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An Isogeometric Error Estimate for Transport Equation in 2D

Aurélien Goudjo[1], Uriel Aguemon[2]

[1]FAculté des Sciences et Techniques (FAST), Université d'Abomey-Calavi (UAC), Abomey-Calavi, Benin
[2]Institut de Mathématiques et de Sciences Physiques (IMSP), Université d'Abomey-Calavi (UAC), Abomey-Calavi, Benin
Email: aguemonuriel@yahoo.fr

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Abstract

In this paper, an isogeometric error estimate for transport equation is obtained in 2D to prove the convergence of isogeometric method. The result that we have obtained, generalizes Ern result, got in finite elements method. For the time discretization, the two stage Heun scheme is used to prove this result. For a polynomial of degree $k \geq 1$, the order of convergence in space is 2 and in time is $k+\frac{1}{2}$.

Keywords

Error Estimate, Isogeometric Method, The Two Stage Heun Scheme, Transport Equation

1. Introduction

Some phenomena of the daily life such as particles transport in an electric field, the signal transport along a wire, evolution of cars on a road [1], and evolution of a pollutant in a narrow channel [2] are modelled by a transport equation. Study of numerical methods for solving this equation is very important to describe, to predict and to control these phenomena.

Isogeometric Analysis has been introduced by Thomas Hughes, Austin Cottrell and Yuri Bazilevs in 2005 [3].

The objectives of Isogeometric Analysis are to generalize and improve upon Finite Element Analysis (FEA) in the following ways:

1) To provide more accurate modeling of complex geometries and to exactly represent common engineering shapes such as circles, cylinders, spheres, ellipsoids, etc.

2) To fix exact geometries at the coarsest level of discretization and eliminate

geometrical errors.

3) To vastly simplify mesh refinement of complex industrial geometries by eliminating the necessity to communicate with the *CAD* (Computer Aided Design) description of geometry.

4) To provide refinement procedures, including classical *h*- and *p*-refinements analogues, and to develop a new refinement procedure called *k*-refinement [4].

The idea of Isogeometric Analysis is to build a geometry model and, rather than develop a finite element model approximating the geometry, directly use the functions describing the geometry in analysis [5] [6]. These functions are B-splines.

Isogeometric Analysis is approached, using continuous or discontinuous Galerkin method. In the context of space semidiscretization by discontinuous Galerkin methods, explicit *RK* schemes are used to approximate in time systems of ordinary differential equations. These schemes have been developed by Cockburn and Shu [7], Cockburn, Lin, and Shu [8], and Cockburn, Hou, and Shu [9] and applied to a wide range of engineering problems [10]. They have been used by Alexandre Ern *et al.* [11] [12], for linear conservation laws using Discontinuous Galerkin Method to prove a convergence result [12]. Authors did a space semidiscretization using the upwind *DG* method. Besides, others tools are fundamental to get this convergence result:

1) Error equation.

2) An energy identity obtained from error equations.

3) A stability estimate using Gronwall lemma, Young inequality and inverse and trace inequalities for finite elements method.

In the literature, there exist many numerical methods to solve transport equation [13] [14]. To our best knowledge, there is no error estimate for transport equation using isogeometric method. In our work, we give such an estimate to generalize results obtained by Alexandre Ern *et al.* [11] [12] in finite elements. In the framework of this dissertation, we want to prove a convergence result using isogeometric method. Among others, unlike finite elements, as far as the space semidiscretization is concerned, we have:

1) Constructed a parametrization of the physical domain, indispensable to describe this domain.

2) Constructed a parametric mesh making a tensor product of knot vectors.

3) Introduced the discrete space on the physical domain, using our parametrization.

Moreover, instead of using inverse and trace inequalities for finite elements method, we will use isogeometric inverse and trace inequalities to obtain our convergence result. As far as the discretization in time is concerned, the explicit two stage Heun scheme is used. Now, we consider the following model:

$$\begin{cases} \partial_t u(x,t) + \nabla \cdot (\beta(x) u(x,t)) = 0, x \in \Omega \subset \mathbb{R}^2, t \in \left[0; t_f\right] \\ u(.,t=0) = u_0 \\ u = 0 \ \text{ in } \partial\Omega^- \times \left[0; t_f\right] \end{cases} \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^2$, $u:\Omega\times\left[0;t_f\right]\mapsto \mathbf{R}^2$ is a scalar-valued function representing the unknown, t_f is a finite time, $\partial\Omega^- = \left\{x\in\partial\Omega, \beta(x)\cdot n(x)<0\right\}$, n is the unit outward normal to the domain boundary, β is the advective velocity, $\beta\in\left[L^\infty(\Omega)\right]^2$, $\nabla\cdot\beta\in L^\infty(\Omega)$ and u_0 is the initial datum.

