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Reflection of Plane Waves from Free Surface of an Initially Stressed Rotating Orthotropic Dissipative Solid Half-Space

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ABSTRACT

The governing equations of an initially stressed rotating orthotropic dissipative medium are solved analytically to obtain the velocity equation which indicates the existence of two quasi-planar waves. The appropriate particular solutions in the half-space satisfy the required boundary conditions at the stress-free surface to obtain the expressions of the reflection coefficients of the reflected quasi-P (qP) and reflected quasi-SV (qSV) waves in closed form for the incidence of qP and qSV waves. A particular model is chosen for numerical computation of these reflection coefficients for a certain range of the angle of incidence. The numerical values of these reflection coefficients are shown graphically against the angle of incidence for different values of initial stress parameter and rotation parameter. The impact of initial stress and rotation parameters on the reflection coefficients is observed significantly.

Keywords: Orthotropic; Dissipative Medium; Initial Stress; Rotation; Plane Waves; Reflection; Reflection Coefficients

1. Introduction

The Earth is considered as an elastic body with various additional parameters, e.g. porosity, initial stress, viscosity, dissipation, temperature, voids, diffusion, etc. Initial stresses in a medium are caused by various reasons such as creep, gravity, external forces, difference in temperatures, etc. The reflection of plane waves at free surface, interface and layers is important in estimating the correct arrival times of plane waves from the source. Various researchers studied the reflection and transmission problems at free surface, interfaces and in layered media [1-12]. The study of the reflection of plane waves in the presence of initial stresses and dissipation finds significant applications in various engineering fields. Following Biot [13] theory of incremental deformation, Selim [14] studied the reflection of plane waves at a free surface of an initially stressed dissipative medium. Recently, Singh and Arora [15] studied the reflection of plane waves from a free surface of an initially stressed transversely isotropic dissipative medium.

In the present paper, we studied the problem on reflection of plane waves at a stress-free surface of an initially stressed rotating orthotropic dissipative solid half-space. The reflection coefficients of reflected waves are computed numerically to observe the effects of initial stress and rotation.

2. Formulation of the Problem and Solution

We consider an initially stressed orthotropic half-space rotating about y-axis $\mathbf{\Omega} = (0, \Omega, 0)$ with $\mathbf{u} = (u, 0, w)$. Following Biot [13] and Schoenberg and Censor [16], the basic dynamical equations of motion in x-z plane for an infinite, initially stressed and rotating medium, in the absence of external body forces are,

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{13}}{\partial z} - P\frac{\partial \omega}{\partial z} = \rho\left(\frac{\partial^2 u}{\partial t^2} - \Omega^2 u + 2\Omega\frac{\partial w}{\partial t}\right) \quad (1)$$

$$\frac{\partial s_{31}}{\partial x} + \frac{\partial s_{33}}{\partial z} - P\frac{\partial \omega}{\partial x} = \rho\left(\frac{\partial^2 w}{\partial t^2} - \Omega^2 w - 2\Omega\frac{\partial u}{\partial t}\right) \quad (2)$$

where ρ is the density, $\omega = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)$ is rotational component, s_{ij} (i, j = 1, 3) are incremental stress components, u and w are the displacement components.

Following Biot [13], the stress-strain relations are:

$$s_{11} = B_{11}\frac{\partial u}{\partial x} + B_{13}\frac{\partial w}{\partial z}$$
$$s_{13} = s_{31} = Q\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad (3)$$
$$s_{33} = (B_{13} - P)\frac{\partial u}{\partial x} + B_{33}\frac{\partial w}{\partial z}$$

where C_{ij} are the incremental elastic coefficients.

For dissipative medium, elastic coefficients are replaced by the complex constants:

$$B_{11} \to B_{11}^R + iB_{11}^I,\ B_{13} \to B_{13}^R + iB_{13}^I,$$
$$B_{33} \to B_{33}^R + iB_{33}^I,\ B_{44} \to B_{44}^R + iB_{44}^I, \quad (4)$$

where, $i=\sqrt{-1}$, $B_{11}^R,\ B_{11}^I,\ B_{13}^R,\ B_{13}^I,\ B_{33}^R,\ B_{33}^I,\ B_{44}^R,\ B_{44}^I$ are real. Following Fung [17], the stress and strain components in dissipative medium are,

$$s_{ij} = \bar{s}_{ij}e^{i\bar{\omega}t}, u_i = \bar{u}_i e^{i\bar{\omega}t}, \quad (5)$$

where (i, j = 1, 3) and $\bar{\omega}$ being the angular frequency.

