u, z^{*} \right\rangle_{\left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n} = 0$,

$\forall u \in L^2\left(0,T;IR^q\right)$ and $z^{*} \neq 0$, which contradicts the fact that the sequence of sensors is G-strategic.

3.4. Remark 3.3

1) The above proposition implies that the required number of sensors is greater than or equal to the largest multiplicity of eigenvalues.

2) By infinitesimally deforming of the domain, the multiplicity of the eigenvalues can be reduced to one [9,10]. Consequently, the regional G-observability on the subregion ω may be possible only by one sensor.

4. Regional Gradient Reconstruction

In this section, we give an approach which allows the reconstruction of the initial state gradient on ω of the system (1). This approach extends the Hilbert Uniqueness Method developed for controllability by Lions [6] and don't take into account what must be the residual initial gradient state on the subregion $\Omega\setminus\omega$. Consider the set

$$F = \left\{ \left(\varphi^0,\varphi^1\right) \in L \big/ \varphi^0 = \varphi^1 = 0 \; on\; \Omega\setminus\omega \right\}$$
$$\cap \left\{ \overline{\nabla}\left(\phi^0,\phi^1\right) \big/ \left(\phi^0,\phi^1\right) \in H_0^1(\Omega)\times H_0^1(\Omega) \right\}$$

where $L = \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n$

for $\left(\varphi^0,\varphi^1\right) \in H_0^1(\Omega)\times H_0^1(\Omega)$, the system

$$\begin{cases} \dfrac{\partial^2\phi(x,t)}{\partial t^2} = A\phi(x,t) & Q \\ \phi(x,0) = \phi^0(x), \dfrac{\partial\phi(x,0)}{\partial t} = \phi^1(x) & \Omega \\ \phi(\zeta,t) = 0 & \Sigma \end{cases} \tag{8}$$

has a unique solution

$$\phi \in C^0\left(0,T;H_0^1(\Omega)\right) \cap C^1\left(0,T;L^2(\Omega)\right).$$

We consider the zone sensor case where the system (1) is observed by the output function

$$z(t) = \sum_{k=1}^{n} \left\langle \frac{\partial y(.,t)}{\partial x_k}, f(x) \right\rangle_{L^2(D)} \tag{9}$$

D is the sensor support, f the function of measure and we consider a semi-norm on F defined by

$$\left(\varphi^0,\varphi^1\right) \in F \mapsto \left\| \left(\varphi^0,\varphi^1\right) \right\|_F^2$$
$$= \int_0^T \left(\sum_{k=1}^{n} \left\langle \frac{\partial\phi(.,t)}{\partial x_k}, f \right\rangle_{L^2(D)} \right)^2 \mathrm{d}t. \tag{10}$$

where $\phi(x,t)$ is the solution of (8).

The reverse system given by

$$\begin{cases} \dfrac{\partial^2\psi(x,t)}{\partial t^2} = A^{*}\psi(x,t) + \sum_{k=1}^{n} \left\langle \dfrac{\partial\phi}{\partial x_k}, f \right\rangle \chi_D f & Q \\ \psi(x,T) = 0, \dfrac{\partial\psi(x,T)}{\partial t} = 0 & \Omega \\ \psi(\zeta,t) = 0 & \Sigma \end{cases} \tag{11}$$

has a unique solution

$$\psi \in C\left(0,T;H_0^1(\Omega)\right) \cap C^1\left(0,T;L^2(\Omega)\right) \text{ [5].}$$

We denote the solution $\psi(x,0)$ by $\psi^0(x)$ and $\dfrac{\partial\psi(x,0)}{\partial t}$ by $\psi^1(x)$.

Let consider the operator

$$\Lambda\left(\varphi^0,\varphi^1\right) = P\left(-\Psi^1,\Psi^0\right)$$

where $P = \overline{\chi}_{\omega}^{*}\overline{\chi}_{\omega}$,

$$\Psi^1 = \left(\psi^1,\psi^1,\cdots,\psi^1\right), \Psi^0 = \left(\psi^0,\psi^0,\cdots,\psi^0\right)$$

and consider the retrograde system which has a unique solution

$$\begin{cases}\dfrac{\partial^2 Z(x,t)}{\partial t^2} = A^* Z(x,t) + \sum_{k=1}^{n}\left\langle \dfrac{\partial y(t)}{\partial x_k}, f\right\rangle \chi_D f & Q\\ Z(x,T)=0,\quad \dfrac{\partial Z(x,T)}{\partial t}=0 & \Omega\\ Z(\zeta,t)=0 & \Sigma\end{cases} \quad (12)$$

$Z \in C\left(0,T;H_0^1(\Omega)\right)\cap C^1\left(0,T;L^2(\Omega)\right)$ [5].

