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Distributed H_∞ Consensus of High-Order Multi-Agents with Nonlinear Dynamics*

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Abstract

This paper deals with the distributed consensus problem of high-order multi-agent systems with nonlinear dynamics subject to external disturbances. The network topology is assumed to be a fixed undirected graph. Some sufficient conditions are derived, under which the consensus can be achieved with a prescribed H_∞ norm bound. It is shown that the parameter matrix in the consensus algorithm can be designed by solving two linear matrix inequalities (LMIs). In particular, if the nonzero eigenvalues of the laplacian matrix according to the network topology are identical, the parameter matrix in the consensus algorithm can be designed by solving one LMI. A numerical example is given to illustrate the proposed results.

Keywords: Consensus, Multi-Agent Systems, Nonlinear Dynamics, External Disturbances

1. Introduction

The consensus problem of multi-agent systems has been researched extensively in recent years. This is because of its widely application in much areas such as flocking [1-2], synchronization of coupled oscillators [3], formation control of mobile robots [4-5], distributed computation [6] and information fusion in wireless sensor networks [7]. The object of consensus control is to design consensus protocol such that the group of agents can asymptotically agree upon certain quantities of interest based on information received from their neighbors.

Most of the work on the consensus problem focuses on the multi-agent systems with first-order dynamics. In particular, [8] deals with the first-order multi-agent systems with switching topologies and time delays in a continuous setting. The fist-order multi-agent systems with switching topologies is investigated in [9] in a discrete-time setting. The consensus problem has also been investigated from many other aspects such as reference signals [10], asynchronous sampling time [11], and so on. Recently, the consensus problem of second-order multi-agent systems has been investigated extensively [12-14]. In particular, the consensus problem of second-order multi-agent systems with nonlinear dynamics was investigated in [15]. The nonlinear dynamics can be taken as the potential functions or the desired final dynamics of the agents. There is also some work on the consensus problems of high-order multi-agent systems [16-17].

Generally speaking, the consensus cannot be achieved accurately if there are external disturbances. To deal with this problem, the H_∞ consensus problem is considered [18-21]. It is shown that for undirected network topologies, the desired parameter matrix in the consensus algorithm can be designed by solving two LMIs, which relate to the system matrix of the agents and the eigenvalues of the laplacian matrix corresponding to the network topology. The H_2 consensus problem was investigated in [22].

In the aforementioned work on the H_∞ or H_2 consensus problem, the nonlinear dynamics was not considered. As is mentioned in [14-15] much multi-agent systems have nonlinear dynamics. Motivated by this, this paper considers the H_∞ consensus problem of high-order multi-agent systems with nonlinear dynamics. To the best of the author's knowledge, this problem has not been considered in the literature. Some sufficient conditions will be derived, under which the consensus can be achieved with a prescribed H_∞ norm bound. It will be shown that the parameter matrix in the consensus algorithm can be designed by solving two LMIs, which relate to the system matrix of the agents and the smallest and

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the biggest nonzero eigenvalue of the laplacian matrix corresponding to the network topology. In particular, if the nonzero eigenvalues of the laplacian matrix according to the network topology are identical, the parameter matrix in the consensus algorithm can be designed by solving one LMI.

