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Continuous Stabilizing of First Order Single Input Nonlinear Systems

Aref Shahmansoorian
Department of Electrical Engineering, Imam Khomeini International University, Qazvin, Iran
E-mail: shahmansoorian@ikiu.ac.ir
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Abstract

In this paper, stabilizability of first order nonlinear systems by a smooth control law is investigated. The main results are presented by the examples and finally summarized in a lemma. The proof for the lemma is according to Sontag's formula. In addition, it is explained that using weak control Lyapunov functions in Sontag's formula generates (possibly nonsmooth) the control law, which globally stabilizes the system-globally asymptotic stability needs more investigation.

Keywords: Control Lyapunov Function, Inverse Optimality, Sontag's Formula

1. Introduction

Consider the following nonlinear system:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in R^n$ is state space vector, $u \in R^m$ is control input vector and $f(x), g(x) \in R^n$.

Definition [1]: A differentiable positive definite and radially unbounded function $V(x): R^n \to R \geq 0$ is called a CLF for the system (1), if for each $x \neq 0$,

$$L_g V(x) = 0 \Rightarrow L_f V(x) < 0 \tag{2}$$

If there exist nonzero points where $\dot{V}(x) = 0$, then $V(x)$ is sometimes referred to as Weak Control Lyapunov Function (WCLF) [2,3].

Assume that $V(x)$ is a CLF for the system (1). It is known that the existence of a CLF for the system (1) is equivalent to the existence of a globally asymptotic stabilizing control law $u = k(x)$, which is continuous everywhere except possibly at $x = 0$ [2]. If $V(x)$ is a CLF for the system (1), then a particular stabilizing control law $u_s(x)$, smooth for all $x \neq 0$, is given by Sontag's formula: Equation (3) [3,4]

It is often desirable to guarantee at least Lipschitz continuity of the control law at $x = 0$ in addition to its smoothness elsewhere [1]. A further characterization of a stabilizing control law $u_s(x)$ for (1) with a given $V(x)$ is continuous at $x = 0$ if and only if the CLF satisfies the small control property [3]. It is well known there is a class of nonlinear systems that can not be stabilized by a continuous time-invariant feedback. Examples of systems which do not admit continuous stabilizing feedback laws are systems which do not satisfy brokett's necessary condition for continuous stabilizability [5,6].

Stabilizability of nonlinear systems is studied in literatures [7,8]. In [3] Brockett defines a necessary condition for stabilizability of nonlinear systems by a continuous feedback. In this paper sufficient condition for stabilizability of single input nonlinear systems by a continuous feedback is introduced.

2. Problem Formulation

Consider the following nonlinear system:

$$u_s(x) = \begin{cases} -\dfrac{L_f V(x) + \sqrt{\left(L_f V(x)\right)^2 + \left(L_g V(x)\left(L_g V(x)\right)^T\right)^2}}{L_g V(x)\left(L_g V(x)\right)^T}(L_g V(x))^T & L_g V(x) \neq 0 \\ 0 & L_g V(x) = 0 \end{cases} \tag{3}$$

$$\dot{x} = f(x) + g(x)u \tag{4}$$

where $x \in R$ is the state space vector, $u \in R$ is the control input vector and $f(x), g(x) \in R$ are smooth.

The question is, "when can the system (1) be stabilized at $x = 0$ by a smooth feedback control law?"

It is clear that when the unforced system is GAS, then the problem is solved. But when the unforced system is unstable or locally stable, the problem depends on the roots of the equation $g(x) = 0$. In the next section all possible situations by using numerical examples will be presented.

3. Examples

Example 1:

Consider the following nonlinear system,

$$\dot{x} = -x^3 + (x-1)u \tag{5}$$

The equation $g(x) = 0$ has a root at $x = 1$, but the unforced system is stable. Thus this system can be stabilized by a continuous control law.

Example 2:

Consider the following nonlinear system,

$$\dot{x} = -x^2 + (x-1)u \tag{6}$$

Although the equation $g(x) = 0$ has a root at $x = 1$, the unforced system solutions with initial states $x(0) > 0$ converge to the origin.

Hence this system can be stabilized by a continuous control law.

Example 3:

Consider the following nonlinear system,

$$\dot{x} = x^3 + (x-1)u \tag{7}$$

The equation $g(x) = 0$ has a root at $x = 1$ but the unforced system solutions with initial states $x(0) > 0$ escape to infinity. For that reason the system can not be stabilized by a continuous control law.