We denote the solution $Z(x,0)$ by $Z^0(x)$ and $\dfrac{\partial Z(x,0)}{\partial t}$ by $Z^1(x)$. Then, the regional gradient observability turns up to solve the equation

$$\Lambda\left(\varphi^0,\varphi^1\right) = P\left(-\bar{Z}^1,\bar{Z}^0\right) \quad (13)$$

where $\bar{Z}^1 = \left(Z^1, Z^1,\cdots,Z^1\right)$ and $\bar{Z}^0 = \left(Z^0,Z^0,\cdots,Z^0\right)$.

4.1. Proposition 4.1

If the sensor (D,f) is G-strategic on ω, then the equation (13) has a unique solution $\left(\varphi^0,\varphi^1\right)$ which is the gradient of the initial state to be observed on ω.

4.2. Proof

1) Let us show first that if the system (1) is G-observable, then (10) defines a norm on F.

Consider a basis (w_j) of the eigenfunctions of A, without loss of generality we suppose that the multiplicity of the eigenvalues are simple, then

$$\left\|\left(\varphi^0,\varphi^1\right)\right\|_F = 0 \Leftrightarrow \sum_{k=1}^{n}\left\langle \frac{\partial\phi}{\partial x_k}, f\right\rangle_{L^2(D)} = 0$$

on $]0,T[$ which is equivalent to

$$\sum_{j=1}^{\infty}\left[\left\langle\phi^0,w_j\right\rangle_{L^2(\Omega)}\cos\sqrt{-\lambda_j}t + \frac{1}{\sqrt{-\lambda_j}}\left\langle\phi^1,w_j\right\rangle_{L^2(\Omega)}\sin\sqrt{-\lambda_j}t\right]\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k}, f\right\rangle = 0$$

The set $\left\{\sin\sqrt{-\lambda_j}t, \cos\sqrt{-\lambda_j}t\right\}_{j\ge 1}$ forms a complete orthogonal set of $L^2(0,T)$, then we obtain

$$\begin{cases}\left\langle\phi^0,w_j\right\rangle_{L^2(\Omega)}\sum_{k=1}^{n}\left\langle\dfrac{\partial w_j}{\partial x_k}, f\right\rangle_{L^2(D)} = 0, \forall j\ge 1\\ \left\langle\phi^1,w_j\right\rangle_{L^2(\Omega)}\sum_{k=1}^{n}\left\langle\dfrac{\partial w_j}{\partial x_k}, f\right\rangle_{L^2(D)} = 0, \forall j\ge 1\end{cases}$$

and since the sensor (D,f) is regionally G-strategic on ω, we have

$$\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k}, f\right\rangle_{L^2(D)} \neq 0, \forall j \ge 1,$$

then $\left\langle\phi^0,w_j\right\rangle_{L^2(\Omega)} = \left\langle\phi^1,w_j\right\rangle_{L^2(\Omega)} = 0,\ \forall j\ge 1$.

Consequently $\phi^0=\phi^1=0$ and thus $\varphi^0=\varphi^1=0$.

Conversely, $\varphi^0=\varphi^1=0 \Rightarrow \phi^0 = c_1$ and $\phi^1 = c_2$ (constants), since

$$\varphi \in C^0\left(0,T;H_0^1(\Omega)\right)\cap C^1\left(0,T;L^2(\Omega)\right)$$

and from $\phi = 0$ on Σ, (10) is a norm.

2) Let denote by F completion of F by the norm (10) and F^* be its dual. We show that Λ is an isomorphism from F into F^*. Indeed, let $\left(\hat{\varphi}^0,\hat{\varphi}^1\right)\in F$ and $\hat{\phi}$ the corresponding solution for the problem (8), multiply the first equation of the system (11) by $\dfrac{\partial\hat{\phi}(x,t)}{\partial x_k}$, and integrate on Q, we obtain

$$\left\langle\frac{\partial\hat{\phi}}{\partial x_k},\frac{\partial^2\psi}{\partial t^2}\right\rangle_{L^2(Q)} = \left\langle\frac{\partial\hat{\phi}}{\partial x_k},A^*\psi\right\rangle_{L^2(Q)} + \left\langle\frac{\partial\hat{\phi}}{\partial x_k},\sum_{l=1}^{n}\left\langle\frac{\partial\phi}{\partial x_l}, f(x)\right\rangle_{L^2(D)} f\chi_D\right\rangle_{L^2(Q)}$$

for the first term, we obtain

$$\left\langle\frac{\partial\hat{\phi}}{\partial x_k},\frac{\partial^2\psi}{\partial t^2}\right\rangle = -\left\langle\frac{\partial\hat{\phi}(.,0)}{\partial x_k},\frac{\partial\psi(.,0)}{\partial t}\right\rangle_{L^2(\Omega)} + \left\langle\frac{\partial}{\partial x_k}\left(\frac{\partial\hat{\phi}(.,0)}{\partial t}\right),\psi(.,0)\right\rangle_{L^2(\Omega)} + \left\langle\frac{\partial}{\partial x_k}\left(A\hat{\phi}\right),\psi\right\rangle_{L^2(Q)}$$

