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Q_K Type Spaces and Bloch Type Spaces on the Unit Ball

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Abstract

Different function spaces have certain inclusion or equivalence relations. In this paper, the author introduces a class of Möbius-invariant Banach spaces $Q_{K,0}(p,q)$ of analytic function on the unit ball of $\mathbb{C}^n$, where $K:(0,\infty)\to[0,\infty)$ are non-decreasing functions and $0<p<\infty$, $\frac{p}{2}-n-1<q<\infty$, studies the inclusion relations between $Q_{K,0}(p,q)$ and a class of $\mathcal{B}_0^{\alpha}$ spaces which was known before, and concludes that $Q_{K,0}(p,q)$ is a subspace of $\mathcal{B}_0^{\frac{q+n+1}{p}}$, and the sufficient and necessary condition on kernel function $K(r)$ such that $Q_{K,0}(p,q)=\mathcal{B}_0^{\frac{q+n+1}{p}}$.

Keywords

Unit Ball, $Q_{K,0}(p,q)$ Space, $\mathcal{B}_0^{\frac{q+n+1}{p}}$ Space, Equivalence Relation

1. Introduction

Q_K spaces were first given by Hasi Wulan and Matts Essen around 2000. In recent years, Q_K type spaces have caused extensive research (cf. [1]-[11]). To study a new kind of function space, we usually need to establish the relationship between that and those known to all. The notion of the spaces Q_K on the unit ball was defined by Xu Wen in his paper [4]. According to Hasi Wulan, Q_K type spaces $Q_K(p,q)$ on unit disk were introduced and investigated, and the conditions on K such that $Q_K(p,q)$ become some known spaces were given (cf. [5]). About multiple variables, the definition of $Q_{K,0}(p,q)$ on unit ball were given by Xu Wen (cf. [6]), and the author has studied the inclusion relations between $Q_K(p,q)$ spaces and $\mathcal{B}^{\frac{q+n+1}{p}}$ spaces on the unit ball (cf. [7]). In this paper, the author introduces the $Q_{K,0}(p,q)$ spaces and $\mathcal{B}_0^{\frac{q+n+1}{p}}$ spaces on the unit ball of $\mathbb{C}^n$, studies the inclusion relationship between them. Firstly, establish the relationship between the norm of the function which belongs to $Q_{K,0}(p,q)$ and the norm $\|f\|_{\mathcal{B}_0^\alpha}$, proof that the $Q_{K,0}(p,q)$ is a subspace of $\mathcal{B}_0^{\frac{q+n+1}{p}}$; and then obtain the necessary and sufficient condition of kernel functions $K(r)$ when $Q_{K,0}(p,q)=\mathcal{B}_0^{\frac{q+n+1}{p}}$.

2. Preliminaries

Let $a\in\mathbb{B}_n$ and φ_a be the involution of $\mathbb{B}_n$ satisfied $\varphi_a(0)=a$. $\mathrm{d}v(z)$ is the volume measure on $\mathbb{B}_n$, normalized so that $v(\mathbb{B}_n)=1$, and $\mathrm{d}\lambda=\frac{\mathrm{d}v(z)}{\left(1-|z|^2\right)^{n+1}}$ is the Möbius invariant volume measure on $\mathbb{B}_n$ (cf. [4]), $\mathrm{d}\sigma$ is the normalized surface measure on $\mathbb{S}_n$, the measure v and σ are related by (cf. [12])

$$\int_{\mathbb{B}_n} f(z)\mathrm{d}v(z)=2n\int_0^1 r^{2n-1}\mathrm{d}r\int_{\mathbb{S}_n} f(r\zeta)\mathrm{d}\sigma(\zeta). \tag{1}$$

Let $\nabla f(z)=\left(\frac{\partial f}{\partial z_1},\frac{\partial f}{\partial z_2},\cdots,\frac{\partial f}{\partial z_n}\right)$ denote the complex gradient of *f*, and $\tilde{\nabla} f(z)=\nabla(f\circ\varphi_z)(0)$ is the invariant gradient of f (cf. [12]). $\tilde{\nabla} f(z)$ and $\nabla f(z)$ are related by ([12])

$$\left(1-|z|^2\right)|\nabla f(z)|\le\left|\tilde{\nabla} f(z)\right|\le\left(1-|z|^2\right)^{\frac{1}{2}}|\nabla f(z)|. \tag{2}$$

The Möbius invariant Green function is defined by $G(z,a)=g(\varphi_a(z))$, where

$$g(z)=\frac{n+1}{2n}\int_{|z|}^1\left(1-t^2\right)^{n-1}t^{-2n+1}\mathrm{d}t. \tag{3}$$

