***Communications and Network*, 2013, 5, 20-24**
doi:10.4236/cn.2013.52B004 Published Online May 2013 (http://www.scirp.org/journal/cn)

Jacket Matrix Based on Modular (3, 5, 6) Lattice Triangular Expansion

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Received 2013

ABSTRACT

A Lattice triangular expansion matrix is presented based on the classical Hadamard matrices, which is defined over the fields of finite characteristic. Also, the modular Lattice and Pentagon expansion matrices are structured from triangular 7×7 matrix, each of the expansion matrices are modular the sides of the shape *p*. The issue for the existence (necessary conditions) of odd and even order matrices of that kind is addressed. The modular Lattice code is highly efficient since it requires only additions, multiplications by constant modulo *p*. The modular 6 Lattice triangular expanded constellation is even possible efficiency to gain advantage from the channel selection and maximum likelihood (ML) decoding in the interference Lattice alignment (IA) system.

Keywords: Element-Wise Inverse; Modulo Jacket Matrix; the Sides of Shape; Lattice Alignment; ML Decoding

1. Introduction

The generalized reverse jacket transforms (GRJT) as multi-phase or multilevel generalizations of the WHT and the even-length DFT were introduced in [1]. With the rapid technological development, many different and generalized forms of signal processing transforms with independent parameters have been proposed. It has been discovered that the new proposed transforms with many parameters have been widely used in various signal processing, CDMA, cooperative relay MIMO system analysis. However, it can be proven that matrices having the abovementioned properties with entries from the field of complex numbers do exist only for even orders [2]. So, it seems the problem of the existence of similar transforms on distinct odd dimension spaces sounds natural. By that motivation, in this Letter, we consider a family of matrices (under the name jacket modulo prime matrices) over the fields of finite characteristic, the properties of which resemble very closely those of the conventional Hadamard matrices.

For basic definitions and notions the reader is referred to [3]. The primary generalized reverse jacket transform $(GRJT)s$, defined in [4], is a permuted version of the $2n-$length DFT, so called mixed-radix representation of integers from the set $\{0,1,...,2n-1\}$, which retains the first n rows and columns unchanged, and reverses the last n rows and columns of the corresponding DFT matrix (a Vandermonde matrix based on a primitive 2nth root of unity on the complex circle).

For the two-user interference channel, one of the best known achievable regions is that introduced by Han and Kobayashi [5]. This achievable region can be naturally generalized to more than two-users. However, a "good" choice for the auxiliary random variables and their joint distribution in the generalization of the Han & Kobayashi coding scheme is not known. In [6], it is shown that a layered lattice coding scheme can result in an improved set of achievable rates than an i.i.d. Gaussian Han & Kobayashi region. The layered lattice coding 1It is known from [7] that i.i.d. Gaussian is in fact a reasonably good choice for the two user Gaussian interference channel. It is therefore somewhat surprising that this is not true for the K > 2 user case scheme in [6] attempts to separate the signal and interference signals into non-interfering levels. Although the scheme in [6] achieves a higher DoF (and a better set of rates at any SNR) than i.i.d. Gaussian Han & Kobayashi -style coding, it does not achieve the same DoF as obtained using the schemes in [8-10]. In order to obtain a better achievable region than in [6], we allow the signal and interference lattices to interact with one another in the case of channels with integer channel gains in [12], and determine algebraic mechanisms of separating signal and interference. Although the scheme in [12] achieves a strictly better set of rates than in [6], it still falls short, in terms of degrees of freedom, than that achieved in [8,9].

2. Center Weighted Hadamard Matrix

In this section, we introduce some definitions and notations. First, we recall the center weighted Hadamard matrix of order 4 in [12]

$$[\mathrm{CWH}]_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\omega & \omega & -1 \\ 1 & \omega & -\omega & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

where ω is a nonzero complex parameter. The inverse of this basic matrix can be easily obtained by element-wise inverse matrix as follows:

$$[\mathrm{CWH}]_4^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\frac{1}{\omega} & \frac{1}{\omega} & -1 \\ 1 & \frac{1}{\omega} & -\frac{1}{\omega} & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Definition 2.1: A matrix $[J]_{N\times N} = (j_{i,k})$ of order N whose entries are complex is called a Jacket matrix, if the element in the (i,k) entry of its inverse matrix is equal to the product of $1/N$ and the inverse of the element in the (k,i) entry of $[J]_{N\times N}$. In other words, if