Using Green formula for the second term, we obtain

$$\left\langle\frac{\partial\hat{\phi}}{\partial x_k},A^*\psi\right\rangle_{L^2(Q)} + \left\langle\frac{\partial\hat{\phi}}{\partial x_k},\sum_{l=1}^{n}\left\langle\frac{\partial\phi}{\partial x_l}, f\right\rangle_{L^2(D)} f\chi_D\right\rangle_{L^2(Q)}$$
$$= \left\langle A\frac{\partial\hat{\phi}}{\partial x_k},\psi\right\rangle_{L^2(Q)} + \int_{\Sigma}\left(\psi(\xi,t)\frac{\partial^2\hat{\phi}(\xi,t)}{\partial\eta_A\partial x_k} - \frac{\partial\hat{\phi}(\xi,t)}{\partial x_k}\frac{\partial\psi(\xi,t)}{\partial\eta_{A^*}}\right)\mathrm{d}\Sigma + \int_0^T\int_{\Omega}\frac{\partial\hat{\phi}(x,t)}{\partial x_k}\sum_{l=1}^{n}\left\langle\frac{\partial\phi}{\partial x_l}, f\right\rangle_{L^2(D)} f\chi_D\,\mathrm{d}x\,\mathrm{d}t$$

and with the boundary conditions, we obtain

$$\left\langle\left(\hat{\varphi}^0,\hat{\varphi}^1\right), \Lambda\left(\varphi^0,\varphi^1\right)\right\rangle = \int_0^T\left(\sum_{k=1}^{n}\left\langle\frac{\partial\hat{\phi}(t)}{\partial x_k}, f\right\rangle\right)\left(\sum_{l=1}^{n}\left\langle\frac{\partial\phi(t)}{\partial x_l}, f\right\rangle\right)\mathrm{d}t$$

Using Cauchy-Schwartz inequality, we have,

$$\forall\left(\varphi^0,\varphi^1\right),\left(\hat{\varphi}^0,\hat{\varphi}^1\right)\in F$$

$$\left\langle\left(\hat{\varphi}^0,\hat{\varphi}^1\right),\Lambda\left(\varphi^0,\varphi^1\right)\right\rangle\le\left\|\left(\hat{\varphi}^0,\hat{\varphi}^1\right)\right\|_F\left\|\left(\varphi^0,\varphi^1\right)\right\|_F$$

Hence, $\forall\left(\varphi^0,\varphi^1\right)\in F$

$$\left\langle\left(\varphi^0,\varphi^1\right),\Lambda\left(\varphi^0,\varphi^1\right)\right\rangle=\left\|\left(\varphi^0,\varphi^1\right)\right\|_{\hat{F}}^2$$

which proves that Λ is an isomorphism and consequently the Equation (13) has a unique solution which corresponds to the state gradient to be observed on the subregion ω.

4.3. Remark 4.2

The previous approach can be established with similar techniques when the output is defined by means of internal or boundary pointwise sensors.

5. Numerical Approach

In this section we give a numerical approach which leads to explicit formulas for $\nabla y^0, \nabla y^1$ on ω. We consider the case where the system (1) is observed by the output equation

$$z(t)=\sum_{k=1}^{n}\left\langle\frac{\partial y(.,t)}{\partial x_k},f\right\rangle_{L^2(D)},t\in\left]0,T\right[$$

