

A Survey on the Stability of 2-D Discrete Systems Described by Fornasini-Marchesini Second Model

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ABSTRACT

A key issue of practical importance in the two-dimensional (2-D) discrete system is stability analysis. Linear state-space models describing 2-D discrete systems have been proposed by several researchers. A popular model, called Fornasini-Marchesini (FM) second model was proposed by Fornasini and Marchesini in 1978. The aim of this paper is to present a survey of the existing literature on the stability of FM second model.

Keywords: 2-D Discrete Systems; FM Second Model; Asymptotic Stability; Lyapunov Methods

1. Introduction

There have been a continuously growing research interests in two-dimensional (2-D) systems due to their applications in various important areas such as multi-dimensional digital filtering, signal processing, seismographic data processing, thermal processes, gas absorption, water stream heating etc. [1-4]. In a 2-D discrete system, information propagates in two independent directions as a result of which the system dynamics may be represented as a function of two independent integer variables. Many researchers have made an attempt to describe the 2-D system dynamics in terms of linear state-space models for 2-D discrete systems [5-7]. The 2-D models that have received considerable attention are Roesser model [5], Fornasini-Marchesini (FM) first model [6] and FM second model [7].

Stability analysis and stabilization are the main issues in the design of any control system. Stability issues of 2-D systems have been considered by many researchers [8-18]. With the introduction of state-space models of 2-D discrete systems, various Lyapunov equations have emerged as powerful tools for the stability analysis and stabilization of 2-D discrete systems. Lyapunov based sufficient conditions for the stability of 2-D discrete systems have been studied in [19-26]. When the dynamics of practical systems are represented using state-space models, errors are inevitable as the actual system parameters would be different than the estimated system parameters, *i.e.*, the model parameters. These errors arise due to the approximations made during the process modeling, differences in presumed and actual process operating points, change in operating conditions, system aging

etc. Control designs based on these models, therefore, may not perform adequately when applied to the actual industrial process and may lead to instability and poor performances. This has motivated the study of robust control for the uncertain 2-D discrete systems. The aim of robust control is to stabilize the system under all admissible parameter uncertainties arising due to the errors around the nominal system. Many significant results on the solvability of robust control problem for the uncertain 2-D discrete systems have been proposed in [27-33].

The issue in robust control design is twofold: first is to design a robust controller to ensure the stability of uncertain systems and the other is to guarantee a certain performance level under the presence of uncertainties. The latter is called as guaranteed cost control problem which has the advantage of providing an upper bound on the closed-loop cost function (performance index). Consequently, a guaranteed cost controller not only stabilizes the uncertain system but also guarantees that the value of closed-loop cost function is not more than the specified upper bound for all admissible parameter uncertainties. Based on this idea, many significant results have been obtained for the uncertain 2-D discrete systems [34-39].

Study and analysis of 2-D discrete systems under the presence of noise is another research area of great interest where it is usually necessary to estimate the state variables from the system measurement data. One of the celebrated approaches is Kalman filtering [40] which is based on two fundamental assumptions that the system under consideration is exactly known and a priori information on the external noises (like white noise, etc.). However, in many practical situations, these assumptions

may be invalid. This has motivated the 2-D signal estimation using H_∞ filtering technique. The advantage of H_∞ filtering is that the noise sources can be arbitrary signals with bounded energy, or bounded average power instead of being Gaussian. Hence, H_∞ filtering tends to be more robust when there exist additional parameter disturbances in models and it is very appropriate in a number of practical situations [41]. The 2-D filtering approach with H_∞ performance measure has been developed in [41-47].

The linear matrix inequalities (LMIs) have been evolved as a powerful technique to formulate various control designs [48]. The advantage of LMI technique is that the problem of testing the stability of a system can be formulated in terms of existence of a certain LMI (e.g., see [49]). Since solving LMIs is a convex optimization problem, such formulations offer an efficient numerical method to deal with the problems that lack an analytical solution. These LMIs can be solved efficiently by Matlab LMI tool box [50].

In this paper, our main focus is on the FM second model which is one of the most investigated models for the study and analysis of 2-D discrete systems. A brief survey of the existing literature on the stability of the 2-D discrete systems described by FM second model has been presented in this paper. The paper is organized as follows: Section 2 presents the description of 2-D system described by FM second model. A brief survey and main results on the stability of FM second model has been discussed in Section 3. Finally, some concluding remarks are given in Section 4.