2. Preliminary Notations and Problem Formulation

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ be a weighted undirected graph of order N, where $\mathcal{V}=\{1,\cdots,N\}$ is the node set. $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is a set of unordered pairs of nodes, and $\mathcal{A}$ is the adjacency matrix. An undirected path is a sequence of edges in a undirected graph of the form $(\mathrm{v}_{i_1},\mathrm{v}_{i_2})$, $(\mathrm{v}_{i_2},\mathrm{v}_{i_3})$, $\cdots$, where $v_{i_1},v_{i_2}\ldots\in\mathcal{V}$ An undirected graph is called connected if for any two nodes of the graph, there exists a path that follows the edges of the graph. The adjacency matrix is a nonnegative matrix $\mathcal{A}_d=[a_{ij}]\in\mathbb{R}^{N\times N}$ satisfying $\mathrm{a}_{ii}=0$ for any $i\in\mathcal{V}$, $\mathrm{a}_{ij}=\mathrm{a}_{ji}>0$, if $(j,i)\in\mathcal{E}$, and $\mathrm{a}_{ij}=0$ if agents j and i are not adjacent. The Laplacian matrix of the graph is defined as $\mathcal{L}=[l_{ij}]\in\mathbb{R}^{N\times N}$ with $l_{ii}=\sum_{j\neq i}a_{ij}$ and $\mathrm{l}_{ij}=-\mathrm{a}_{ij},\forall i\neq j$. We can see that $\mathcal{L}$ satisfies $\mathcal{L}\mathbf{1}=0$ and $\mathbf{1}^T\mathcal{L}=0$ where $\mathbf{1}=[1,\cdots,1]^T$. For matrices M and N, $M\otimes N$ denotes their Kronecker product. It is well known that if the undirected network topology $\mathcal{G}$ is connected, the lapalacian matrix corresponding to $\mathcal{G}$ has $N-1$ positive eigenvalues and a simple zero eigenvalue.

Consider a group of N agents with the following dynamics:

$$\dot{x}_i=Ax_i+Bu_i+B_1\omega_i+B_2 f(x_i), \tag{1}$$

where $x_i(t)\in\mathbb{R}^m$, $u_i(t)\in\mathbb{R}^p$ are, respectively, the state, the control input of agent i, $\omega_i(t)\in\mathbb{R}^m$ is the external disturbance which belongs to $\mathcal{L}_2[0,\infty)$, and $f(x_i)\in\mathbb{R}^m$ is a nonlinear function.

Assumption 1: There exists a positive scalar α such that

$$\begin{aligned}&[f(x_i)-f(x_j)]^T[f(x_i)-f(x_j)]\\&\le\alpha(x_i-x_j)^T(x_i-x_j),\\&\forall x_i,x_j\in\mathbb{R}^m.\end{aligned}$$

Remark 1: Assumption 1 is similar to the Assumption 1 in [14]. It is a Lipschitz-type condition satisfied by many systems.

Definition 1: We say algorithm u_i solves the consensus problem if

$$x_i-\sum_{j=1}^N\frac{x_j}{N}\to 0,\ t\to\infty,\ \forall i\in\mathcal{V}.$$

Definition 2: We say algorithm u_i solves the consensus problem with H_∞ norm bound γ if the following two conditions are satisfied:

1) Algorithm u_i solves the consensus problem if $\omega\equiv 0$;

2) If $z_0=0$, the following inequality is satisfied:

$$\int_0^\infty\|z\|^2dt<\gamma^2\int_0^\infty\|\omega\|^2dt,$$

where

$$z_i=x_i-\sum_{j=1}^N\frac{x_j}{N},$$

$$z=\begin{bmatrix}z_1^T & \cdots & z_N^T\end{bmatrix}^T,$$

$$\omega=\begin{bmatrix}\omega_1^T & \cdots & \omega_N^T\end{bmatrix}^T,$$

and z_0 is the initial value of z.

The object of the H_∞ consensus control is to design consensus algorithms such that the consensus problem is solved for a prescribed H_∞ norm bound.

Lemma 1: (Schur complement [23]) Let S be a symmetric matrix of partitioned form $S=[S_{ij}]$ with $S_{11}\in\mathbb{R}^{r\times r}$, $S_{12}\in\mathbb{R}^{r\times(n-r)}$ and $S_{22}\in\mathbb{R}^{(n-r)\times(n-r)}$ Then, $S<0$ if and only if $S_{11}<0$, $S_{22}-S_{21}S_{11}^{-1}S_{12}<0$, or equivalently $S_{22}<0$, $S_{11}-S_{12}S_{22}^{-1}S_{21}<0$.