5.1. Proposition 5.1

If the sensor (D,f) is G-strategic on ω, then the initial gradients ∇y^0 and ∇y^0 may be approached by $\hat{\nabla}y^0$ and $\hat{\nabla}y^1$ respectively

$$\hat{\nabla}y^0\approx\begin{cases}\sum_{j=1}^{M}\left[\frac{2}{T\left(\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k},f\right\rangle\right)^2}\sum_{l=1}^{n}\sum_{m=1}^{M}\langle w_m,f\rangle\right.\\ \left.\int_0^T\sum_{i=1}^{n}\left\langle\frac{\partial y}{\partial x_i},f\right\rangle\cos\sqrt{-\lambda_m}t\,\mathrm{d}t\left\langle w_m,\frac{\partial w_j}{\partial x_l}\right\rangle\right]\nabla w_j & \text{on } \omega\\ 0 & \text{on } \Omega\backslash\omega\end{cases}\tag{14}$$

$$\hat{\nabla}y^1\approx\begin{cases}\sum_{j=1}^{M}\left[\frac{-2\lambda_j}{T\left(\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k},f\right\rangle\right)^2}\sum_{l=1}^{n}\sum_{m=1}^{M}\langle w_m,f\rangle\right.\\ \left.\int_0^T\sum_{i=1}^{n}\left\langle\frac{\partial y}{\partial x_i},f\right\rangle\frac{\sin\sqrt{-\lambda_m}t}{\sqrt{-\lambda_m}}\mathrm{d}t\left\langle w_m,\frac{\partial w_j}{\partial x_l}\right\rangle\right]\nabla w_j & \text{on } \omega\\ 0 & \text{on } \Omega\backslash\omega\end{cases}\tag{15}$$

where M is an order of truncation.

5.2. Proof

In the previous section, it has been seen that the regional reconstruction of the initial state gradient on ω turns up to solve the Equation (13). For that consider the functional

$$\begin{aligned}&\Phi\left(\varphi^0,\varphi^1\right)\\&=\frac{1}{2}\left\langle\Lambda\left(\varphi^0,\varphi^1\right),\left(\varphi^0,\varphi^1\right)\right\rangle-\left\langle P\left(-\bar{Z}^1,\bar{Z}^0\right),\left(\varphi^0,\varphi^1\right)\right\rangle\\&=\frac{1}{2}\int_0^T\left(\sum_{k=1}^{n}\left\langle\frac{\partial\varphi}{\partial x_k},f\right\rangle\right)^2\mathrm{d}t-\left\langle-\bar{Z}^1,\varphi^0\right\rangle-\left\langle\bar{Z}^0,\varphi^1\right\rangle\end{aligned}$$

And solving Equation (13) turns up to minimize Φ with respect to $\left(\varphi^0,\varphi^1\right)$.

After development and when $T\to+\infty$, we obtain

$$\begin{aligned}&\lim_{T\to+\infty}\frac{1}{2T}\int_0^T\left(\sum_{k=1}^{n}\left\langle\frac{\partial\phi(.,t)}{\partial x_k},f\right\rangle_{L^2(D)}\right)^2\mathrm{d}t\\&=\frac{1}{4}\sum_{j=1}^{+\infty}\left[\left\langle\phi^0,w_j\right\rangle^2-\frac{1}{\lambda_j}\left\langle\phi^1,w_j\right\rangle^2\right]\left(\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k},f\right\rangle\right)^2\end{aligned}$$

For T large enough, we have

$$\begin{aligned}&\frac{1}{2}\int_0^T\left(\sum_{k=1}^{n}\left\langle\frac{\partial\phi(.,t)}{\partial x_k},f\right\rangle_{L^2(D)}\right)^2\mathrm{d}t\\&\approx\frac{T}{4}\sum_{j=1}^{+\infty}\left[\left\langle\phi^0,w_j\right\rangle^2-\frac{1}{\lambda_j}\left\langle\phi^1,w_j\right\rangle^2\right]\left(\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k},f\right\rangle\right)^2\end{aligned}$$

On the other hand, we have

$$\phi^0(x)=\sum_{j=1}^{+\infty}\left\langle\phi^0,w_j\right\rangle w_j(x)$$

and

$$\phi^1(x)=\sum_{j=1}^{+\infty}\left\langle\phi^1,w_j\right\rangle w_j(x)$$

since $\left(\varphi^0,\varphi^1\right)=\bar{\nabla}\left(\phi^0,\phi^1\right)$, then

$$\varphi^0(x)=\sum_{j=1}^{+\infty}\left\langle\phi^0,w_j\right\rangle\left(\frac{\partial w_j}{\partial x_1},\frac{\partial w_j}{\partial x_2},\cdots,\frac{\partial w_j}{\partial x_n}\right)\ \text{on }\omega\tag{16}$$

and

$$\varphi^1(x)=\sum_{j=1}^{+\infty}\left\langle\phi^1,w_j\right\rangle\left(\frac{\partial w_j}{\partial x_1},\frac{\partial w_j}{\partial x_2},\cdots,\frac{\partial w_j}{\partial x_n}\right)\ \text{on }\omega\tag{17}$$

we obtain

$$\left\langle-\bar{Z}^1,\varphi^0\right\rangle=\sum_{k=1}^{n}\sum_{j=1}^{+\infty}\left\langle\phi^0,w_j\right\rangle\left\langle-Z^1,\frac{\partial w_j}{\partial x_k}\right\rangle$$

and

$$\left\langle \bar{Z}^0, \varphi^1 \right\rangle = \sum_{k=1}^{n} \sum_{j=1}^{+\infty} \left\langle \phi^1, w_j \right\rangle \left\langle Z^0, \frac{\partial w_j}{\partial x_k} \right\rangle$$