Throughout the paper the following notations are used: The Closed unit disc is represented by \bar{U} , $\partial\bar{U}$ represents the unit circle, U^2 denotes the closed unit bidisc. The superscript T stands for matrix transposition, R^n denotes real vector space of dimension n , $R^{n \times m}$ is the set of $n \times m$ real matrices, $\mathbf{0}$ denotes null matrix or null vector of appropriate dimension, I_n is the $n \times n$ identity matrix, $\text{diag}(\dots)$ stands for a block diagonal matrix, $\mathbf{G} > \mathbf{0}$ stands for the matrix \mathbf{G} is positive definite, $\det(\cdot)$ denotes determinant of a matrix, $\|\cdot\|$ denotes induced matrix norm, $\rho(\cdot)$ stands for spectral radius of a matrix, $\sigma(\cdot)$ stands for spectrum of a matrix, for a matrix pair (\mathbf{A}, \mathbf{B}) , $\sigma(\mathbf{A}, \mathbf{B})$ denote the set of all its generalized eigenvalues *i.e.* $\sigma(\mathbf{A}, \mathbf{B}) := \{\lambda \in \mathbb{R} : \det(\mathbf{A} - \lambda\mathbf{B}) = 0\}$, where \mathbb{R} is the set of complex numbers. Further, $|\mathbf{B}|$ stands for the matrix $\begin{bmatrix} |b_{ij}| \end{bmatrix}$, and $\mathbf{A} \otimes \mathbf{B}$ represents Kronecker product of the matrices \mathbf{A} and \mathbf{B} .

2. Description of FM Second Model

Consider the following 2-D discrete system represented by FM second model [7]:

$$\mathbf{x}(i+1, j+1) = \mathbf{A}_1 \mathbf{x}(i, j+1) + \mathbf{A}_2 \mathbf{x}(i+1, j) + \mathbf{B}_1 \mathbf{u}(i, j+1) + \mathbf{B}_2 \mathbf{u}(i+1, j), \quad (1a)$$

$$z(i, j) = \mathbf{C} \mathbf{x}(i, j) + \mathbf{D} \mathbf{u}(i, j), \quad (1b)$$

$$i \geq 0, j \geq 0 \quad (1c)$$

where $\mathbf{x}(i, j)$ is an $n \times 1$ state vector, $\mathbf{A}_1 \in R^{n \times n}$, $\mathbf{A}_2 \in R^{n \times n}$, $\mathbf{u}(i, j)$ is $m \times 1$ input vector, $\mathbf{B}_1 \in R^{n \times m}$, $\mathbf{B}_2 \in R^{n \times m}$, z is a scalar output, $\mathbf{C} \in R^{1 \times n}$ and $\mathbf{D} \in R^{1 \times m}$. It is understood that the above system has a finite set of initial conditions [2] *i.e.*, there exist two positive integers r_1 and r_2 such that

$$\mathbf{x}(i, 0) = \mathbf{0}, i \geq r_1, \mathbf{x}(0, j) = \mathbf{0}, j \geq r_2 \quad (1d)$$

The equilibrium $\mathbf{x}(i, j) = 0$ of system (1) is said to be globally asymptotically stable [2] if

$$\lim_{i \rightarrow \infty \text{ and/or } j \rightarrow \infty} \mathbf{x}(i, j) = \lim_{i+j \rightarrow \infty} \mathbf{x}(i, j) = 0 \quad (2)$$

The transfer function of system (1) is given as

$$H(z_1, z_2) = \mathbf{C} (\mathbf{I}_n - z_1 \mathbf{A}_1 - z_2 \mathbf{A}_2)^{-1} \times (z_1 \mathbf{B}_1 + z_2 \mathbf{B}_2) + \mathbf{D} \quad (3)$$

If we define,

$$N(z_1, z_2) = \det(\mathbf{I}_n - z_1 \mathbf{A}_1 - z_2 \mathbf{A}_2), \quad (4a)$$

then the state-space model (1) is asymptotically stable [7] if and only if

$$N(z_1, z_2) \neq 0 \text{ for all } (z_1, z_2) \in \bar{U}^2 \quad (4b)$$

where $\bar{U}^2 = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}$.