Lemma 2: For matrices A, B, C, D with appropriate dimensions, one has

$$(A\otimes B)^T=A^T\otimes B^T,\ (CD)=(A\otimes C)(B\otimes D)$$

and $(A+B)\otimes C=A\otimes C+B\otimes C$

3. Results

In this section, the H_∞ consensus problem of multi-agent systems with nonlinear dynamics will be investigated. Considers the following state feedback consensus algorithm:

$$u_i(t)=K\sum_{j=1}^N a_{ij}(x_i-x_j) \tag{2}$$

With (2), system (1) becomes

$$\dot{x}_i=Ax_i+BK\sum_{j=1}^N a_{ij}(x_i-x_j)+B_1\omega_i+B_2 f(x_i), \tag{3}$$

which can be written in a compact form as

$$\begin{aligned}\dot{x}=&(I_N\otimes A+L\otimes BK)x+(I_N\otimes B_1)\omega\\&+(I_N\otimes B_2)f,\end{aligned} \tag{4}$$

where $x=\begin{bmatrix}x_1^T & \cdots & x_N^T\end{bmatrix}^T$, and $f=\begin{bmatrix}f^T(x_1) & \cdots & f^T(x_N)\end{bmatrix}^T$.

By the definition of z we have the consensus is achieved if and only if $z\to 0$ as $t\to\infty$. It is easy to see that

$$z = \left(H \otimes I_m\right)x, \tag{5}$$

where $H \in \mathbb{R}^{N\times N}$ with $H_{ij} = \begin{cases} \dfrac{N-1}{N}, & i = j \\ -\dfrac{1}{N}, & i \neq j. \end{cases}$

It can be seen that $H = I_N - \mathbf{1}\mathbf{1}^T/N$, $H^2 = H$, $\mathbf{1}_N^T H = \mathbf{0}_N^T$, $H\mathbf{1}_N = \mathbf{0}_N$ and $H\mathcal{L} = \mathcal{L}H = \mathcal{L}$.

Lemma 3: There exists an orthogonal matrix $U \in \mathbb{R}^{N\times N}$ with last column $\mathbf{1}_N/\sqrt{N}$ H such that

$$U^T HU = \begin{bmatrix} I_{N-1} & \mathbf{0}_{n-1} \\ * & 0 \end{bmatrix}, \quad U^T LU = \begin{bmatrix} \Delta & \mathbf{0}_{N-1} \\ * & 0 \end{bmatrix}.$$

From (4) and (5) we have

$$\begin{aligned} \dot{z} &= \left(H \otimes I_m\right)\dot{x} \\ &= \left(H \otimes A + L \otimes BK\right)x + \left(H \otimes B_1\right)\omega \\ &\quad + \left(H \otimes B_2\right) f \\ &= \left(H \otimes A + L \otimes BK\right)z + \left(H \otimes B_1\right)\omega \\ &\quad + \left(H \otimes B_2\right) f. \end{aligned} \tag{6}$$

Define $U \triangleq \begin{bmatrix} U_1 & \dfrac{\mathbf{1}_N}{\sqrt{N}} \end{bmatrix}$, from Lemma 3 we have $U_1^T HU_1 = I_{N-1}$ and $U_1^T LU_1 = \Delta.$ Define $\delta \triangleq \left(U^T \otimes I_m\right)z \triangleq \begin{bmatrix} \delta_1^T & \cdots & \delta_N^T \end{bmatrix}^T$, we have

$$\begin{aligned} \dot{\delta} &= \left(U^T \otimes I_m\right)\left(H \otimes A + L \otimes BK\right)\left(U \otimes I_m\right)z \\ &\quad + \left(U^T \otimes I_m\right)\left(H \otimes B_1\right)\omega + \left(U^T \otimes I_m\right)\left(H \otimes B_2\right) f \\ &= \left(\begin{bmatrix} I_{N-1} & \mathbf{0}_{n-1} \\ * & 0 \end{bmatrix} \otimes A + \begin{bmatrix} \Delta & \mathbf{0}_{N-1} \\ * & 0 \end{bmatrix} \otimes BK\right)\delta \\ &\quad + \left(\begin{bmatrix} U_1^T H \\ \mathbf{0}_{N-1}^T \end{bmatrix} \otimes B_1\right)\omega + \left(\begin{bmatrix} U_1^T H \\ \mathbf{0}_{N-1}^T \end{bmatrix} \otimes B_2\right) f. \end{aligned} \tag{7}$$