The minimization of (13) is equivalent to solve the two following problems

$$\operatorname*{Inf}_{\varphi^0} \sum_{j=1}^{+\infty} \left\{ \frac{T}{4} \left\langle \phi^0, w_j \right\rangle_{L^2(\Omega)}^2 \left(\sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle_{L^2(D)} \right)^2 - \left\langle \phi^0, w_j \right\rangle_{L^2(\Omega)} \sum_{k=1}^{n} \left\langle -Z^1, \frac{\partial w_j}{\partial x_k} \right\rangle_{L^2(\omega)} \right\}$$

and

$$\operatorname*{Inf}_{\varphi^1} \sum_{j=1}^{+\infty} \left\{ \frac{-T}{4\lambda_j} \left\langle \phi^1, w_j \right\rangle_{L^2(\Omega)}^2 \left(\sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle_{L^2(D)} \right)^2 - \left\langle \phi^1, w_j \right\rangle_{L^2(\Omega)} \sum_{k=1}^{n} \left\langle Z^0, \frac{\partial w_j}{\partial x_k} \right\rangle_{L^2(\omega)}$$

which solutions are, $\forall j \geq 1$

$$\left\langle \phi^0, w_j \right\rangle_{L^2(\Omega)} = -\frac{2}{T} \frac{\left\langle \bar{Z}^1, \nabla w_j \right\rangle_{(L^2(\omega))^n}}{\left(\sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle_{L^2(D)} \right)^2} \quad (18)$$

and

$$\left\langle \phi^1, w_j \right\rangle_{L^2(\Omega)} = -\frac{2\lambda_j}{T} \frac{\left\langle \bar{Z}^0, \nabla w_j \right\rangle_{(L^2(\omega))^n}}{\left(\sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle_{L^2(D)} \right)^2} \quad (19)$$

Now, let $Z(x,t) = \sum_{m \geq 1} Z_m(t) w_m(x)$ be the solution of the system (12) with

$$Z_m(t) = \frac{\langle w_m, f \rangle}{\sqrt{-\lambda_m}} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle \sin\sqrt{-\lambda_m}(s-t)\,\mathrm{d}s.$$

Thus

$$Z^0 = Z(x,0) = \sum_{m \geq 1} \frac{\langle w_m, f \rangle}{\sqrt{-\lambda_m}} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle \sin\sqrt{-\lambda_m}\,s\mathrm{d}s \;\; w_m(x).$$

and

$$Z^1 = \frac{\partial Z}{\partial t}(x,0) = -\sum_{m \geq 1} \langle w_m, f \rangle \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle \cos\sqrt{-\lambda_m}\,s\,\mathrm{d}s\; w_m(x).$$

then, we obtain

$$\left\langle \bar{Z}^0, \nabla w_j \right\rangle_{L^2(\omega)} = \sum_{l=1}^{n} \sum_{m=1}^{+\infty} \frac{\langle w_m, f \rangle_{L^2(D)}}{\sqrt{-\lambda_m}} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle \sin\sqrt{-\lambda_m}\,s\,\mathrm{d}s \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle_{\omega}$$

and

$$\left\langle \bar{Z}^1, \nabla w_j \right\rangle_{L^2(\omega)} = -\sum_{l=1}^{n} \sum_{m=1}^{+\infty} \langle w_m, f \rangle_{L^2(D)} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(s)}{\partial x_i}, f \right\rangle \cos\sqrt{-\lambda_m}\,s\,\mathrm{d}s \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle_{\omega}$$

With these developments, according to (18) and (19), we obtain. $\forall j \geq 1$

$$\left\langle \phi^0, w_j \right\rangle_{L^2(\Omega)} = \frac{2}{T\left(\sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle \right)^2} \sum_{l=1}^{n} \sum_{m=1}^{+\infty} \langle w_m, f \rangle \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle \cos\sqrt{-\lambda_m}\,s\mathrm{d}s \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle_{\omega}$$

and

$$\left\langle \phi^1, w_j \right\rangle_{L^2(\Omega)} = \frac{-2\lambda_j}{T\left(\sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle \right)^2} \sum_{l=1}^{n} \sum_{m=1}^{+\infty} \frac{\langle w_m, f \rangle}{\sqrt{-\lambda_m}} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle \sin\sqrt{-\lambda_m}\,s\,\mathrm{d}s \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle_{\omega}.$$