Example 4:

Consider the following nonlinear system,

$$\dot{x} = -x^2 + (x^2-1)u \tag{8}$$

The equation $g(x) = 0$ has roots at $x = 1$ and $x = -1$. For the root $x = 1$ the argument is as example 2, but for the initial states $x(0) < 0$ the unforced system solutions escape to infinity. Because of that the system can not be stabilized by a continuous control law.

Example 5:

Consider the following nonlinear system,

$$\dot{x} = x^3 + x^2 u \tag{9}$$

The unforced system is unstable, and the equation $g(x) = 0$ has the root $x = 0$. This system can be stabilized by a smooth control law (*i.e.* $u = -2x$). Actually when the equation $g(x) = 0$ has only the root $x = 0$, the system can be stabilized by a control law which is smooth everywhere except possibly at $x = 0$

Example 6:

Consider the following nonlinear system,

$$\dot{x} = -x^2 + x^3 u \tag{10}$$

This system can not be stabilized by a smooth control law. Nevertheless, this system can be stabilized by a control law, which is continuous at every nonzero x and is right-continuous at $x = 0$. The control law,

$$u^* = \begin{cases} \dfrac{1 - Sgn(x)\sqrt{x^2+1}}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{11}$$

globally asymptotically stabilizes the system and this control law is right-continuous at $x = 0$.

The reason is that the unforced system solution with initial states $x(0) > 0$ is stable.

Example 7:

Consider the following nonlinear system,

$$\dot{x} = x^2(x+1) + x^4 u \tag{12}$$

The equation $g(x) = 0$ has the root $x = 0$. The unforced system is unstable, but with the initial states $-1 < x(0) < 0$ the unforced system solutions converge to the origin. Therefore the system can be stabilized by a control law, which is continuous at every nonzero x, and at $x = 0$ is left continuous.

4. The Existence of WCLF

Actually the single input system (1) when the unforced system is not stable and the equation $g(x) = 0$ has real nonzero root(s) has not CLF and has only WCLF. The existence of WCLF is not the sufficient condition for the existence of a globally asymptotic stabilizing control law which is continuous everywhere except possibly at $x = 0$. Furthermore using WCLF in Sontag's formula generates a (possibly nonsmooth) control law, which guarantees asymptotic stability-globally asymptotic stability need to more investigation.

Example 8:

Consider the following second order nonlinear system,

$$\begin{aligned} \dot{x}_1 &= x_2 + (x_1^3 + 2x_1x_2^2 - x_1)u \\ \dot{x}_2 &= -x_1 + (x_2^3 - x_2)u \end{aligned} \tag{13}$$

It can be proved that the function $V(x) = x_1^2 + x_2^2$ is a WCLF for the system. With all initial states interior the circle $x_1^2 + x_2^2 = 1$ and a smooth stabilizing control law, the state trajectories converge to the origin. Globally asymptotic stabilizing by smooth control law is not possible.

Example 9:

Consider the following second order nonlinear system,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1\left(\frac{\pi}{2}+\arctan\left(5x_1\right)\right)-\frac{5x_1^2}{2\left(1+25x_1^2\right)}+4x_2+3u \quad (14)$$

In [2] the function:

$$V\left(x\right)=\frac{\pi}{2}x_1^2+x_2^2$$

is used as a CLF in the Sonag's formula. It can be verified the function:

$$V\left(x\right)=\frac{\pi}{2}x_1^2+2x_2^2$$

is a WCLF for the system. Using this WCLF in the Sontag's formula (Equation (3)) yields a discontinuous control law which does not globally asymptotically stabilize the system. The state trajectory with this control law and the initial state $x(0)=\left[3-2\right]^T$ converges to the point (2.946,0). In **Figure 1** the state trajectory is shown.

5. The Main Results

From the above examples the following lemma can be suggested.