It can be seen that $\delta_N \equiv 0$. So $\delta \to 0$ if and only if $\delta_i \to 0$, $i = 1,\cdots,N-1$. Define $\bar{\delta} \triangleq \begin{bmatrix} \delta_1^T & \cdots & \delta_{N-1}^T \end{bmatrix}^T$. From (7) we have

$$\begin{aligned} \dot{\bar{\delta}} &= \left(I_{N-1} \otimes A + \Delta \otimes BK\right)\bar{\delta} \\ &\quad + \left(U_1^T H \otimes B_1\right)\omega + \left(U_1^T H \otimes B_2\right) f. \end{aligned} \tag{8}$$

Note that the eigenvalues of Δ are $\lambda_2,\cdots,\lambda_N$. There exists an orthogonal matrix $F \in \mathbb{R}^{(N-1)\times(N-1)}$ so that $F^T \Delta F = diag\{\lambda_2,\cdots,\lambda_N\}$.

Define $\xi \triangleq \left(F^T \otimes I_m\right)\bar{\delta} \triangleq \begin{bmatrix} \xi_1^T & \cdots & \xi_{N-1}^T \end{bmatrix}^T$ we have

$$\begin{aligned} \dot{\xi} &= \left(I_{N-1} \otimes A + diag\{\lambda_2,\cdots,\lambda_N\} \otimes BK\right)\xi \\ &\quad + \left(F^T U_1^T H \otimes B_1\right)\omega + \left(F^T U_1^T H \otimes B_2\right) f. \end{aligned} \tag{9}$$

Noting that $z^T z = \delta^T \delta = \bar{\delta}^T \bar{\delta} = \xi^T \xi$, we conclude that algorithm (2) solves the consensus problem with H_∞ norm bound γ if and only if system (9) is asymptotically stable with $\left\|T_{\omega\xi}\right\|_\infty < \gamma$, where $\left\|T_{\omega\xi}\right\|_\infty$ denotes the H_∞ norm of the transfer function matrix from ω to ξ.

Theorem 1: Suppose the undirected graph $\mathcal{G}$ is connected and the nonzero eigenvalues of $\mathcal{L}$ are $\lambda_2,\cdots,\lambda_N$. Using algorithm (2), the consensus is achieved with H_∞ norm bound γ if there exists a symmetric positive definite matrix X and a matrix W such that the LMIs

$$\begin{bmatrix} \Xi & B_1 & XB_2^T & X \\ B_1^T & -\gamma^2 I_m & 0 & 0 \\ B_2 & 0 & -I_m & 0 \\ X & 0 & 0 & -\dfrac{1}{\alpha+1}I_m \end{bmatrix} < 0, \tag{10}$$

$i = 2, N$, hold, where

$$\Xi = AX + XA^T + \lambda_i\left(BW + W^T B^T\right).$$

In this case, the parameter matrix in (2) can be chosen as $K = WX^{-1}$.

Proof: Assume that the undirected graph $\mathcal{G}$ is connected, we have $\lambda_2 > 0$. Suppose there exists a symmetric positive definite matrix X and a matrix W such that (10) hold. Define $WP \triangleq X^{-1}$, $K \triangleq WX^{-1}$. Pre- and post-multiply both sides of (10) by $\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}$ one can get

$$\begin{bmatrix} \Psi & PB_1 & B_2^T & I_m \\ B_1^T P & -\gamma^2 I_m & 0 & 0 \\ B_2 & 0 & -I_m & 0 \\ I_m & 0 & 0 & -\dfrac{1}{\alpha+1}I_m \end{bmatrix} < 0, \tag{11}$$