We replace that in the relation (16) and (17), we obtain

$$\varphi^0(x) = \sum_{j=1}^{+\infty} \frac{2}{T\left(\sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle \right)^2} \sum_{l=1}^{n} \sum_{m=1}^{+\infty} \langle w_m, f \rangle \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(s)}{\partial x_i}, f \right\rangle \cos\sqrt{-\lambda_m}\,s\mathrm{d}s \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle \nabla w_j \text{ on } \omega$$

and

$$\varphi^1(x) = \sum_{j=1}^{+\infty} \frac{-2\lambda_j}{T\left(\sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle \right)^2} \sum_{l=1}^{n} \sum_{m=1}^{+\infty} \frac{\langle w_m, f \rangle}{\sqrt{-\lambda_m}} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(s)}{\partial x_i}, f \right\rangle \sin\sqrt{-\lambda_m}\,s\mathrm{d}s \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle \nabla w_j \text{ on } \omega$$

We consider a truncation up to order $M\ (M \in IN^*)$, then we obtain the relation (14) and (15).

We define a final error

$$\xi^2 = \left\| \bar{\nabla} y^0 - \hat{\nabla} y^0 \right\|_{L^2(\omega)}^2 + \left\| \bar{\nabla} y^1 - \hat{\nabla} y^1 \right\|_{L^2(\omega)}^2.$$

The good choice of M will be such that $\xi \leq \varepsilon$

$(\varepsilon > 0)$, and we have the following algorithm:

Algorithm

Step 1: Data: The region ω, the sensor location D and ε.

Step 2: Choose a low truncation order M.

Step 3: Computation of $\hat{\nabla}y^0$ and $\hat{\nabla}y^1$ by the formulae (14) and (15).

Step 4: If $\xi < \varepsilon$ then stop, otherwise.

Step 5: $M \leftarrow M+1$ and return to step 3.

5.3. Remark 5.2

if y^0 and y^1 are regular enough, we have a regular system state, so measurements may be taken with pointwise sensor. In this case we obtain similar formulaes as in the previous proposition given by

$$\hat{\nabla}y^0 \approx \begin{cases} \sum_{j=1}^{M}\left[\dfrac{2}{T\left(\sum_{k=1}^{n}\dfrac{\partial w_j(b)}{\partial x_k}\right)^2}\sum_{l=1}^{n}\sum_{m=1}^{M} w_m(b) \int_0^T \sum_{k=1}^{n}\dfrac{\partial y(b,t)}{\partial x_k}\cos\sqrt{-\lambda_m}t\mathrm{d}t\left\langle w_m, \dfrac{\partial w_j}{\partial x_l}\right\rangle\right]\nabla w_j & \text{on } \omega \\ 0 & \text{on } \Omega\backslash\omega \end{cases} \quad (20)$$

$$\hat{\nabla}y^1 \approx \begin{cases} \sum_{j=1}^{M}\left[\dfrac{-2\lambda_j}{T\left(\sum_{k=1}^{n}\dfrac{\partial w_j(b)}{\partial x_k}\right)^2}\sum_{l=1}^{n}\sum_{m=1}^{M} \dfrac{w_m(b)}{\sqrt{-\lambda_m}} \int_0^T \sum_{k=1}^{n}\dfrac{\partial y(b,t)}{\partial x_k}\sin\sqrt{-\lambda_m}t\mathrm{d}t\left\langle w_m, \dfrac{\partial w_j}{\partial x_l}\right\rangle\right]\nabla w_j & on\,\omega \\ 0 & on\ \Omega\backslash\omega \end{cases} \quad (21)$$

6. Simulations

6.1. Example

In this section we develop a numerical example that leads to results related to the choice of the subregion, the sensor location and the initial state gradient.

On $\Omega =]0,1[$, we consider the one dimensional system.

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = \dfrac{\partial^2 y(x,t)}{\partial x^2} &]0,1[\times]0,T[\\ y(x,0) = y^0(x) &]0,1[\\ \dfrac{\partial y(x,0)}{\partial t} = y^1(x) &]0,1[\\ y(0,t) = y(1,t) = 0 &]0,1[\end{cases} \quad (22)$$

Measurements are given by the output function

$$z(t) = y'(b,t) \quad (23)$$

The previous system is G-observable on $[0,1]$ [7] if and only if

$$b \notin S = \bigcup_{m=1}^{\infty}\left\{\frac{2k+1}{2m} \Big/ k \in [0, m-1] \cap IN\right\}$$

We denote that numerically an irrational number does not exist but it can be considered as irrational if truncation number exceeds the desired precision.