With the help of Equations (4) and (5), the Equation (3) becomes,

$$\bar{s}_{11} = \left(B_{11}^R + iB_{11}^I\right)\frac{\partial \bar{u}}{\partial x} + \left(B_{13}^R + iB_{13}^I\right)\frac{\partial \bar{w}}{\partial z},$$
$$\bar{s}_{31} = \bar{s}_{13} = \left(Q^R + iQ^I\right)\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right), \quad (6)$$
$$\bar{s}_{33} = \left(B_{13}^R + iB_{13}^I - P\right)\frac{\partial \bar{u}}{\partial x} + \left(B_{33}^R + iB_{33}^I\right)\frac{\partial \bar{w}}{\partial z},$$

The displacement vector $\mathbf{U}^{(n)} = \left(u^{(n)}, 0, w^{(n)}\right)$ is given by $\mathbf{U}^{(n)} = A_n \mathbf{d}^{(n)} e^{i\omega_n}$, where (n) assigns an arbitrary direction of propagation of waves, $\mathbf{d}^n = \left(d_1^{(n)}, d_3^{(n)}\right)$ is the unit displacement vector and

$$\omega_n = k_n\left[c_n t - \left(\mathbf{X}\cdot\mathbf{\Gamma}^{(n)}\right)\right]$$

is the phase factor, in which $\mathbf{\Gamma}^{(n)} = \left(\Gamma_1^n, \Gamma_3^{(n)}\right)$ is the unit propagation vector, c_n is the velocity of propagation, $\mathbf{X} = (x, z)$, and k_n is corresponding wave number, which is related to the angular frequency by $\bar{\omega} = k_n c_n$. The displacement components $u^{(n)}$ and $w^{(n)}$ are written as

$$\begin{pmatrix} u^n \\ w^n \end{pmatrix} = \begin{pmatrix} A_n d_1^n \\ A_n d_3^n \end{pmatrix} e^{\left[-ik_n\left(x\Gamma_1^n + z\Gamma_3^n - c_n t\right)\right]} \quad (7)$$

Making use of Equations (6) and (7) into the Equations (1) and (2), we obtain a system of two homogeneous equations, which has non-trivial solution if

$$A\left(\rho c_n^2\right)^2 + B\left(\rho c_n^2\right) + C = 0, \quad (8)$$

where,

$$A = \left[\frac{\Omega^2}{\omega^2} + 1\right]^2,\ B = \left[\frac{\Omega^2}{\omega^2} - 1\right](D_1 + D_3),\ C = D_1 D_3 - D_2^2,$$

$$D_1 = \left\{B_{11}^R \Gamma_1^{(n)^2} + \left(Q^R + \frac{P}{2}\right)\cdot\Gamma_3^{(n)^2} + i\left[B_{11}^I \Gamma_1^{(n)^2} + Q^I \Gamma_3^{(n)^2}\right]\right\},$$

$$D_2 = \left[\left(B_{13}^R + Q^R - \frac{P}{2}\right)\Gamma_1^{(n)}\Gamma_3^{(n)} + i\left(B_{13}^I + Q^I\right)\Gamma_1^{(n)}\Gamma_3^{(n)}\right],$$

$$D_3 = \left\{\left[\left(Q^R - \frac{P}{2}\right)\Gamma_1^{(n)^2} + B_{33}^R \Gamma_3^{(n)^2}\right] + i\left[Q^I \Gamma_1^{(n)^2} + B_{33}^I \Gamma_3^{(n)^2}\right]\right\},$$

The roots $c_n^2 = \frac{-B + \sqrt{B^2 - 4AC}}{2\rho A}, c_n^2 = \frac{-B - \sqrt{B^2 - 4AC}}{2\rho A}$, correspond to quasi-P (qP) waves and quasi-SV (qSV) waves respectively. These two roots give the square of velocities of propagation as well as damping. Real parts of the right hand sides correspond to phase velocities and the respective imaginary parts correspond to damping velocities of qP and qSV waves, respectively. It is observed that both c_1^2 and c_2^2 depend on initial stresses, rotation, damping and direction of propagation $\mathbf{\Gamma}^n$. In the absence of initial stresses, rotation and damping, the above analysis corresponds to the case of orthotropic elastic solid.