$i = 2, N$, hold, where

$$\Psi = PA + A^T P + \lambda_i\left(PBK + K^T B^T P\right).$$

For λ_i, $i = 3,\cdots,N-1$, there exists $0 \le \mu_i \le 1$ such that $\lambda_i = \mu\lambda_2 + (1-\mu)\lambda_N$. It is easy to see that (11) also holds for $i = 3,\cdots,N-1$. By Lemma 1 we know that (11) holds if and only if

$$\begin{bmatrix} \Psi + B_2^T B_2 + (\alpha+1)I_m & PB_1 \\ B_1^T P & -\gamma^2 I_m \end{bmatrix} < 0 \tag{12}$$

holds. Define

$$\Omega \triangleq \begin{bmatrix} \Gamma & I_{N-1} \otimes (PB_1) \\ I_{N-1} \otimes (PB_1)^T & -\gamma^2 I_{(N-1)\times m} \end{bmatrix},$$

where

$$\Gamma = I_{(N-1)\times m} + diag[\lambda_2, \cdots, \lambda_N] \otimes (PBK + K^T B^T P) + I_{N-1} \otimes (B_2^T B_2) + I_{N-1} \otimes (PA + A^T P + \alpha I_m).$$

From (12) we know that $\Omega < 0$.

Next prove that the consensus is achieved if $\omega \equiv 0$. If $\omega \equiv 0$, (9) becomes

$$\dot{\xi} = (I_{N-1} \otimes A + diag\{\lambda_2, \cdots, \lambda_N\} \otimes BK)\xi + (F^T U_1^T H \otimes B_2) f. \quad (13)$$

Consider the Lyapunov function

$$V(\xi) = \xi^T (I_{N-1} \otimes P)\xi.$$

Because *P* symmetric positive definite, we have $V(\xi)$ is symmetric positive definite with respect to ξ. Taking derivative of $V(\xi)$ along (13), we have

$$\begin{aligned} \dot{V}(\xi) &= 2\xi^T (I_{N-1} \otimes P)(I_{N-1} \otimes A)\xi \\ &\quad + 2\xi^T (I_{N-1} \otimes P)(diag[\lambda_2, \cdots, \lambda_N] \otimes BK)\xi \\ &\quad + 2\xi^T (I_{N-1} \otimes P)(F^T U_1^T H \otimes B_2) f \\ &= \xi^T \{I_{N-1} \otimes (PA + A^T P) \\ &\quad + diag[\lambda_2, \cdots, \lambda_N] \otimes (PBK + K^T B^T P)\}\xi \\ &\quad + 2\xi^T (F^T U_1^T H \otimes B_2) f. \end{aligned} \quad (14)$$

Because *U* is an orthogonal matrix, one has $I_N = U^T U = \begin{bmatrix} U_1 & \frac{\mathbf{1}_N}{\sqrt{N}} \end{bmatrix}^T \begin{bmatrix} U_1 & \frac{\mathbf{1}_N}{\sqrt{N}} \end{bmatrix}$ It follows that $U_1^T U_1 = I_{N-1}$. Then we have

$$\begin{aligned} & 2\xi^T (F^T U_1^T H \otimes B_2) f \\ & \le \xi^T [(F^T U_1^T U_1 F) \otimes (B_2^T B_2)]\xi \\ & \quad + f^T [(H^2) \otimes I_m] f \\ & = \xi^T [I_{N-1} \otimes (B_2^T B_2)]\xi + f^T [H \otimes I_m] f. \end{aligned} \quad (15)$$

Notice that

$$\begin{aligned} & f^T [H \otimes I_m] f \\ & = \sum_{i=1}^{N} \sum_{j>i} \frac{1}{N} [f(x_i) - f(x_j)]^T [f(x_i) - f(x_j)] \\ & \le \alpha \sum_{i=1}^{N} \sum_{j>i} \frac{1}{N} (x_i - x_j)^T (x_i - x_j) \\ & = \alpha x^T [H \otimes I_m] x \\ & = \alpha z^T z = \alpha \xi^T \xi. \end{aligned} \quad (16)$$