Let $T = 5$, and the sensor is located at $b = 0.48505$. The initial gradient to be reconstructed is given by

$$\nabla y^0 = A_1 \sin 3\pi x$$

and

$$\nabla y^1 = (1 + A_2)[\sin 4\pi x + 4\pi x \cos 4\pi x]$$

The coefficients A_1, A_2 are chosen such that the numerical scheme be stable, and in order to obtain a reasonable amplitude of ∇y^0 and ∇y^1 let us take $A_1 = 0.015$ and $A_2 = -0.01$.

Applying the previous algorithm, using the formulae (20), (21) we respectively obtain the **Figures 1** and **2** for $\omega =]0.46, 0.66[$ and respectively the **Figures 3** and **4** for $\omega =]0,1[$.

The estimated gradient is obtained with error $\xi = 0.81 \times 10^{-4}$ on $\omega =]0.46, 0.66[$.

For $\omega =]0,1[$, the gradient is reconstructed with error $\xi = 4.04912 \times 10^{-2}$ on $\omega =]0,1[$.

6.2. Simulating Conjectures

Now we show numerically how the error grows with respect to the subregion area. It means that the larger the region is, the greater the error is. The obtained results are presented in **Table 1**.

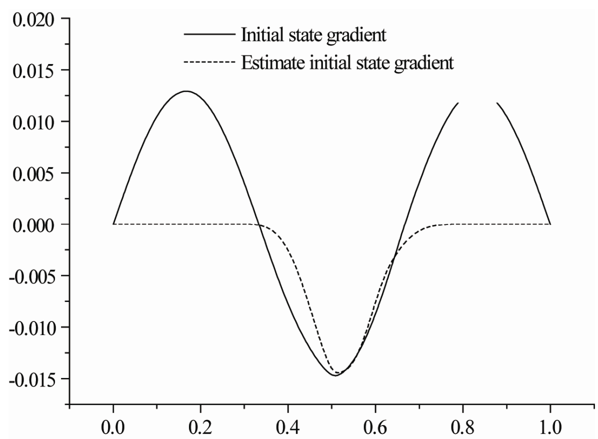

Figure 1. Iheb2.eps: initial state gradient ∇y^0 (continuous line) and estimate initial state gradient bar $(\nabla)y^0$ (dashed line).

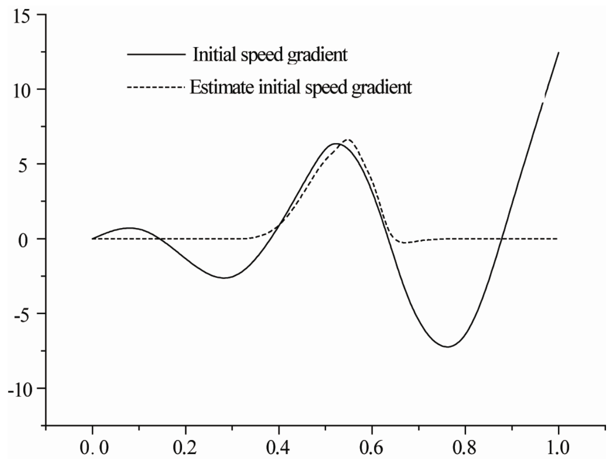

Figure 2. Nouha2.eps: initial speed gradient ∇y^1 (continuous line) and estimate initial speed gradient bar $(\nabla)y^1$ (dashed line).

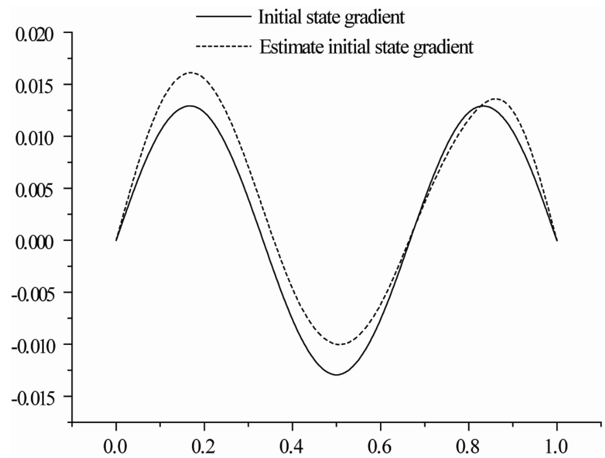

Figure 3. Iheb1.eps: initial state gradient ∇y^0 (continuous line) and estimate initial state gradient bar $(\nabla)y^0$ (dashed line).