3. A Brief Survey

The problem of asymptotic stability of system (1) has been studied by many researchers [51-59]. Lyapunov based sufficient condition for the stability of system (1) has been investigated in [51] and it is proposed that the system (1) is asymptotically stable if there exist an $n \times n$ symmetric matrix $\mathbf{P} > \mathbf{0}$ such that

$$\begin{bmatrix} \alpha \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \beta \mathbf{P} \end{bmatrix} - [\mathbf{A}_1 \quad \mathbf{A}_2]^T \mathbf{P} [\mathbf{A}_1 \quad \mathbf{A}_2] > \mathbf{0}, \quad (5a)$$

provided

$$\alpha > 0, \beta > 0, \alpha + \beta = 1 \quad (5b)$$

Reference [52] presents a stability test for system (1) which states that

$$N(z_1, z_2) \neq 0 \text{ in } \bar{U}^2 \text{ if } \|\mathbf{A}_1\| + \|\mathbf{A}_2\| < 1, \quad (6)$$

where $N(z_1, z_2)$ is defined in (4a). Further, based on the 2-D Lyapunov equation approach, the problem of stability margin has also been studied in [52].

Studies in [53] has illustrated that there are a large

number of systems that are stable but their stability cannot be assured by (5). That is, no values of α and β can be found to satisfy (5) for a large number of systems to confirm their stability. The result proposed in [51] is made into more generalized form in [53] and it has been proposed that the system (1) is asymptotically stable if there exist $n \times n$ symmetric matrices $\mathbf{P} > \mathbf{0}$, $\mathbf{W}_1 > \mathbf{0}$, $\mathbf{W}_2 > \mathbf{0}$, $\mathbf{R} > \mathbf{0}$ such that

$$\begin{bmatrix} \mathbf{P}^{T/2} \mathbf{W}_1 \mathbf{P}^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{T/2} \mathbf{W}_2 \mathbf{P}^{1/2} \end{bmatrix} - [\mathbf{A}_1 \quad \mathbf{A}_2]^T \mathbf{P}^{T/2} \mathbf{R} \mathbf{P}^{1/2} [\mathbf{A}_1 \quad \mathbf{A}_2] > \mathbf{0}, \quad (7a)$$

and

$$(\mathbf{R} - \mathbf{W}_1 - \mathbf{W}_2) \geq \mathbf{0}. \quad (7b)$$

As noted in [53], (5) can be recovered as a special case of (7). It is also mentioned in [53], that without loss of generality \mathbf{R} can be assigned to \mathbf{I}_n and an equivalent condition of stability can be given as: The system (1) is asymptotically stable if there exist $n \times n$ symmetric matrices $\mathbf{P} > \mathbf{0}$, $\mathbf{W}_1 > \mathbf{0}$ and $\mathbf{W}_2 > \mathbf{0}$ such that

$$\begin{bmatrix} \mathbf{P}^{T/2} \mathbf{W}_1 \mathbf{P}^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{T/2} \mathbf{W}_2 \mathbf{P}^{1/2} \end{bmatrix} - [\mathbf{A}_1 \quad \mathbf{A}_2]^T \mathbf{P} [\mathbf{A}_1 \quad \mathbf{A}_2] > \mathbf{0}, \quad (8a)$$

and

$$(\mathbf{I}_n - \mathbf{W}_1 - \mathbf{W}_2) \geq \mathbf{0}. \quad (8b)$$

In [54], sufficient conditions to guarantee the asymptotic stability of system (1) are presented. The first criterion states that for system (1) to be asymptotically stable it is sufficient that

$$1) \quad \det(\mathbf{I}_n - e^{j\omega_1} \mathbf{A}_1 - e^{j\omega_2} \mathbf{A}_2) \neq 0, \quad (9a)$$

and

2) there exist an $n \times n$ symmetric matrix $\mathbf{P} > \mathbf{0}$ such that

$$\begin{bmatrix} \alpha \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \beta \mathbf{P} \end{bmatrix} - [\mathbf{A}_1 \quad \mathbf{A}_2]^T \mathbf{P} [\mathbf{A}_1 \quad \mathbf{A}_2] \geq \mathbf{0}, \quad (9b)$$

provided

$$\alpha > 0, \beta > 0, \alpha + \beta = 1.$$

Here ω_1 and ω_2 are the horizontal and vertical radian frequencies, respectively.