$$[J]_{N\times N} = \begin{pmatrix} j_{0,0} & j_{0,1} & \cdots & j_{0,N-1} \\ j_{1,0} & j_{1,1} & \cdots & j_{1,N-1} \\ \cdots & \cdots & \cdots & \cdots \\ j_{N-1,0} & j_{N-1,1} & \cdots & j_{N-1,N-1} \end{pmatrix}$$

and its inverse

$$[J]_{N\times N}^{-1} = \begin{pmatrix} \frac{1}{j_{0,0}} & \frac{1}{j_{0,1}} & \cdots & \frac{1}{j_{0,N-1}} \\ \frac{1}{j_{1,0}} & \frac{1}{j_{1,1}} & \cdots & \frac{1}{j_{1,N-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{j_{N-1,0}} & \frac{1}{j_{N-1,1}} & \cdots & \frac{1}{j_{N-1,N-1}} \end{pmatrix}$$

then is called a Jacket matrix.

From the definition of Jacket matrices, it is easy to see that any Hadamard matrices of order are Jacket matrices. In addition, the center weighted Hadamard (CWH) is also a Jacket matrix.

We can find that Jacket matrices have reciprocal orthogonality and reciprocal relation. The basic Jacket matrix of order 3 is defined as

$$[J]_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

where ω is the third primitive root of unity. The inverse of J_3 is

$$[J]_3^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{\omega} & \frac{1}{\omega^2} \\ 1 & \frac{1}{\omega^2} & \frac{1}{\omega} \end{pmatrix}$$

which satisfies

$$[J]_3[J]_3^{-1} = [I]_3$$

where I_n is the identity matrix of order n. From (10), it is easy to see that the inverse of $[J]_3^{-1}$ can be easily obtained from the forward matrix J_3 by taking the inverse of each entry J_3 of and then transposing the resulting matrix. Hence the Jacket transform has following two advantages:

1) Element-wise inverse orthogonality.
2) The entries of the forward and the inverse transforms have a reciprocal relationship.

3. Jacket Matrix over Finite Characteristic Fields

Without loss of generality we may focus on the fields $GF(p)$, where p is a prime and define the notion of the jacket modulo prime matrix over them.

Definition 3.1: A jacket modulo prime (JMP) matrix J of order n over $GF(p)$ is an $n\times n$ non-singular matrix of $\pm 1s$ that field such that

$$JJ^T = nI_n \tag{1}$$

where I_n is the identity matrix of order n.

As usual, the notation M^T is used for the transpose matrix of a given matrix M. We shall use also the notation $JMP(p)$ for the set of jacket modulo prime matrices over $GF(p)$.

Example 1: Triangular 7×7 matrix (**Figure 1**)

Let J_n, where $n = pk+4$ and $k = 1,2,...,$ be a square matrix of order n consisting of with the following description. Its first row and column consist entirely of $1s$; its last row and column consist of $-1s$ with exception of the corner entries, and all other entries are equal to 1 with the exception of those on the main diagonal. For instance, $J_7(p=3, k=1)$ looks as:

$$J_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

also

$$J_7^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

The inner product of a pair of rows equals either to $pk = 3\times 1 = 3$, i.e. in $GF(3)$ the following matrix equation holds:

$$J_7 J_7^T = 7 I_7 \quad (2)$$

Clearly, $J_7^{-1} = J_7 = J_7^T$, where J_7^T is the transpose matrix of J_7. So, J_7 is an orthogonal matrix over the filed $GF(p=3)$. We stress once again that in this example the operations are taken modulo 3. Thus, J_n is a JMP matrix over $GF(P=3)$.

Example 2: Extended Lattice Triangular 10×10 matrix (**Figure 2**)

Similarly, by the same way as shown in the example 1, the 10×10 matrix can be expressed as follow

$$L_{10} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix},$$

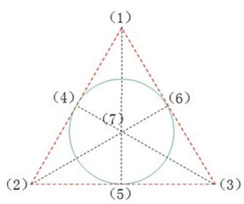

Figure 1. Triangular and circular internally tangent.