3. Reflection of Plane Waves from Free Surface

We consider an initially stressed rotating orthotropic dissipative half-space occupying the region $z \geq 0$ (**Figure 1**). In this section, we shall drive the closed form expressions for the reflection coefficients for incident qP or qSV waves.

The displacement components of incident and reflected waves are as,

$$u(x, z, t) = \sum_{j=1}^{4} A_j d_1^{(j)} e^{i\omega_j},\ w = (x, z, t) = \sum_{j=1}^{4} A_j d_3^{(j)} e^{i\omega_j} \quad (9)$$

where,

$$\omega_1 = k_1\left[c_1 t - \left(\sin e_1 x - \cos e_1 z\right)\right],$$
$$\omega_2 = k_2\left[c_2 t - \left(\sin e_2 x - \cos e_2 z\right)\right],$$
$$\omega_3 = k_3\left[c_3 t - \left(\sin e_3 x + \cos e_3 z\right)\right], \quad (10)$$
$$\omega_4 = k_4\left[c_4 t - \left(\sin e_4 x + \cos e_4 z\right)\right],$$

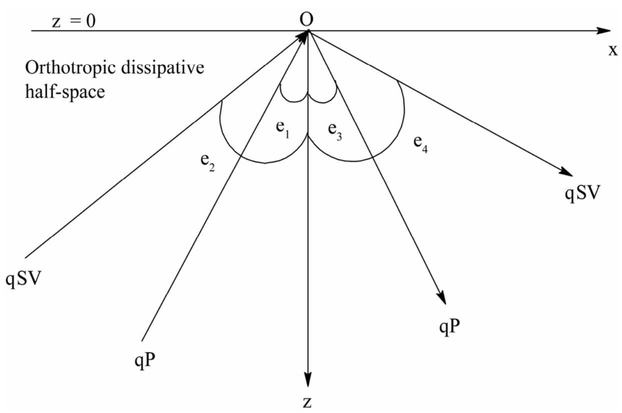

Figure 1. Geometry of the problem.

Here, subscripts 1, 2, 3 and 4 correspond to incident qP wave, incident qSV wave, reflected qP wave and reflected qSV wave, respectively.

In the x-z plane, the displacement and stress components due to the incident qP wave $\left(\Gamma_1^{(1)} = \sin e_1,\ \Gamma_3^{(1)} = -\cos e_1\right)$ are written as

$$u^{(1)} = A_1 d_1^{(1)} e^{i\omega_1},\ w^{(1)} = A_1 d_3^{(1)} e^{i\omega_1},$$
$$s_{13}^{(1)} = iA_1 Q_1 k_1 \left[d_1^{(1)} \cos e_1 - d_3^{(1)} \sin e_1 \right] e^{i\omega_1} \quad (11)$$
$$s_{33}^{(1)} = iA_1 k_1 \left(Q_3 d_3^{(1)} \cos e_1 - Q_2 d_1^{(1)} \sin e_1 \right) e^{i\omega_1}$$

where,

$$Q_1 = Q^R + iQ^I,\ Q_2 = B_{13}^R + iB_{13}^I,\ Q_3 = B_{33}^R + iB_{33}^I.$$

In the x-z plane, the displacement and stress components due to the incident qSV wave $\left(\Gamma_1^{(2)} = \sin e_2,\ \Gamma_3^{(2)} = -\cos e_2\right)$ are written as

$$u^{(2)} = A_2 d_1^{(2)} e^{i\omega_2},\ w^{(2)} = A_2 d_3^{(2)} e^{i\omega_2},$$
$$s_{13}^{(2)} = iA_2 Q_1 k_2 [d_1^{(2)} \cos e_2 - d_3^{(2)} \sin e_2] e^{i\omega_2} \quad (12)$$
$$s_{33}^{(2)} = iA_2 k_2 \left(Q_3 d_3^{(2)} \cos e_2 - Q_2 d_1^{(2)} \sin e_2 \right) e^{i\omega_2}$$