From (14)-(16) we have

$$\begin{aligned} & \dot{V}(\xi) \\ & \le \xi^T \{I_{N-1} \otimes (PA + A^T P) \\ & \quad + diag[\lambda_2, \cdots, \lambda_N] \otimes (PBK + K^T B^T P)\}\xi \\ & \quad + \xi^T [I_{N-1} \otimes (B_2^T B_2)]\xi + \alpha \xi^T \xi \\ & = \xi^T \{I_{N-1} \otimes (PA + A^T P + B_2^T B_2 + \alpha I_m) \\ & \quad + diag[\lambda_2, \cdots, \lambda_N] \otimes (PBK + K^T B^T P)\}\xi. \end{aligned} \quad (17)$$

It follows from (12) that $\Psi + B_2^T B_2 + (\alpha + 1) I_m < 0$, which, together with (17) implies that $\dot{V}(\xi)$ is negative definite with respect to ξ. It then follows that $\xi \to 0$ asymptotically. From the analysis above we know the consensus can be achieved.

Assume that $\omega \not\equiv 0$. Taking derivative of $V(\xi)$ along (9), we have

$$\begin{aligned} \dot{V}(\xi) &= 2\xi^T (I_{N-1} \otimes P)\{(I_{N-1} \otimes A \\ &\quad + diag[\lambda_2, \cdots, \lambda_N] \otimes BK)\xi + (F^T U_1^T H \otimes B_1)\omega \\ &\quad + (F^T U_1^T H \otimes B_2) f\} \\ &\le \xi^T \{I_{N-1} \otimes (PA + A^T P + B_2^T B_2 + \alpha I_m) \\ &\quad + diag[\lambda_2, \cdots, \lambda_N] \otimes (PBK + K^T B^T P)\}\xi \\ &\quad + 2\xi^T (F^T U_1^T H) \otimes (PB_1)\omega. \end{aligned} \quad (18)$$

Assume that $z_0 = 0$, which implies that $\xi_0 = 0$, where ξ_0 is the initial state of ξ. It follows that

$$\begin{aligned} & \int_0^\infty \|z\|^2 dt - \gamma^2 \int_0^\infty \|\omega\|^2 dt \\ & = \int_0^\infty \left[\|\xi\|^2 - \gamma^2 \|\omega\|^2 + \dot{V}(\xi)\right] dt - V(\xi) + V_0 \\ & \le \int_0^\infty \{\xi^T \xi - \gamma^2 \omega^T \omega \\ & \quad + \xi^T \{I_{N-1} \otimes (PA + A^T P + B_2^T B_2 + \alpha I_m) \\ & \quad + diag[\lambda_2, \cdots, \lambda_N] \otimes (PBK + K^T B^T P)\}\xi \\ & \quad + 2\xi^T (F^T U_1^T H) \otimes (PB_1)\omega\} dt \\ & = \int_0^\infty \eta^T \Pi \eta dt, \end{aligned} \quad (19)$$

where $\eta = \begin{bmatrix} \xi^T & \omega^T \end{bmatrix}^T$, $V_0 = 0$ is the initial value of $V(\xi)$, and

$$\Pi \triangleq \begin{bmatrix} \Gamma & (F^T U_1^T H) \otimes (PB_1) \\ (HU_1 F) \otimes (PB_1)^T & -\gamma^2 I_{Nm} \end{bmatrix}.$$

By Lemma 1 we know that $\Pi < 0$ if and only if

$$\begin{aligned}
&\Gamma + diag\left[\lambda_2, \cdots, \lambda_N\right] \otimes \left(PBK + K^T B^T P\right) \\
&+ \frac{1}{\gamma^2}\left[\left(F^T U_1^T H\right) \otimes (PB_1)\right]\left[\left(HU_1 F\right) \otimes \left(PB_1\right)^T\right] \\
&= \Gamma + diag\left[\lambda_2, \cdots, \lambda_N\right] \otimes \left(PBK + K^T B^T P\right) \\
&+ \frac{1}{\gamma^2}\left[I_{N-1} \otimes \left(PB_1 B_1^T P\right)\right] < 0.
\end{aligned} \tag{20}$$