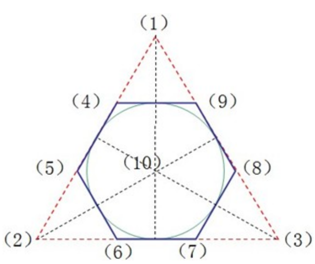

Figure 2. Lattice and circular internally tangent.

The inverse of J_{10} can be easily calculate as

$$L_{10}^{-1} = \frac{1}{10}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

Clearly, that

$$L_{10} = L_{10}^{-1} = L_{10}^T \quad (3)$$

$$L_{10} L_{10}^{-1} = 10 I_{10} \quad (\text{in } GF6) \quad (4)$$

By the Definition 2.1, $L_{10}, L_{10}^{-1}, L_{10}^T$ are also Jacket matrices over $GF(6)$.

Example 3: Extended Pentagon Triangular 9×9 matrix

Also, the Pentagon Triangular 9×9 matrix can be structured as p_9

$$p_9 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

Clearly, that

$$P_9 = P_9^{-1} = P_9^T \quad (5)$$

$$P_9 P_9^{-1} = 9 I_9 \quad (\text{in } GF5) \quad (6)$$

By the Definition 2.1, P_9, P_9^{-1}, P_9^T are also Jacket matrices over $GF(5)$

Over these 3 examples, the modular (5,6) Jacket matrix J_n is constructed based on the triangular (7×7) matrix, where $n = p + 4$, $p \in 5, 6$ is the number of sides for the shape (that's meaning pentagon, lattice). Note that this scheme is highly efficient since it requires only additions, multiplications by constant modulo p, and it is even possible to gain advantage from interference alignment.

4. Lattice Alignment Application

In this section, we consider 3-pairs interference system, where each transmitter T_i and receiver R_i equipped with one antenna, respectively. The channel coefficients $H_{i,j}$ define links from transmitter *i* to the receiver *j*, where $i, j \in 1, 2, 3$.

4.1. Channel Selection with Lattice Constellation

Motivated by the advantage of having a structured interference, we propose an approximate lattice alignment scheme in which the precoders are designed to best align the receiving lattices. However, we accept the fact that lattice alignment may not be perfect (due to infeasible configurations and imperfect CSI effects) and try to model and minimize the effect of the residual lattice alignment errors. The lattice alignment error is given by

$$\mathrm{e}_a = | u_j^H h_{ij} v_j - a_L | \quad (7)$$

where a_L is the lattice coordinate as shown in the **Figures 3** and **4**. As a result, the design parameters $\{u_i, v_i, a_L\}$ in (7) are chosen to minimize the effects of the lattice alignment errors.

The error is smaller, the channel state information (CSI) is better. The optimal is $\mathrm{e}_a = 0$ .The conditional error probability given u_j can be upper bounded as follows:

$$P(e \mid u_j) = P\{u_j^H h_{ij} v_j \neq a_L\}$$
$$= \sum_{u_j^H h_{ij} v_j \in \Lambda} P\{\| u_j^H h_{ij} v_j - a_L \|^2 < \| \mathbb{Z} \|^2\}$$

where $\mathbb{Z}$ is the maximized distance in the constellation, on the other hand $\mathbb{Z}$ is the length of side.

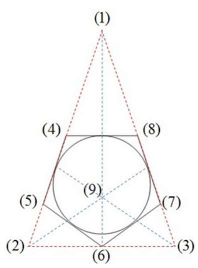

Figure 3. Pentagon and circular internally tangent.

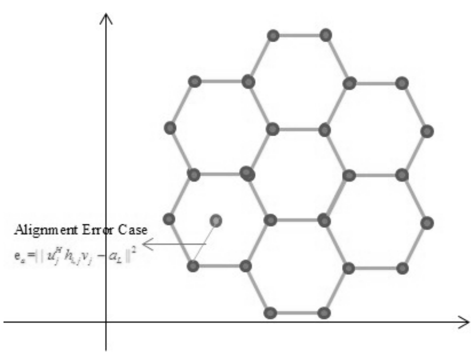

Figure 4. Lattice alignment constellation with imperfect CSI.