In the x-z plane, the displacement and stress components due to the reflected qP wave $\left(\Gamma_1^{(3)} = \sin e_3,\ \Gamma_3^{(3)} = \cos e_3\right)$ are written as

$$u^{(3)} = A_3 d_1^{(3)} e^{i\omega_3},\ w^{(3)} = A_3 d_3^{(3)} e^{i\omega_3},$$
$$s_{13}^{(3)} = -iA_3 Q_1 k_3 [d_1^{(3)} \cos e_3 + d_3^{(3)} \sin e_3] e^{i\omega_3}, \quad (13)$$
$$s_{33}^{(3)} = -iA_3 k_3 \left(Q_3 d_3^{(3)} \cos e_3 + Q_2 d_1^{(3)} \sin e_3 \right) e^{i\omega_3},$$

In the x-z plane, the displacement and stress components due to the reflected qSV wave $\left(\Gamma_1^{(4)} = \sin e_4,\ \Gamma_3^{(4)} = \cos e_4\right)$ are written as

$$u^{(4)} = A_4 d_1^{(4)} e^{i\omega_4},\ w^{(4)} = A_4 d_3^{(4)} e^{i\omega_4},$$
$$s_{13}^{(4)} = -iA_4 Q_1 k_4 [d_1^{(4)} \cos e_4 + d_3^{(4)} \sin e_4] e^{i\omega_4}, \quad (14)$$
$$s_{33}^{(4)} = -iA_4 k_4 \left(Q_3 d_3^{(4)} \cos e_4 + Q_2 d_1^{(4)} \sin e_4 \right) e^{i\omega_4},$$

The boundary conditions required to be satisfied at the free surface z = 0,

$$\Delta f_x = s_{13}^{(n)} + e_{13}^{(n)} P = 0,\ \Delta f_z = s_{33}^{(n)} = 0, \quad (15)$$

The above boundary conditions are written as

$$s_{13}^{(1)} + s_{13}^{(2)} + s_{13}^{(3)} + s_{13}^{(4)} + P\left(e_{13}^{(1)} + e_{13}^{(2)} + e_{13}^{(3)} + e_{13}^{(4)} \right) = 0,$$
$$s_{33}^{(1)} + s_{33}^{(2)} + s_{33}^{(3)} + s_{33}^{(4)} = 0, \quad (16)$$

The Equations (11) to (14) will satisfy the boundary conditions (16), if the following Snell's law holds

$$\frac{\sin e_1}{c_1} = \frac{\sin e_2}{c_2} = \frac{\sin e_3}{c_3} = \frac{\sin e_4}{c_4}, \quad (17)$$

with the following relations

$$A_1 \delta_1 + A_2 \delta_2 + A_3 \delta_3 + A_4 \delta_4 = 0, \quad (18)$$

$$A_1 \delta_5 + A_2 \delta_6 + A_3 \delta_7 + A_4 \delta_8 = 0, \quad (19)$$

where

$$\delta_1 = k_1 L \left(d_1^{(1)} \cos e_1 - d_3^{(1)} \sin e_1 \right),$$
$$\delta_2 = k_2 L \left(d_1^{(2)} \cos e_2 - d_3^{(2)} \sin e_2 \right),$$
$$\delta_3 = -k_3 L \left(d_1^{(3)} \cos e_3 + d_3^{(3)} \sin e_3 \right),$$
$$\delta_4 = -k_4 L \left(d_3^{(4)} \sin e_4 + d_1^{(4)} \cos e_4 \right), \quad (20)$$
$$\delta_5 = k_1 \left(Q_3 d_3^{(1)} \cos e_1 - Q_2 d_1^{(1)} \sin e_1 \right),$$
$$\delta_6 = k_2 \left(Q_3 d_3^{(2)} \cos e_2 - Q_2 d_1^{(2)} \sin e_2 \right),$$
$$\delta_7 = -k_3 \left(Q_3 d_3^{(3)} \cos e_3 + Q_2 d_1^{(3)} \sin e_3 \right),$$
$$\delta_8 = -k_4 \left(Q_2 d_1^{(4)} \sin e_4 + Q_3 d_3^{(4)} \cos e_4 \right),$$

and $L = Q_1 + P/2$.

