

Inference and Properties of Mixture Two Extreme Lower Bound Distributions

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Abstract

In this paper, we discuss the mixture model of two extreme lower bound distributions. First, some properties we obtain of the model with hazard function are discussed. In addition, the estimates of the unknown parameters via the EM algorithm are obtained. The performance of the findings in the paper is showed by demonstrating some numerical illustrations through Monte Carlo simulation.

Keywords

Mixture Extreme Lower Bound Distribution, Reliability, Estimation, EM Algorithm, Monte Carlo Simulation

1. Introduction

Recently, the extreme value distribution is becoming increasingly important in engineering statistics as a suitable model to represent phenomena with usually large maximum observations. In engineering circles, this distribution is often called the extreme lower bound model. It is one of the pioneers of extreme value statistics. The extreme lower bound distribution is one of the probability distributions used to model extreme events. The generalization of the standard extreme lower bound distribution has been introduced by Nadarajah and Kotz [1] and Abd-Elfattah [2]. There are over fifty applications ranging from accelerated life testing through to earthquakes, floods, rain fall, queues in supermarkets, sea currents, wind speeds and track race records, see Kotz and Nadarajah [3]. Mixture models play an important role in many practical applications. For example, direct applications of finite mixture models are in fisheries research, economics, medicine, psychology, palaeoanthropology, botany, agriculture, zoology, life testing and reliability. Direct applications

include outliers, Gaussian sums, cluster analysis, latent structure models, modeling prior densities, empirical Bayes method and nonparametric density estimation. In many applications, the available data can be considered as data coming from a mixture population of two or more distributions. This data enables us to mix statistical distributions to get a new distribution which has the properties of its components. For an excellent survey of estimation techniques, discussion and applications, mixture distribution have been considered extensively by many authors, see Titterington [4], Maclachlan and Basford [5], Lindsay [6], Maclachlan and Krishnan [7] and Maclachlan and Peel [8]. Recently, there are many authors [4] [9] [10] who discuss the mixture models, Mohie El-Din *et al.* [11] [12] [13]. In this paper, we discuss some important measures of two extreme lower bound distributions. Also, we estimate the vector of unknown parameters of a mixture model via the EM algorithm proposed by Dempster *et al.* [9]. Further, we carry out some simulated illustrations using Monte Carlo method.

2. Description of the Model

The mixture of two extreme lower bound distributions has its pdf as

$$f(x, \Theta) = H_1 f_1(x, \Theta_1) + H_2 f_2(x, \Theta_2) \quad : H_1 + H_2 = 1 \quad (2.1)$$

where $\Theta = (H_1, \Theta_j)$, $\Theta_j = (c_j, \beta_j, \theta_j)$, $j = 1, 2$ and $f(x, \Theta_j)$, the density function of j th component, is given by

$$f_j(x) = \left(\frac{c_j}{\beta_j}\right) \left(\frac{x - \theta_j}{\beta_j}\right)^{-c_j - 1} \exp\left[-\left(\frac{x - \theta_j}{\beta_j}\right)^{-c_j}\right], \quad (x > \theta, c > 0, \beta > 0). \quad (2.2)$$

The *cdf* of the mixture of two extreme lower bound distributions is given by

$$F(x, \Theta) = H_1 F_1(x, \Theta_1) + H_2 F_2(x, \Theta_2) \quad : H_1 + H_2 = 1 \quad (2.3)$$

where $F_j(x, \Theta)$, the *cdf* j th component, is given by

$$F_j(x) = \exp\left[-\left(\frac{x - \theta_j}{\beta_j}\right)^{-c_j}\right], \quad (x > \theta, c > 0, \beta > 0). \quad (2.4)$$

Such that, We study this case, when c and β are the parameters unknown and θ is the parameter known.

3. Properties

In this section we obtain some properties for two extreme lower bound distribution by extending the corresponding results of the two parameters extreme lower bound distribution where (2.3) θ is known as follow.