4.2. Lattice Alignment in 3-pairs Interference Chanel

In the conventional works, the perfect IA requirements for all $k \in K$ are summarized as

$$\mathrm{U}_j^H \mathrm{H}_{i,j} \mathrm{V}_j = 0 , \quad (8)$$

$$\mathrm{rank}(\mathrm{U}_i^H \mathrm{H}_{i,i} \mathrm{V}_i) = d_i . \quad (9)$$

Eq. (8) guarantees that all the interfering signals at destination $l \in K$ are aligned in a subspace of $N_k - d_i$ dimensions and can be zero-forced by Z_j. Eq. (9) guarantees that destination $k \in K$ is able to decode all d_j intended data streams successfully. When both equations (8) and (9) are satisfied, the interference alignment is feasible for the given DoF.

We will work with a many-to-one Gaussian interference channel with 3 users, where interference is only present at receiver 1. The desired symbols of receiver $k \in 1, 2, 3$ can be estimated as

$$\mathrm{y}_i = \overbrace{\mathrm{u}_i^H \mathrm{h}_{i,i} x_i}^{\text{desired signal}} + \overbrace{\sum_{i \neq j}^{k} \mathrm{u}_j^H h_{i,j} x_j}^{\text{interference signal}} + \mathrm{u}_i^H \mathrm{n}_i$$

where $h_{i,j}$ is the $n \times n$ channel matrix from transmitter *j* to receiver *i*, x_j is the transmitted symbols and n_i is the additive white Gaussian noise with variance σ^2 .

At receiver 1, there is interference from users 2 and 3. By suitably choosing v_2 and v_3 in such a way that

$$h_{12} v_2 = h_{13} v_3 .$$

We can perform lattice alignment of the interfering signals from users 2 and 3 at the first receiver's as follow:

$$L(h_{12} v_2 = h_{13} v_3) = L(h_{13} v_3)$$

where $L(J_n)$ is the lattice generated by the matrix J_n . Then the desired signal belongs to the lattice $\Lambda_1 = h_{11} v_1 a_L$, while the sum of the interfering signals $L[h_{12} v_2 (x_2 + x_3)]$ is aligned in the lattice $\Lambda_2 = h_{12} v_2 a_L$, where a_L is the Lattice alignment coordinate which will be introduced in the next section. Then the received signal at the receiver 1 can be rewritten as:

$$y_1 = \overline{x}_1 + \overline{x}_2 + n_1$$

where $\overline{x}_1 = h_{11} v_1 s_1$ and $\overline{x}_2 = L[h_{21} v_2 (x_2 + x_3)]$ belong to the Lattice constellation coordinate.

After successfully channel selection, we wish to decode the desired signal x_i^k at stage-II as illustrated in **Figure 5**. The desired signal is detected given by

$$\tilde{y}_i^k = | u_i^H y_i^k - u_i^H \tilde{x}_1 - u_j^H \tilde{x}_2 | \quad (10)$$

Compared with the alignment error and ML decoding algorithm in first and second stage, both of them are the Euclidean distance. Let's focus on:

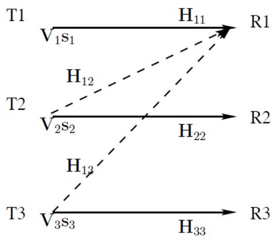

Figure 5. 3-piars interference channel.

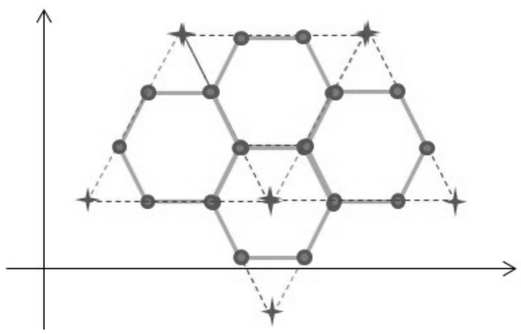

Figure 6. ML decoding based on Lattice expansion triangular.

$$\begin{aligned}\tilde{y}_i^k &= | u_i^H y_i^k - (u_i^H h_{i,i} v_i x_i + \sum_{i \neq j}^{k} u_j^H h_{i,j} v_j x_j) | \\ &= | u_i^H y_i^k - \tilde{x}_i - e_a x_j \pm a_L x_j |\end{aligned} \quad (11)$$

where a_L is given as the coordinate in each constellation, $\tilde{x}_i = u_i^H h_{i,i} v_i x_i$. We can find that the ML decoding mapping constellation should include the lattice constellation.