Definition 1 Let $K:(0,\infty)\to[0,\infty)$ is a right-continuous, non-decreasing function, for $0<p<\infty$, $\frac{p}{2}-n-1<q<\infty$, we say that a holomorphic function *f* belongs to the space $Q_{K,0}(p,q)$ if

$$\lim_{|a|\to 1}\int_{\mathbb{B}_n}\left|\tilde{\nabla} f(z)\right|^p\left(1-|z|^2\right)^{q+n+1-p}K(G(z,a))\mathrm{d}\lambda(z)=0. \tag{4}$$

Definition 2 $\mathcal{B}_0^\alpha$ space is defined by

$$\mathcal{B}_0^\alpha=\left\{f\in H(\mathbb{B}_n):\lim_{|a|\to 1}\left(1-|z|^2\right)^{\alpha-1}\left|\tilde{\nabla} f(z)\right|=0\right\}. \tag{5}$$

The constant C can represent different values in different places in this paper.

3. Main Results

In this paper, the author demonstrates that $Q_{K,0}(p,q)$ is a subspace of $\mathcal{B}_0^{\frac{q+n+1}{p}}$

as the first main result and it is of great help for the second one.

Theorem 1. Let $0<p<\infty$, $\frac{p}{2}-n-1<q<\infty$, then $Q_{K,0}(p,q)\subset \mathcal{B}_0^{\frac{q+n+1}{p}}$.

Proof Let $E(a,r_0)=\{z\in\mathbb{B}_n,|\varphi_a(z)|<r_0\}$, then

$$\begin{aligned}
&\int_{\mathbb{B}_n}|\tilde{\nabla}f(z)|^p(1-|z|^2)^{q+n+1-p}K(G(z,a))\mathrm{d}\lambda(z)\\
&\ge\int_{E(a,r_0)}|\tilde{\nabla}f(z)|^p(1-|z|^2)^{q+n+1-p}K(g(\varphi_a(z)))\mathrm{d}\lambda(z)\\
&=\int_{|z|<r_0}|\tilde{\nabla}(f\circ\varphi_a)(z)|^p(1-|\varphi_a(z)|^2)^{q+n+1-p}K(g(z))\mathrm{d}\lambda(z)\\
&\ge K(g(r_0))\int_{|z|<r_0}(1-|z|^2)^p|\nabla(f\circ\varphi_a)(z)|^p(1-|\varphi_a(z)|^2)^{q+n+1-p}\frac{\mathrm{d}v(z)}{(1-|z|^2)^{n+1}}\\
&\ge C\int_{|z|<r_0}(1-|\varphi_a(z)|^2)^{q+n+1-p}|\nabla(f\circ\varphi_a)(z)|^p\mathrm{d}v(z)
\end{aligned}$$

We have $\left(1-|\varphi_a(z)|^2\right)=\frac{(1-|z|^2)(1-|a|^2)}{|1-\langle z,a\rangle|^2}$, when $|z|\le r_0$,

$\frac{1-r_0^2}{(1+r_0)^2}\le\frac{(1-|z|^2)}{|1-\langle z,a\rangle|^2}\le\frac{1}{(1-r_0)^2}$, and since $|\nabla f(z)|^p$ is subharmonic, that

$$\begin{aligned}
&\int_{\mathbb{B}_n}|\tilde{\nabla}f(z)|^p(1-|z|^2)^{q+n+1-p}K(G(z,a))\mathrm{d}\lambda(z)\\
&\ge C(1-|a|^2)^{q+n+1-p}\int_{|z|<r_0}|\nabla(f\circ\varphi_a)(z)|^p\mathrm{d}v(z)\\
&=C(1-|a|^2)^{q+n+1-p}\int_0^{r_0}r^{2n-1}\mathrm{d}r\int_{S_n}|\nabla(f\circ\varphi_a)(r\varsigma)|^p\mathrm{d}\sigma(\varsigma)\\
&\ge C(1-|a|^2)^{q+n+1-p}|\nabla(f\circ\varphi_a)(z)|^p\\
&=C(1-|a|^2)^{q+n+1-p}|\tilde{\nabla}f(a)|^p
\end{aligned}$$

Thus, we have $\lim_{|a|\to1}(1-|a|^2)^{q+n+1-p}|\tilde{\nabla}f(a)|^p=0$ when $f\in Q_{K,0}(p,q)$, then $f\in\mathcal{B}_0^{\frac{q+n+1}{p}}$.

The following result is the further study on the equivalence between $Q_{K,0}(p,q)$ and $\mathcal{B}_0^{\frac{q+n+1}{p}}$.

Theorem 2. Let $0<p<\infty$, $\frac{p}{2}-n-1<q<\infty$, $Q_{K,0}(p,q)=\mathcal{B}_0^{\frac{q+n+1}{p}}$ if and only if

$$\int_0^1(1-r^2)^{-n-1}r^{2n-1}K(g(r))\mathrm{d}r<\infty. \tag{6}$$

Proof Sufficiency: By theorem 1, we only need to show that $\mathcal{B}_0^{\frac{q+n+1}{p}}\subset Q_{K,0}(p,q)$.