Also by Lemma 1 we know that (20) is equivalent to $\Omega < 0$, which has been proved in the above analysis. So we have that Π is symmetric negative definite. It follows from (19) that

$$\int_0^\infty \|z\|^2 dt - \gamma^2 \int_0^\infty \|\omega\|^2 dt \le 0.$$

Therefore, the consensus is achieved with H_∞ norm bound γ. The proof is completed.

Sometimes, the laplacian matrix has $N-1$ identical nonzero eigenvalues, *i. e.* $0 < \lambda_2 = \cdots = \lambda_N$. Take the complete graph for example. Consider the complete graph with *N* nodes. The laplacian matrix is chosen as

$$\mathcal{L} = \begin{bmatrix} \frac{N-1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ -\frac{1}{N} & \frac{N-1}{N} & \cdots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{N} & -\frac{1}{N} & \cdots & \frac{N-1}{N} \end{bmatrix}.$$

By some calculations we have the eigenvalues of $\mathcal{L}$ are $0, \frac{1}{N-1}, \cdots, \frac{1}{N-1}$. In this case, we have the following corollary.

Corollary 1: Suppose the undirected graph $\mathcal{G}$ is connected and the nonzero eigenvalues of $\mathcal{L}$ satisfy $\lambda_2 = \cdots = \lambda_N$. Using algorithm (2), the consensus is achieved with H_∞ norm bound γ if there exists a symmetric positive definite matrix *X* and a matrix *W* such that the LMI

$$\begin{bmatrix} \Xi_1 & B_1 & XB_2^T & X \\ B_1^T & -\gamma^2 I_m & 0 & 0 \\ B_2 & 0 & -I_m & 0 \\ X & 0 & 0 & -\frac{1}{\alpha+1} I_m \end{bmatrix} < 0, \tag{21}$$

holds, where $\Xi_1 = AX + XA^T + BM + M^T B^T$. In this case, the parameter matrix in (2) can be chosen as $K = WX^{-1}$.

Proof: Assume that there exists a symmetric positive definite matrix *X* and a matrix *W* such that (21) holds. Define $XM \triangleq \frac{1}{\lambda_2} W$. It follows that

$$\begin{bmatrix} \Xi & B_1 & XB_2^T & X \\ B_1^T & -\gamma^2 I_m & 0 & 0 \\ B_2 & 0 & -I_m & 0 \\ X & 0 & 0 & -\frac{1}{\alpha+1} I_m \end{bmatrix} < 0$$

Holds for $i = 2$. From Theorem 1 we have the consensus is achieved with H_∞ norm bound γ, and the parameter matrix can be chosen as $K = WX^{-1}$.

Remark 2: From Corollary 1 one has that if the nonzero eigenvalues of the laplacian matrix are identical, the H_∞ performance is determined by λ_2 and the system matrices of the agents. It has no relationship with the number of the agents.

4. A Numerical Example

Consider a multi-agent systems consisted of *N* nodes with the following second-order

$$\dot{x}_i = v_i + \omega_i,\ \dot{v}_i = u_i + \omega_i + \sin\left(0.5 x_i\right),$$

where $\omega_i = \sin\left(\frac{1}{t+1}\right)$ is the external disturbance. This multi-agent system can be written in the form of (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The communication topology is given in **Figure 1**. The laplacian matrix is chosen as

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}$ are 0, 0.5858, 2 and 3.4142. Solving the LMIs in (10) with $\lambda_2 = 0.5858$, $\lambda_4 = 3.4142$, $\alpha = 0.5$ and $\gamma = 1.29$, we can get

$$X = \begin{bmatrix} 0.0455 & -0.6688 \\ -0.6688 & 23.2996 \end{bmatrix},\ W = \begin{bmatrix} 0.1 & -2605.7 \end{bmatrix}.$$

From Theorem 1 we know that *K* can be chosen as

$$K = WX^{-1} = \begin{bmatrix} -2844.2 & -193.5 \end{bmatrix}.$$

Figure 2 shows the trajectory of the external disturbance. **Figures 3** and **4** show, respectively, the position and velocity responses of nodes $1-4$.