The second criterion states that the system (1) is asymptotically stable if there exist $n \times n$ symmetric matrices $\mathbf{P} > \mathbf{0}$, $\mathbf{W}_1 > \mathbf{0}$ and $\mathbf{W}_2 > \mathbf{0}$ such that

$$\begin{bmatrix} \mathbf{P}^{T/2} \mathbf{W}_1 \mathbf{P}^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{T/2} \mathbf{W}_2 \mathbf{P}^{1/2} \end{bmatrix} - [\mathbf{A}_1 \quad \mathbf{A}_2]^T \mathbf{P} [\mathbf{A}_1 \quad \mathbf{A}_2] \geq \mathbf{0} \quad (10a)$$

and

$$(\mathbf{I}_n - \mathbf{W}_1 - \mathbf{W}_2) > \mathbf{0}. \quad (10b)$$

At this point, readers are advised to observe the differences between (8) and (10). Furthermore, the relationship between 2-D Lyapunov approach and stability margin has also been investigated in [54].

Another 2-D Lyapunov equation, which is in a more general form, for asymptotic stability of system (1) has been presented in [55]. According to [55], the system (1) is asymptotically stable provided there exist $n \times n$ symmetric matrices $\mathbf{R}_1 > \mathbf{0}$, $\mathbf{R}_2 > \mathbf{0}$ such that

$$\begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{bmatrix} - [\mathbf{A}_1 \quad \mathbf{A}_2]^T (\mathbf{R}_1 + \mathbf{R}_2) [\mathbf{A}_1 \quad \mathbf{A}_2] > \mathbf{0}. \quad (11)$$

Further, it has been shown that the Lyapunov matrix Inequality (11) can be expressed in a succinct form using parallel addition of positive definite matrices and an equivalent condition of stability can be given as: The system (1) is asymptotically stable if there exist a pair of $n \times n$ positive definite matrices \mathbf{P}_1 , \mathbf{P}_2 such that

$$\mathbf{P}_1 : \mathbf{P}_2 - \mathbf{A}_1 \mathbf{P}_1 \mathbf{A}_1^T - \mathbf{A}_2 \mathbf{P}_2 \mathbf{A}_2^T > \mathbf{0}, \quad (12)$$

where $\mathbf{P}_1 : \mathbf{P}_2 = (\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1})^{-1} = \mathbf{P}_1 (\mathbf{P}_1 + \mathbf{P}_2)^{-1} \mathbf{P}_2$, $\mathbf{P}_1 = \mathbf{R}_1^{-1}$, $\mathbf{P}_2 = \mathbf{R}_2^{-1}$ and $\mathbf{P}_1 : \mathbf{P}_2$ is known as parallel addition of \mathbf{P}_1 and \mathbf{P}_2 . It is interesting to note that as a rough approximation, the terms $\mathbf{A}_1 \mathbf{P}_1 \mathbf{A}_1^T$ and $\mathbf{A}_2 \mathbf{P}_2 \mathbf{A}_2^T$ makes out the next time energy along the one-dimensional dynamics; it then follows that the sum of total next time energy should be less than the modified present energy represented by $\mathbf{P}_1 : \mathbf{P}_2$ for the 2-D system (1) to be asymptotically stable. It has been further illustrated that in spite of its outward beauty, the Inequality (12) is rather complicated even for smaller values of n .

In [56], the estimation of stability robustness for the FM second model has been studied and it is mentioned that the use of the stability bounds through the 2-D Lyapunov approach is limited in application. Studies in [57] explore numerically efficient stability test methods for 2-D discrete systems based on matrix pencil approach. The necessary and sufficient conditions for the stability 2-D system have been formulated as a problem of solving the generalized eigenvalues of a constant matrix pair. As stated in [57], the system (1) is stable if and only if

$$N(z_1, 0) \text{ is stable} \quad (13a)$$

and

$$N(z_1, z_2) \neq 0 \text{ for all } z_1 \in \partial \bar{U}, z_2 \in \bar{U} \quad (13b)$$