3.1. The Expected Value and Variance

The expected value of the *pdf* of the two extreme lower bound distribution obtain in (2.1) and (2.3) is

$$E(x) = H_1\beta_1\Gamma\left(1 - \frac{1}{c_1}\right) + H_2\beta_2\Gamma\left(1 - \frac{1}{c_2}\right) \tag{3.5}$$

and the variance is given by

$$\begin{aligned} \text{Var}(x) = & H_1\beta_1^2 \left[\Gamma\left(1 - \frac{2}{c_1}\right) - H_1\Gamma^2\left(1 - \frac{1}{c_1}\right) \right] \\ & + H_2\beta_2^2 \left[\Gamma\left(1 - \frac{2}{c_2}\right) - H_2\Gamma^2\left(1 - \frac{1}{c_2}\right) \right] \\ & - 2H_1H_2\beta_1\beta_2\Gamma\left(1 - \frac{1}{c_1}\right)\Gamma\left(1 - \frac{1}{c_2}\right) \end{aligned} \tag{3.6}$$

3.2. Mode and Median

The mode of the mixture of two extreme lower bound distribution is obtained by solving the following nonlinear equation with respect to x

$$\sum_{j=0}^2 H_j \left(\frac{x - \theta_j}{\beta_j}\right)^{-(2+c_j)} e^{-\left(\frac{x - \theta_j}{\beta_j}\right)^{-c_j}} \left[-(1 + c_j) + c_j \left(\frac{x - \theta_j}{\beta_j}\right)^{-c_j} \right] \tag{3.7}$$

By using (2.4) and (3.5), the median of the mixture of two extreme lower bound distribution is obtained by solving the following nonlinear equation with respect to x

$$H_1 e^{-\left(\frac{x - \theta_1}{\beta_1}\right)^{-c_1}} + H_2 e^{-\left(\frac{x - \theta_2}{\beta_2}\right)^{-c_2}} = 0.5 \tag{3.8}$$

From **Table 1**, we obtain the median and the mode of the mixture two extreme lower bound distribution based on different choices of the parameters H_i and Θ_j for each $j = 1, 2$ from this table we observe that the mode is slightly affected by the variation in the values of the mixing proportion H_1 , while one mode is stable in the bimodal case. In addition, for unimodal case, the median increases when H_1 increases. From the bimodal case, we observe that the median decreases when H_1 increases.

3.3. Reliability and Failure Rate Function

The reliability function of the mixture two extreme lower bound distribution is given by

Table 1. The median and the mode of the mixture of two extreme lower bound distribution.

Bimodel case			
$\Theta = (H_1, c_1, \beta_2, c_2, \beta_2)$	(0.2, 2.5, 2, 1, 2.9)	(0.4, 2.5, 2, 1, 2.9)	(0.6, 2.5, 2, 1, 2.9)
Median	1.0286	0.9201	0.7852
Mode	0.326, 0.885	0.326, 0.863	0.326, 0.798
Unimodel case			
$\Theta = (H_1, c_1, \beta_2, c_2, \beta_2)$	(0.2, 1, 2, 2, 3)	(0.4, 1, 2, 2, 3)	(0.6, 1, 2, 2, 3)
Median	0.635	0.733	0.882
Mode	0.459	0.469	0.882

$$R(x) = H_1 \left[1 - e^{-\left(\frac{x-\theta_1}{\beta_1}\right)^{-c_1}} \right] + H_2 \left[1 - e^{-\left(\frac{x-\theta_2}{\beta_2}\right)^{-c_2}} \right] \tag{3.9}$$

By using (2.4) and (3.5) it can be seen the failure rate function (hazard rate function) of the mixture two extreme lower bound distribution is given by

$$r(x) = \frac{H_1 \frac{c_1}{\beta_1} \left(\frac{x-\theta_1}{\beta_1}\right)^{-(1+c_1)} e^{-\left(\frac{x-\theta_1}{\beta_1}\right)^{-c_1}} + H_2 \frac{c_2}{\beta_2} \left(\frac{x-\theta_2}{\beta_2}\right)^{-(1+c_2)} e^{-\left(\frac{x-\theta_2}{\beta_2}\right)^{-c_2}}}{R(x)} \tag{3.10}$$