Figure 1. The communication topology of nodes 1-4.

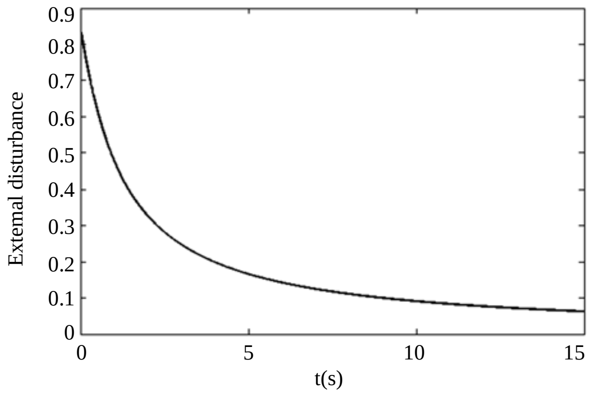

Figure 2. Positions of nodes 1-4.

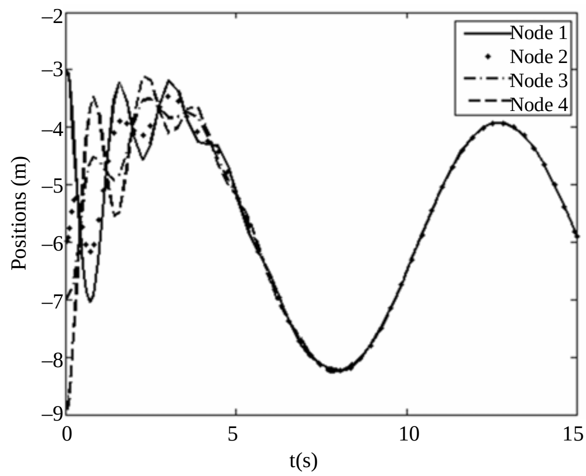

Figure 3. Positions of nodes 1-4.

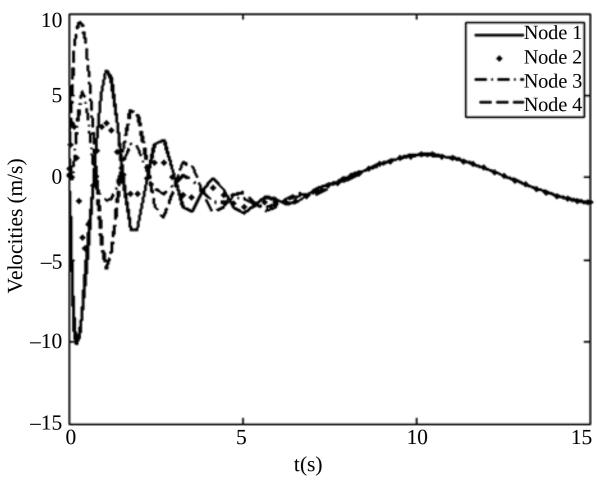

Figure 4. Velocities of nodes 1-4.

5. Conclusions

The H_∞ consensus problem has been investigated in this paper, for the high-order multi-agent systems with nonlinear dynamics. Sufficient conditions have been given in the forms of LMIs, under which the H_∞ consensus problem can be solved. The parameter matrix in the consensus algorithm can be designed by solving two LMIs. If the nonzero eigenvalues of the laplacian matrix according to the network topology are identical, the parameter matrix in the consensus algorithm can be designed by solving one LMI. The numerical simulation confirmed the proposed results.

6. References

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