Since $\int_0^1(1-r^2)^{-n-1}r^{2n-1}K(g(r))\mathrm{d}r<\infty$, for given $\varepsilon>0$, then there exists $r_0:0<r_0<1$, such that

$$\int_{r_0}^{1}\left(1-r^2\right)^{-n-1} r^{2n-1} K\left(g\left(r\right)\right)\mathrm{d}r < \varepsilon .$$

Let $E\left(a,r_0\right)=\left\{z\in\mathbb{B}_n,\left|\varphi_a\left(z\right)\right|<r_0\right\}$, for any $f\in\mathcal{B}^{\frac{q+n+1}{p}}$, $z\in\mathbb{B}_n\setminus E\left(a,r_0\right)$, we have

$$\begin{aligned}
&\int_{\mathbb{B}_n\setminus E(a,r_0)}\left|\tilde{\nabla}f\left(z\right)\right|^p\left(1-\left|z\right|^2\right)^{q+n+1-p}K\left(G\left(z,a\right)\right)\mathrm{d}\lambda\left(z\right)\\
&\le\left\|f\right\|^p_{\mathcal{B}^{\frac{q+n+1}{p}}}\int_{\mathbb{B}_n\setminus E(a,r_0)}K\left(G\left(z,a\right)\right)\mathrm{d}\lambda\left(z\right)\\
&\le\left\|f\right\|^p_{\mathcal{B}^{\frac{q+n+1}{p}}}\int_{r_0<|z|<1}\left(1-\left|z\right|^2\right)^{-n-1}K\left(g\left(z\right)\right)\mathrm{d}v\left(z\right)\\
&\le\left\|f\right\|^p_{\mathcal{B}^{\frac{q+n+1}{p}}}\int_{r_0}^{1}\left(1-r^2\right)^{-n-1}r^{2n-1}K\left(g\left(r\right)\right)\mathrm{d}r\int_{\mathbb{S}_n}\mathrm{d}\sigma\left(\varsigma\right)\\
&<\varepsilon\left\|f\right\|^p_{\mathcal{B}^{\frac{q+n+1}{p}}}
\end{aligned}\tag{7}$$

And when $z\in E\left(a,r_0\right)$, we have

$$\begin{aligned}
&\lim_{|a|\to1}\int_{E(a,r_0)}\left|\tilde{\nabla}f\left(z\right)\right|^p\left(1-\left|z\right|^2\right)^{q+n+1-p}K\left(G\left(z,a\right)\right)\mathrm{d}\lambda\left(z\right)\\
&=\lim_{|a|\to1}\int_{|z|<r_0}\left|\tilde{\nabla}\left(f\circ\varphi_a\right)\left(z\right)\right|^p\left(1-\left|\varphi_a\left(z\right)\right|^2\right)^{q+n+1-p}K\left(g\left(z\right)\right)\mathrm{d}\lambda\left(z\right)\\
&\le\lim_{|a|\to1}\sup_{|z|<r_0}\left(1-\left|\varphi_a\left(z\right)\right|^2\right)^{q+n+1-p}\left|\tilde{\nabla}\left(f\circ\varphi_a\right)\left(z\right)\right|^p\int_{|z|<r_0}K\left(g\left(z\right)\right)\left(1-\left|z\right|^2\right)^{-n-1}\mathrm{d}V\left(z\right)\\
&=\lim_{|a|\to1}\sup_{|z|<r_0}\left(1-\left|\varphi_a\left(z\right)\right|^2\right)^{q+n+1-p}\left|\tilde{\nabla}\left(f\circ\varphi_a\right)\left(z\right)\right|^p 2n\\
&\quad\times\int_0^{r_0}\left(1-r^2\right)^{-n-1}r^{2n-1}K\left(g\left(r\right)\right)\mathrm{d}r\int_{\mathbb{S}_n}\mathrm{d}\sigma\left(\varsigma\right)\\
&\le C\lim_{|a|\to1}\sup_{|z|<r_0}\left(1-\left|\varphi_a\left(z\right)\right|^2\right)^{q+n+1-p}\left|\tilde{\nabla}\left(f\circ\varphi_a\right)\left(z\right)\right|^p
\end{aligned}$$