Here the stability of $N(z_1, 0)$ is equivalent to $\rho(\mathbf{A}_1) < 1$. If we define the matrices

$$\mathbf{K}_0 = \mathbf{A}_1 \otimes \mathbf{A}_2^T, \quad (14a)$$

$$\mathbf{K}_1 = \mathbf{A}_1 \otimes \mathbf{A}_1^T + \mathbf{A}_2 \otimes \mathbf{A}_2^T - \mathbf{I}_n, \quad (14b)$$

$$\mathbf{K}_2 = \mathbf{A}_2 \otimes \mathbf{A}_1^T, \quad (14c)$$

then, the necessary and sufficient condition for the stability is as follows: The 2-D system (1) is stable if and only if $\rho(\mathbf{A}_1) < 1$ and additionally one of the following conditions holds:

$$1) \rho(\mathbf{A}_1 + z_2 \mathbf{A}_2) < 1, \text{ for all } z_2 \in \bar{\mathbf{U}}. \quad (15a)$$

$$2) \rho(\mathbf{A}_1 + e^{j\theta} \mathbf{A}_2) < 1, \text{ for all } \theta \in [0, 2\pi]. \quad (15b)$$

$$3) \sigma(\mathbf{A}_1 + z_2 \mathbf{A}_2) \cap \partial \bar{\mathbf{U}} = \emptyset, \text{ for all } z_2 \in \partial \bar{\mathbf{U}}$$

and

$$\rho(\mathbf{A}_1 + \mathbf{A}_2) < 1. \quad (15c)$$

$$4) \rho(\mathbf{A}_1 + \mathbf{A}_2) < 1$$

and

$$\sigma(\mathbf{U}, \mathbf{V}) \cap \partial \bar{\mathbf{U}} = \emptyset, \quad (15d)$$

where

$$\mathbf{U} := \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{K}_0 & -\mathbf{K}_1 \end{bmatrix},$$

and

$$\mathbf{V} := \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2 \end{bmatrix}.$$

Further, the authors in [57] have also claimed that the above stability tests tend to provide high computational accuracy.

An LMI based necessary and sufficient condition for the positive FM second model has been presented in [58]. If all the elements of system matrices \mathbf{A}_1 and \mathbf{A}_2 are positive then the system (1) will be asymptotically stable if and only if one of the following equivalent conditions holds:

1) The LMI

$$\text{block diag} \left[2\mathbf{P} - \sum_{k=1}^2 (\mathbf{A}_k^T \mathbf{P} + \mathbf{P} \mathbf{A}_k), \mathbf{P} \right] > \mathbf{0}, \quad (16a)$$

is feasible with respect to the diagonal matrix \mathbf{P} .

2) The LMI

$$\text{block diag} \left[\mathbf{P} - \sum_{k=1}^2 \sum_{l=1}^2 (\mathbf{A}_k^T \mathbf{P} \mathbf{A}_l), \mathbf{P} \right] > \mathbf{0}, \quad (16b)$$

is feasible with respect to the diagonal matrix \mathbf{P} .

The Lyapunov based sufficient condition for the stability of system (1) under shift delays has been discussed in [59]. It has been proposed that system (1) under shift delays is asymptotically stable if there exist $n \times n$ symmetric matrices $\mathbf{P} > \mathbf{0}$, $\mathbf{Q} > \mathbf{0}$, $\mathbf{Q}_1 > \mathbf{0}$, $\mathbf{Q}_2 > \mathbf{0}$ such that

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_{1d} & \mathbf{A}_2 & \mathbf{A}_{2k} \end{bmatrix}^T \mathbf{P} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_{1d} & \mathbf{A}_2 & \mathbf{A}_{2k} \end{bmatrix} - \begin{bmatrix} \mathbf{P} - \mathbf{Q} - \mathbf{Q}_1 - \mathbf{Q}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_2 \end{bmatrix} < \mathbf{0}, \quad (17)$$

where $\mathbf{A}_{1d} \in R^{n \times n}$ and $\mathbf{A}_{2k} \in R^{n \times n}$ are delay matrices. Based on condition (17), the problem of robust stability and stabilization of 2-D discrete shift-delayed system described by the FM second model has also been addressed.

In [60], the problem of robust guaranteed cost control for 2-D discrete system under shift delays has been considered and sufficient condition for the existence of robust guaranteed cost controller via static-state feedback has been derived.

In [61], some technical errors that have occurred in the main results of [60] are pointed out and corrected.

4. Concluding Remarks

A review on the stability of 2-D discrete systems described by FM second model has been presented in this paper. The Lyapunov based approach has emerged as a popular approach to study the stability properties of such systems. The 2-D Lyapunov based stability conditions discussed so far in literature are only sufficient conditions. The Lyapunov based necessary and sufficient condition for the stability of 2-D discrete systems remains an open and challenging problem.

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