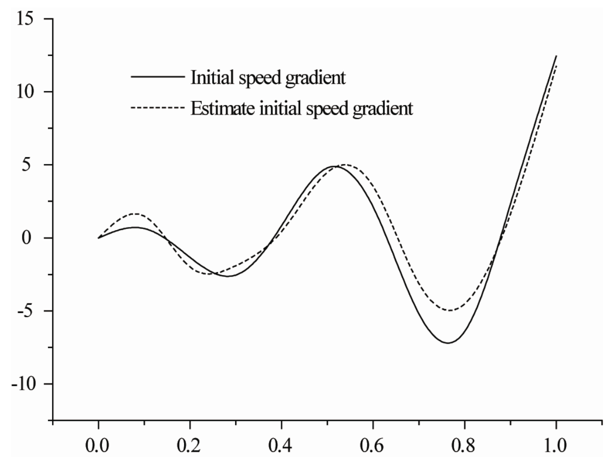

Figure 4. Nouha1.eps: initial speed gradient ∇y^1 (continuous line) and estimate initial speed gradient bar $(\nabla)y^1$ (dashed line).

Table 1. Evolution error with respect to the area of the subregion.

ω	ξ
$]0,1[$	4.04912×10^{-2}
$]0.14,0.94[$	1.00315×10^{-2}
$]0.24,0.84[$	2.04316×10^{-3}
$]0.34,0.74[$	5.61831×10^{-4}
$]0.46,0.66[$	8.16314×10^{-5}

Table 2. Evolution error with respect to the initial state gradient amplitude.

A_1	ξ
0.5	1.56181×10^{-2}
0.1	2.38512×10^{-3}
0.08	9.02815×10^{-4}
0.04	1.52361×10^{-4}
0.015	8.16314×10^{-5}

Also how both the error decreases with respect to the amplitude A_1 of the initial state gradient. For this let take the subregion $\omega =]0.46, 0.66[$ and $A_2 = -0.01$. We note that the reconstruction error depends on the amplitude of initial state gradient. It means that the greater the amplitude is, the greater the error is. The obtained results are presented in **Table 2**.

7. Conclusion

Gradient Observability on a subregion interior to the spatial evolution domain of hyperbolic system is considered. A relation between this notion and the sensors structure is established and numerical approach for its reconstruction is given. This allows the computation of the initial state gradient without the knowledge of the system state. Illustrations by numerical simulations show the efficiency of the approach. Interesting questions remain open, the case where the subregion ω is part of the boundary of the system domain. This question is under consideration.

REFERENCES

[1] E. Zerrik, H. Bourray and A. El Jai, "Regional Flux Reconstruction for Parabolic Systems," *International Journal of Systems Science*, Vol. 34, No. 12-13, 2003, pp. 641-650. doi:10.1080/00207720310001601028

[2] E. Zerrik and H. Bourray, "Regional Gradient Observability for Parabolic Systems," *International Journal of Applied Mathematics and Computer Science*, Vol. 13, 2003, pp. 139-150.

[3] A. El Jai, M. C. Simon and E. Zerrik, "Regional Ob- servability and Sensor Structures," *International Journal on Sensors and Actuators*, Vol. 39, No. 2, 1993, pp. 95- 102. doi:10.1016/0924-4247(93)80204-T

[4] E. Zerrik, "Regional Analysis of Distributed Parameter Systems," Ph.D, Thesis, University Med V, Rabat, 1993.

[5] J. L. Lions and E. Magenes, "Problème aux Limites non Homogènes et Applications," Dunod, Paris, 1968.

[6] J. L. Lions, "Contrôlabilité Exacte. Perturbation et Stabilisation des Systèmes Distribués," Dunod, Paris, 1988.

[7] A. El Jai and A. J. Pritchard, "Sensors and Actuators in the Analysis of Distributed Systems," In: E. Horwood, Ed., *Series in Applied Mathematics*, John Wiley, Hoboken, 1988.

[8] V. Trenoguine, "Analyse Fonctionnelle," Edition M.I.R., Moscou, 1985.

[9] A. M. Micheletti, "Perturbazione dello Sopettro di un Elliptic di Tipo Variazionale in Relazione ad uma Variazione del Campo," (in Italian), *Ricerche di Matematica*, Vol. XXV Fasc II, 1976.

[10] A. El Jai and S. El Yacoubi, "On the Number of Actuators in Distributed System," *International Journal of Applied Mathematics and Computer Science*, Vol. 3, No. 4, 1993, pp. 673-686.