For incident qP wave ($A_2 = 0$), we obtain from equations (18) and (19),

$$\frac{A_3}{A_1} = \frac{\delta_4 \delta_5 - \delta_1 \delta_8}{\delta_3 \delta_8 - \delta_4 \delta_7},\quad \frac{A_4}{A_1} = -\frac{\delta_3 \delta_5 - \delta_1 \delta_7}{\delta_3 \delta_8 - \delta_4 \delta_7}, \quad (21)$$

For incident qSV wave ($A_1 = 0$), we obtain from equations (18) and (19),

$$\frac{A_3}{A_2} = \frac{\delta_4 \delta_6 - \delta_2 \delta_8}{\delta_3 \delta_8 - \delta_4 \delta_7},\quad \frac{A_4}{A_2} = -\frac{\delta_3 \delta_6 - \delta_2 \delta_7}{\delta_3 \delta_8 - \delta_4 \delta_7} \quad (22)$$

For isotropic case, $B_{11} = \lambda + 2\mu + P$, $B_{13} = \lambda + P$, $B_{33} = \lambda + 2\mu$, $Q = \mu$, $P = -S_{11}$, $\Omega = 0$, then the above theoretical derivations reduce to those obtained by Selim [14]

4. Numerical Example

For numerical purpose, a particular example of the material (Zinc) is chosen with the following physical constants,

$$B_{11}^R = 1.628 \times 10^{10}\,\mathrm{N \cdot m^{-2}},\ B_{33}^R = 1.562 \times 10^{10}\,\mathrm{N \cdot m^{-2}},$$
$$B_{13}^R = 0.508 \times 10^{10}\,\mathrm{N \cdot m^{-2}},\ Q^R = 0.385 \times 10^{10}\,\mathrm{N \cdot m^{-2}},$$
$$B_{11}^I = 1.025 \times 10^{10}\,\mathrm{N \cdot m^{-2}},\ B_{33}^I = 0.250 \times 10^{10}\,\mathrm{N \cdot m^{-2}},$$
$$B_{13}^I = 0.225 \times 10^{10}\,\mathrm{N \cdot m^{-2}},\ Q^I = 0.125 \times 10^{10}\,\mathrm{N \cdot m^{-2}},$$
$$\rho = 7.14 \times 10^3\,\mathrm{N \cdot m^{-2}}.$$

From Equations (21) and (22), the reflection coefficients of reflected qP and qSV waves are computed for the incident qP and qSV waves. The numerical values of the reflection coefficients of reflected qP and qSV waves are shown graphically in **Figures 2** and **3** for incident qP wave and in **Figures 4** and **5** for incident qSV wave.

In **Figure 2**, from comparison of solid line with dashed lines, it is observed that the reflection coefficients of qP and qSV waves change due to the presence of initial stresses at each angle of incidence of qP wave except