Which can be written as

$$r(x) = k(x)r_1(x) + (1-k(x))r_2(x) \tag{3.11}$$

where

$$k(x) = \frac{H_1 R_1(x)}{H_1 R_1(x) + H_2 R_2(x)}, r_j(x) = \frac{f_j(x)}{R_j(x)}$$

and

$$R_j(x) = 1 - e^{-\left(\frac{x-\theta_j}{\beta_j}\right)^{-c_j}}$$

The failure rate function of the mixture two extreme lower bound distribution given in (3. 10) satisfies the following limits

$$\lim_{x \rightarrow \infty} r(x) = 0 \text{ and } \lim_{x \rightarrow 0} r(x) = 0 \tag{3.12}$$

4. Estimation via EM Algorithm

The EM algorithm provides a simple computational method for fitting mixture models. We use the EM algorithm to estimate the parameters of the pdf of the mixture two extreme lower bound distribution which given in (2.1) and (2.3). We focus in this section, the Maximum likelihood fitting of two extreme lower bound distributions mixture via the EM algorithm. Maclachlan and Peel [9], the essential nature of the algorithm is the alternation of expectation and maximization steps.

$$\xi_1 = \sum_{j=1}^n \frac{H_j^m \left(\frac{y_j - \theta_j}{\beta_j}\right)^{-(1+c_j)} e^{-\left(\frac{y_j - \theta_j}{\beta_j}\right)^{-c_j}}}{\sum_{j=1}^2 H_j^m \left(\frac{y_j - \theta_j}{\beta_j}\right)^{-(1+c_j)} e^{-\left(\frac{y_j - \theta_j}{\beta_j}\right)^{-c_j}}} \tag{4.13}$$

$$\xi_2 = \sum_{j=1}^n \frac{H_j^m \beta_j^{c_j} \left(\frac{y_j - \theta_j}{\beta_j}\right)^{-(1+2c_j)} e^{-\left(\frac{y_j - \theta_j}{\beta_j}\right)^{-c_j}}}{\sum_{j=1}^2 H_j^m \left(\frac{y_j - \theta_j}{\beta_j}\right)^{-(1+c_j)} e^{-\left(\frac{y_j - \theta_j}{\beta_j}\right)^{-c_j}}} \tag{4.14}$$

then, Concerning the E-step on the $(m + 1)$ iteration, the updated estimate of

the j mixing proportion H_j is given by

$$H_j^{(m+1)} = \frac{1}{n} \xi_1 \tag{4.15}$$

From (4.13) we obtain the M-step of the $(m+1)$ iteration, the updated estimates $c_j^{(m+1)}$ and $\beta_j^{(m+1)}$ for each $j=1,2$ are obtained, respectively, by solving the following systems of equations

$$c_j^{(m+1)} = \left(\frac{\xi_1}{\xi_2} \right)^{\frac{1}{c_j}} \tag{4.16}$$

and

$$\sum_{j=1}^n T_j \left[\frac{1}{c_j} + \log \beta_j - \log (y_j - \theta_j) + \left(\frac{y_j - \theta_j}{\beta_j} \right)^{-c_j} \log \left(\frac{y_j - \theta_j}{\beta_j} \right) \right] = 0 \tag{4.17}$$

where

$$T_j = \frac{H_j^m \left(\frac{y_j - \theta_j}{\beta_j} \right)^{-(1+c_j)} e^{-\left(\frac{y_j - \theta_j}{\beta_j} \right)^{-c_j}}}{\sum_{j=1}^2 H_j^m \left(\frac{y_j - \theta_j}{\beta_j} \right)^{-(1+c_j)} e^{-\left(\frac{y_j - \theta_j}{\beta_j} \right)^{-c_j}}}, \quad j=1,2 \text{ and } H_1 + H_2 = 1. \tag{4.18}$$

The estimates of H, c_1, β_1, c_2 and β_2 are obtained by solving (4.15), (4.17) and (4.18). Equations (4.15) and (4.17) are written explicitly but Equation (4.18) has to be solved numerically with random choices of the initial values.

5. Numerical Illustration

In order to calculate the estimates of the five parameters H, c_1, β_1, c_2 and β_2 where θ_1 and θ_2 are known that appear in the pdf of the mixture two extreme lower bound distribution given in (2.1) and (2.3) by using EM algorithm in a Monte Carlo simulation as follows:

Generate random sample of size $n = 50$ and 100 from the mixture two extreme lower bound distribution with for each choice of the parameters $H, c_1, \beta_1, \theta_1, c_2, \theta_2$ and β_2 . Some of choices cover the unimodal case and other cover the bimodal case.