$\left(1-\left|\varphi_a\left(z\right)\right|^2\right)=\dfrac{\left(1-\left|z\right|^2\right)\left(1-\left|a\right|^2\right)}{\left|1-\left\langle z,a\right\rangle\right|^2}$, and $\dfrac{1-r_0^2}{\left(1+r_0\right)^2}\le\dfrac{\left(1-\left|z\right|^2\right)}{\left|1-\left\langle z,a\right\rangle\right|^2}\le\dfrac{1}{\left(1-r_0\right)^2}$ when $\left|z\right|\le r_0$, so

$$\lim_{|a|\to1}\left(1-\left|\varphi_a\left(z\right)\right|^2\right)^{\frac{q+n+1-p}{p}}\left|\tilde{\nabla}\left(f\circ\varphi_a\right)\left(z\right)\right|=0,$$

thus

$$\lim_{|a|\to1}\int_{E(a,r_0)}\left|\tilde{\nabla}f\left(z\right)\right|^p\left(1-\left|z\right|^2\right)^{q+n+1-p}K\left(G\left(z,a\right)\right)\mathrm{d}\lambda\left(z\right)=0,$$

By formula(7), then we have $\lim_{|a|\to1}\int_{\mathbb{B}_n}\left|\tilde{\nabla}f\left(z\right)\right|^p\left(1-\left|z\right|^2\right)^{q+n+1-p}K\left(G\left(z,a\right)\right)\mathrm{d}\lambda\left(z\right)=0$, *i.e.* $f\in Q_{K,0}\left(p,q\right)$. It means $\mathcal{B}_0^{\frac{q+n+1}{p}}\subset Q_{K,0}\left(p,q\right)$.

Necessary: We only need to show that if $\int_0^1\left(1-r^2\right)^{-n-1}r^{2n-1}K\left(g\left(r\right)\right)\mathrm{d}r=\infty$, there exists a function $f \in \mathcal{B}_0^{\frac{q+n+1}{p}}$, but $f \notin Q_{K,0}(p,q)$.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be an n-tuple of non-negative integers, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ satisfied $|\alpha| = 2^N$ where N is a integer. Let $f = |\alpha|^{\frac{q+n+1-p}{p}} z^\alpha$, it is easy to show that $f \in \mathcal{B}^{\frac{q+n+1}{p}}$, and by the proof of theorem 3 in [7], we know that $\int_{\mathbb{S}_n} J(r\varsigma)^{\frac{p}{2}} \mathrm{d}\sigma(\varsigma) \geq C(1-r)^{-(q+n+1)+\frac{p}{2}}$ when $r \in \left[\frac{3}{4}, 1\right)$, which

$$J(r\varsigma) = r^{2|\alpha|-2} |\alpha|^{\frac{2(q+n+1-p)}{p}} \left(\alpha_1^2 \left|\varsigma_1^{\alpha_1-1}\varsigma_2^{\alpha_2}\cdots\varsigma_n^{\alpha_n}\right|^2 + \cdots \right.$$
$$\left. + \alpha_n^2 \left|\varsigma_1^{\alpha_1}\varsigma_2^{\alpha_2}\cdots\varsigma_n^{\alpha_n-1}\right|^2 - r^2 |\alpha|^2 \left|\varsigma^\alpha\right|^2 \right)$$

thus

$$\begin{aligned}
&\int_{\mathbb{B}_n} \left|\tilde{\nabla} f(z)\right|^p \left(1-|z|^2\right)^{q+n+1-p} K(G(z,a)) \mathrm{d}\lambda(z) \\
&\geq \int_{\mathbb{B}_n} \left(1-|z|^2\right)^{\frac{p}{2}} (J(z))^{\frac{p}{2}} \left(1-|z|^2\right)^{q+n+1-p} K(g(z)) \mathrm{d}\lambda(z) \\
&= 2n\int_0^1 \left(1-r^2\right)^{q-\frac{p}{2}} r^{2n-1} K(g(r)) \mathrm{d}r \int_{\mathbb{S}_n} J(r\varsigma)^{\frac{p}{2}} \mathrm{d}\sigma(\varsigma) \\
&\geq C\int_{\frac{3}{4}}^1 \left(1-r^2\right)^{-n-1} r^{2n-1} K(g(r)) \mathrm{d}r
\end{aligned}$$

Since the conclusion of theorem 1 in [7], we have

$$\int_0^{\frac{3}{4}} \left(1-r^2\right)^{-n-1} r^{2n-1} K(g(r)) \mathrm{d}r \leq C\int_0^1 \left(1-r^2\right)^{2-n-1} r^{2n-1} K(g(r)) \mathrm{d}r < \infty,$$

Then if $\int_0^1 \left(1-r^2\right)^{-n-1} r^{2n-1} K(g(r)) \mathrm{d}r = \infty$, we can get

$$\int_{\mathbb{B}_n} \left|\tilde{\nabla} f(z)\right|^p \left(1-|z|^2\right)^{q+n+1-p} K(G(z,a)) \mathrm{d}\lambda(z) = \infty,$$

which shows that $f \notin Q_{K,0}(p,q)$, the theorem is proved.

With the above conclusion, further study in this field of operator theory on $Q_{K,0}(p,q)$ can be conducted in the future.

Founding

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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