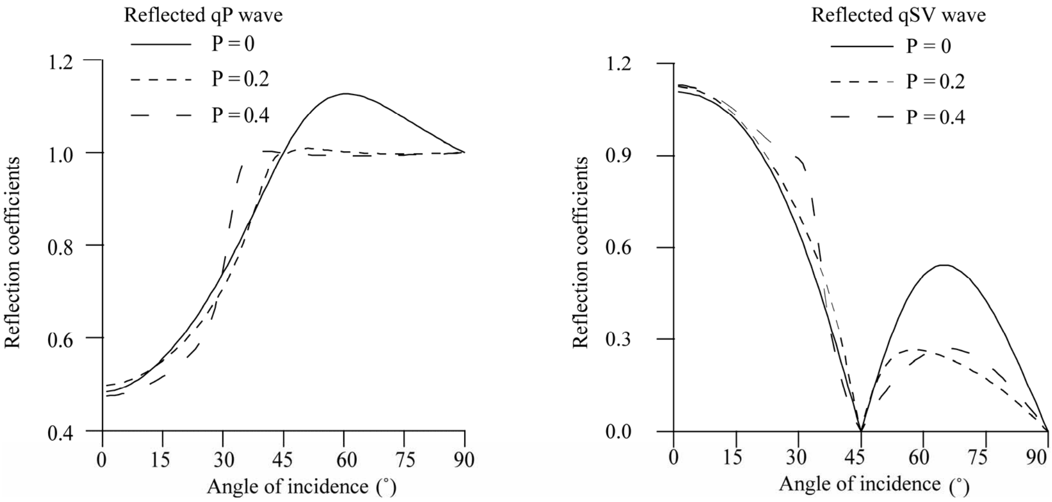

Figure 2. Effect of initial stresses on the reflection coefficients of qP and qSV waves for incidence of qP wave.

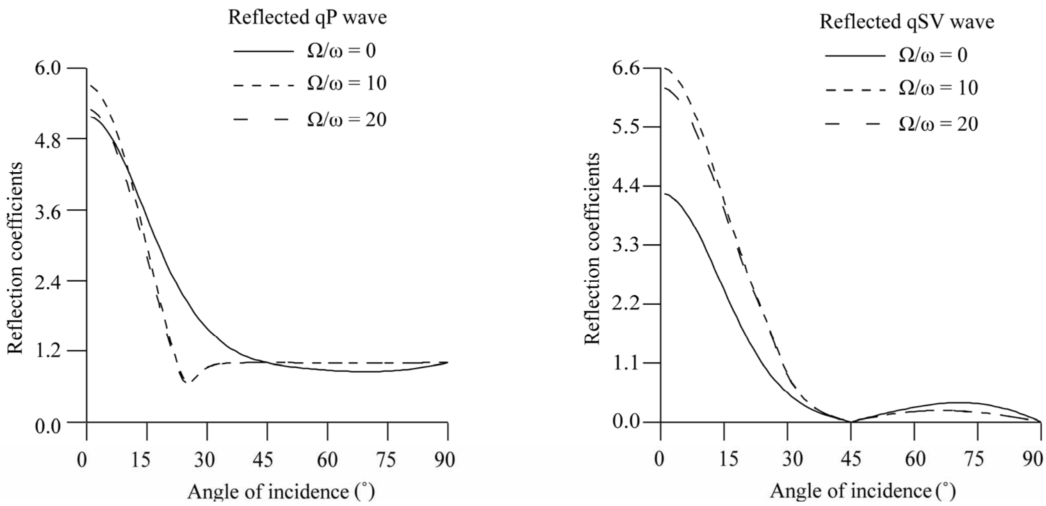

Figure 3. Effect of rotation parameter on the reflection coefficients of qP and qSV waves for incidence of qP wave.