Lemma 1: Consider the following single input first order system,

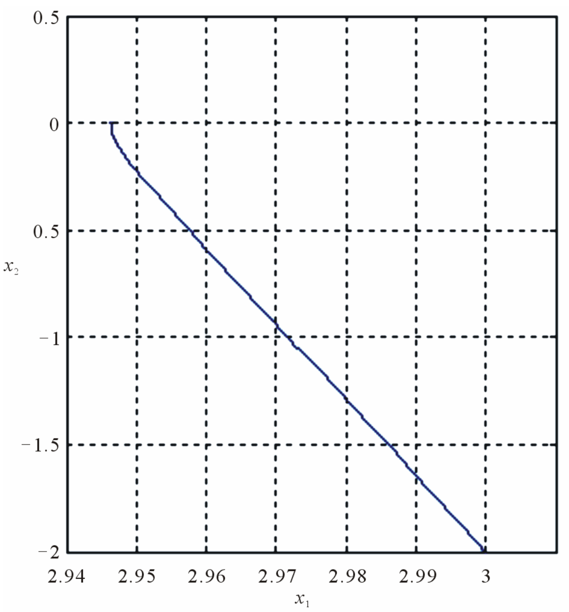

Figure 1. State trajectory for the example 9.

$$\dot{x}=f\left(x\right)+g\left(x\right)u \quad (15)$$

where $x\in R$ is state variable, $u\in R$ is control input vector and $f\left(x\right)\in R$ and $g\left(x\right)\in R$ are smooth. Assume the unforced system,

$$\dot{x}=f\left(x\right) \quad (16)$$

is locally asymptotically stable and its domain of attraction is $x\left(0\right)\in\left(a,b\right)$ and the roots of the equation $g\left(x\right)=0$ belong to the interval $\left(a,b\right)$. The system can be stabilized by a continuous control law.

Proof: Using $V\left(x\right)=\frac{1}{2}x^2$ as a WCLF for the system, the Sontag's formula gives:

$$u^*\left(x\right)=\begin{cases}-\dfrac{f\left(x\right)+Sgn\left(x\right)\sqrt{\left(f\left(x\right)\right)^2+x^2\left(g\left(x\right)\right)^4}}{g\left(x\right)}, & g\left(x\right)\neq 0\\ 0, & g\left(x\right)=0\end{cases} \quad (17)$$

It can be shown that this control law globally asymptotically stabilizes the system (15). Assume $g\left(x\right)=0$ has a nonzero root $x=c$ such that $a<c<b$. According to the assumption of lemma $xf\left(x\right)<0$, $\forall x\in\left(a,b\right)$, $x\neq 0$. Thus we have:

$$\lim_{x\to c}u^*\left(x\right)=\lim_{x\to c}\frac{x^2\left(g\left(x\right)\right)^3}{f\left(x\right)-Sign\left(x\right)\sqrt{\left(f\left(x\right)\right)^2+x^2\left(g\left(x\right)\right)^4}}$$
$$=\lim_{x\to c}\frac{x^3\left(g\left(x\right)\right)^3}{xf\left(x\right)-\left|x\right|\sqrt{\left(f\left(x\right)\right)^2+x^2\left(g\left(x\right)\right)^4}}=0$$

This proves the continuity of the control law (17). When the equation $g\left(x\right)\in R$ has root(s) at $x=0$. Then it is clear that:

$$\lim_{x\to 0}\frac{f\left(x\right)}{\left(g\left(x\right)\right)^2}$$

is equal to zero or infinity (when $f\left(x\right)$ and $g\left(x\right)$ are smooth and the system (16) is locally stable, this limit can not be equal to a nonzero finite value). Using this fact, it can be proved that:

$$\lim_{x\to 0}u^*\left(x\right)=\lim_{x\to 0}\frac{x^2\left(g\left(x\right)\right)}{\dfrac{f\left(x\right)}{\left(g\left(x\right)\right)^2}-Sign\left(x\right)\sqrt{\left(\dfrac{f\left(x\right)}{\left(g\left(x\right)\right)^2}\right)^2+x^2}}$$
$$=0$$

Remark 1: If the unforced system is unstable and the unforced system solutions with initial states $x\left(0\right)\in\left(0,b\right)=0$, $\left(x\left(0\right)\in\left(a,0\right)\right)$ converge to the origin, then the system can be stabilized by a control law which is continuous at every nonzero x and right/left-continuous at $x = 0$.

6. Conclusions

The stabilizability of affine single input first order systems by a continuous control law is investigated. It is demonstrated that sometimes a stabilizing control law can be defined that is right/left-continuous at the origin. $x = 0$. In addition, using WCLF in Sontag's formula generates a (possibly nonsmooth) control law, which globally stabilizes the system and globally asymptotic stability needs more investigation.

7. References

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