The random samples of the mixtures are generated with respect to two uniform variables u_1 and u_2 . If $u_1 < H_1$, then use u_2 to generate a random variable x from the mixture two extreme lower bound distribution by using (3.5) as $x = F_1^{-1}(u_2)$, but if $u_1 < H_1$, then $x = F_2^{-1}(u_2)$.

The bias and the mean square errors of the estimates are calculated based on 10,000 Monte Carlo simulation and the results are illustrated in **Table 2** and **Table 3**. We see that in most of the considered cases, the mean square errors of the estimated parameters decrease as n increase.

6. Conclusion

In this paper, the behaviors of the mode and median of the mixture two extreme

Table 2. Bais of the estimate of $\hat{\Theta}$ based on EM algorithm.

Bimodal						
$\Theta = (H_1, c_1, \beta_1, c_2, \beta_2)$	n	\hat{H}_1	\hat{c}_1	$\hat{\beta}_1$	\hat{c}_2	$\hat{\beta}_2$
(0.2, 2.5, 2, 1, 2.9)	50	-0.116	-1.204	-0.313	0.299	-1.203
	100	-0.056	-1.189	-0.333	0.313	-1.242
(0.4, 2.5, 2, 1, 2.9)	50	-0.068	-0.770	-0.411	0.729	-1.309
	100	-0.001	-0.757	-0.421	0.743	-1.333
(0.6, 2.5, 2, 1, 2.9)	50	0.006	-0.614	-0.398	0.894	-1.270
	100	0.015	-0.611	-0.379	0.886	-1.295
Unimodal						
$\Theta = (H_1, c_1, \beta_1, c_2, \beta_2)$	n	\hat{H}_1	\hat{c}_1	$\hat{\beta}_1$	\hat{c}_2	$\hat{\beta}_2$
(0.2, 1, 2, 2, 3)	50	-0.042	0.864	0.487	-0.136	-0.514
	100	-0.025	0.850	0.423	-0.131	-0.542
(0.4, 1, 2, 2, 3)	50	0.011	0.565	-0.019	-0.483	-1.025
	100	0.005	0.569	-0.015	-0.438	-0.101
(0.6, 1, 2, 2, 3)	50	0.024	0.465	-0.089	-0.613	-1.170
	100	0.006	0.494	-0.074	-0.573	-1.132

Table 3. MSE of $\hat{\Theta}$ based on EM algorithm.

Unimodal						
$\Theta = (H_1, c_1, \beta_1, c_2, \beta_2)$	n	\hat{H}_1	\hat{c}_1	$\hat{\beta}_1$	\hat{c}_2	$\hat{\beta}_2$
(0.2, 2.5, 2, 1, 2.9)	50	0.025	1.0002	0.1003	0.098	1.003
	100	0.006	1.0001	0.120	0.095	1.0001
(0.4, 2.5, 2, 1, 2.9)	50	0.202	1.127	0.1003	0.200	2.005
	100	0.309	1.0007	0.1001	0.199	2.002
(0.6, 2.5, 2, 1, 2.9)	50	0.002	0.377	0.108	0.798	2.002
	100	0.0005	0.369	0.1008	0.789	2.000
Unimodal						
$\Theta = (H_1, c_1, \beta_1, c_2, \beta_2)$	n	\hat{H}_1	\hat{c}_1	$\hat{\beta}_1$	\hat{c}_2	$\hat{\beta}_2$
(0.2, 1, 2, 2, 3)	50	0.0021	0.7002	0.2384	0.0189	0.2005
	100	0.0005	0.7003	0.1798	0.0183	0.2004
(0.4, 1, 2, 2, 3)	50	0.0084	0.5987	0.0725	0.0482	0.5000
	100	0.0024	0.5169	0.0248	0.0430	0.5023
(0.6, 1, 2, 2, 3)	50	0.0020	0.2145	0.0083	0.3001	1.000
	100	0.0002	0.2003	0.0061	0.3000	0.9998

lower bound distribution are investigated, based on different choices of the parameters. Also, the behaviors of the failure rate function are discussed through some different graphs. In addition, the estimation of the unknown parameters is obtained using the EM algorithm. Finally, a Monte Carlo simulation based on 10,000 runs is carried out.

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