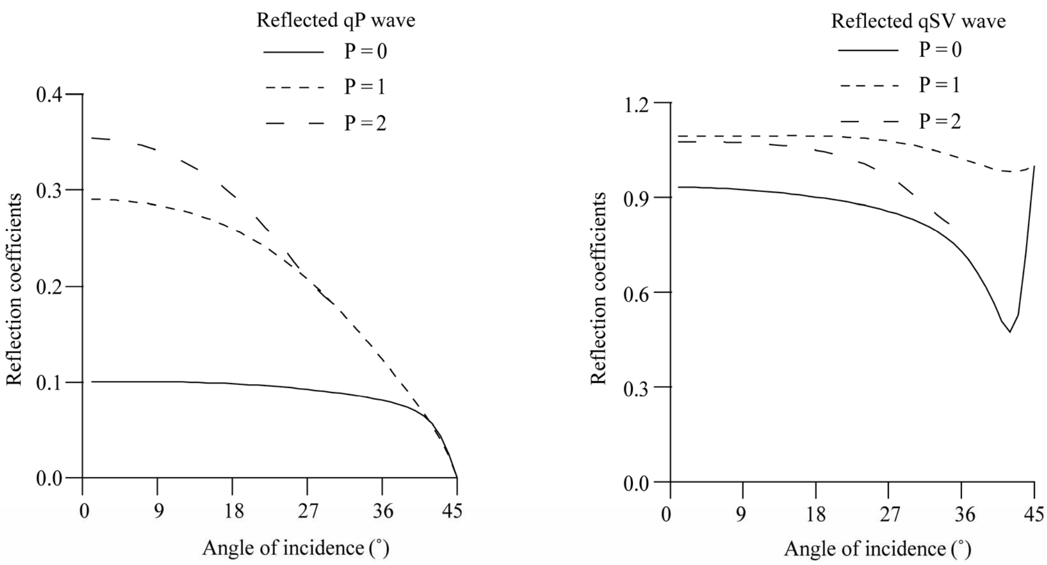

Figure 4. Effect of initial stresses on the reflection coefficients of qP and qSV waves for incidence of qSV wave.

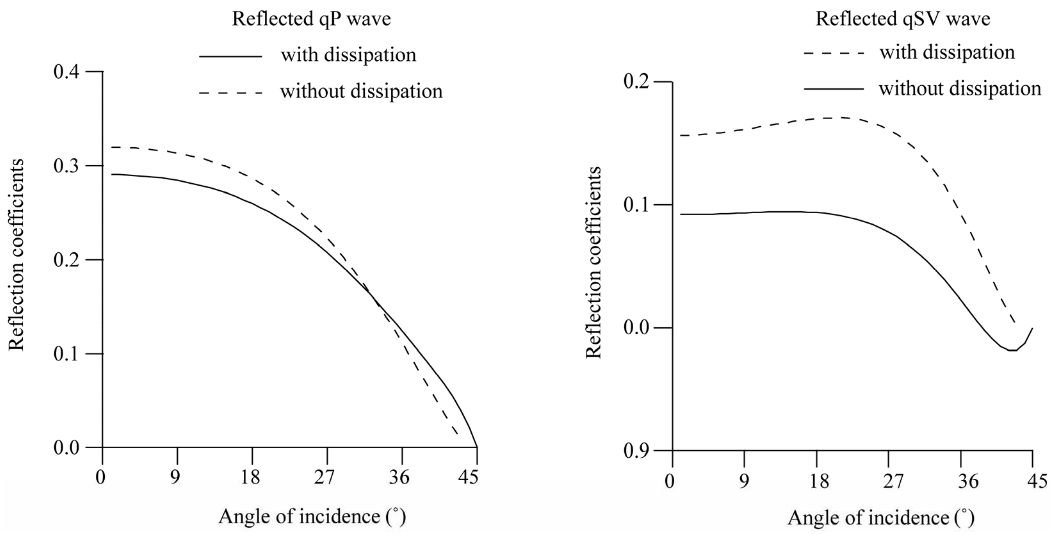

Figure 5. Effect of dissipation on the reflection coefficients of qP and qSV waves for incidence of qSV wave.

grazing incidence. The effect of initial stresses is observed maximum in the range $45^\circ < e_1 < 90^\circ$.

From **Figure 3**, it is observed that the reflection coefficients of qP and qSV waves change due to the presence of rotation in the medium at each angle of incidence of qP wave except grazing incidence.

In **Figure 4**, from comparison of solid line with dashed lines, it is observed that the reflection coefficients of qP and qSV waves change due to the presence of initial stresses in the medium at each angle of incidence of qSV wave except grazing incidence.

From **Figure 5**, it is observed that the reflection coefficients of qP and qSV waves change due to the presence of dissipation in the medium at each angle of incidence of qSV wave.

5. Conclusions

The reflection from the stress-free surface of an initially stressed rotating orthotropic dissipative medium is considered. The expressions for the reflection coefficients of reflected qP and qSV waves are obtained in closed form for the incidence of qP and qSV waves. For a particular material, these coefficients are computed and depicted graphically against the angle of incidence for different values of initial stress and rotation parameters. From the figures, it observed that 1) the initial stresses affect significantly the reflection coefficients of reflected qP and qSV waves; 2) the rotation parameter also affects significantly the reflection coefficients of qP and qSV waves; 3) Reflection coefficients are also affected due to the presence of dissipation.

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