Also, in the **Figures 4** and **6**, the area of the lattice expansion triangular is greater than the lattice. It's also satisfied the formula in (7).

5. Conclusions

Clearly, the Lattice triangular expansion matrix, which can be given for an arbitrary field of finite characteristic, is presented based on the conventional real Hadamard matrices. In this paper, we have also addressed a necessary condition for the existence and presented a construction of odd and even order JMP matrices. The modular Lattice and Pentagon expansion matrices are structured by the triangular 7×7 matrix, and modular the sides of the shape *p*.The modular Lattice is highly efficient since it requires only additions, multiplications by constant modulo *p*. The modular 6 Lattice triangular expanded constellation is even possible to gain advantage from the channel selection and maximum likelihood (ML) decoding in the interference alignment (IA) system.

6. Acknowledgements

This work was supported by World Class University, R32-2012-000-20014-0, Basic Science Research Program 2010-0020942, MEST 2012-002521 and NRF China-Korea International Corsearch (D00066, I00026).

REFERENCES

[1] M. H. Lee, B. S. Rajan and J. Y. Park, "A Generalized Reverse JacketTransform," *IEEE Transform Circuits System II, Analog Digital Signal Processing,* Vol. 48, No. 7, 2001, pp. 684-691.

[2] M. H. Lee and Y. L. Borissov, "Proof of Non-existence of Borderedjacket Matrices of Odd Order Over Some Fields," *Electronic Letters,*Vol. 46, No. 5, 2010, pp. 349-351. doi:10.1049/el.2010.2991

[3] R. E. Blahut, "Theory and Practice of Error Control Codes,"Addison- Wesley, New York, 1983.

[4] M. H. Lee, B. S. Rajan and J. Y. Park, "A Generalized Reverse Jackettransform," *IEEE Transformation Circuits System II, Analog Digital Signal Processing,* Vol. 48, No. 7, 2001, pp. 684-691.

[5] T. S. Han and K. Kobayashi, "A New Achievable Rate Region for Theinterference Channel," *IEEE Transaction Information Theory,* Vol. 27, No. 1, 1981, pp.49-60. doi:10.1109/TIT.1981.1056307

[6] S. Sridharan, A. Jafarian, S. Vishwanath, S. A. Jafar and S. Shamai, "A Layered Lattice Coding Scheme for a Class of Three User Gaussian Interference Channels," *IEEE Transaction Information Theory,*2008.

http://arxiv.org/abs/0809.4316.

[7] R. H. Etkin, D. N. C. Tse and H. Wang, "Gaussian Interference Channel Capacity to within One Bit," Vol. 54, No. 12, 2008, pp. 5534-5562.

[8] A. S. Motahari, S. O. Gharan and A. K. Khandani, "Real InterferenceAlignment with Real Numbers," *IEEE Transaction Information Theory*, 2009, p. 1208.

[9] R. Etkin and E. Ordentlich, "On the Degrees-of-freedom of the K-userGaussian Interference Channel,"2009.

http://arxiv.org/abs/0901.1695.

[10] V. R. Cadambe, S. A. Jafar and C. Wang, "Interference Alignment with Asymmetric Complex Signaling - settling the Host,".

[11] A. Jafarian, J. Jose and S. Vishwanath, "Lattice Alignment Using Algebraic Alignment," *Allerton Conference on Community Control and Computing*, 2009.

[12] M. H. Lee, "The Center Weighted Hadamard Transform," *IEEE Transaction Circuits System*, Vol. 36, No. 9, pp. 1247-1249.

[13] R. Zamir and M. Feder, "On Lattice Quantization Noise," *IEEE Transaction Information Theory*, Vol. IT-42, 1996, pp. 1152-1159. doi:10.1109/18.508838

[14] U. Erez, S. Litsyn and R. Zamir, "Lattices Which Are Good for (almost)Everything," *IEEE Transaction Information Theory*, Vol. IT-51, 2005, pp. 3401-3416. doi:10.1109/TIT.2005.855591

[15] T. M. Cover and J. A. Thomas, "Elements of Information Theory," New York: Wiley, 1991.