Let us introduce some notations and assumptions:

- Assume β is a Lipschitz continuous functions *i.e.*

$$\exists L_\beta, \forall (x,y)\in\Omega^2, \left\|\beta(x)-\beta(y)\right\|\le L_\beta\left\|x-y\right\|.$$

where $\left\|x-y\right\|$ denotes the Euclidean norm of $(x-y)$ in $\mathbf{R}^2$.

- We set $\tau_c := \dfrac{1}{\max\left\{\left\|\beta'\right\|_{L^\infty(\Omega)}; L_\beta\right\}}$ and $\beta_c := \left\|\beta\right\|_{\left[L^\infty(\Omega)\right]^2}$.
- We set $\tau_\star := \min\left(t_f, \tau_c\right)$.
- Assume $h\le \beta_c\tau_\star$.
- $\forall x\in\mathbb{R}$, $x^\ominus := \frac{1}{2}\left(|x|-x\right)$ and $x^\oplus := \frac{1}{2}\left(|x|+x\right)$.
- Let $l\in\mathbb{N}$, we consider the space

$$C^l(V) := C^l\left(\left[0,t_f\right],V\right), \tag{2}$$

where V is a Hilbert space and equipped with the scalar product defined by:

$$(u,s)_V = (u,s)_{L^2(\Omega)} + \left(\nabla\cdot(\beta u), \nabla\cdot(\beta s)\right)_{L^2(\Omega)}, \forall (u,s)\in V^2. \tag{3}$$

The associated norm is:

$$\left\|u\right\|_V^2 = \left\|u\right\|_{L^2(\Omega)}^2 + \left\|\nabla\cdot(\beta u)\right\|_{L^2(\Omega)}^2, \forall u\in V. \text{ [11] (page 39)} \tag{4}$$

This paper is organized as follows. In the first section, we will describe univariate B-splines. In the second one, we will describe bivariate B-splines and geometry of the physical domain. In the third one, we present main results of this work. In the fourth one, we will state inverse and isogeometric inequalities. In the fifth one, we will talk about the functional setting and space semidiscretization. In the sixth one, we will look into the explicit two stage Heun scheme analysis.

2. Univariate B-Splines

Definition 1. *Let* $x_1\le x_2\le\cdots\le x_m$ *be an increasing sequence of reals, B-splines functions of degree k are defined by Cox-de Boor-Mansfield recursion formula* [15]:

$$\begin{cases} \text{For } 1\le i\le m-1 \\ N_{i,0}(t)=1 \quad \text{if } t\in\left[x_i, x_{i+1}\right[\\ N_{i,0}(t)=0 \qquad \text{otherwise} \end{cases} \tag{5}$$

$$\begin{cases} \text{For } k\ge 1 \text{ and } 1\le i\le m-k-1 \\ N_{i,k}(t) = \dfrac{t-x_i}{x_{i+k}-x_i}N_{i,k-1}(t) + \dfrac{x_{i+k+1}-t}{x_{i+k+1}-x_{i+1}}N_{i+1,k-1}(t), \end{cases} \tag{6}$$

with the convention $\frac{x}{0} := 0$ for all real number *x*.

The set $(x_i)_{i=1}^{m}$ $(1 \le i \le m)$ is called knots vector.

Now, we want to look into bivariate B-splines, obtained from univariate B-splines.

3. Bivariate B-Splines and Geometry of the Physical Domain

The definition of bivariate B-splines follows easily through a tensor-product construction. Let us focus on the two-dimensional case. Notably, let us consider the unit square $\hat{\Omega} = [0;1]^2 \subset \mathbb{R}^2$. Mimicking the one-dimensional case, given integers p_l and n_l for $l = 1,2$. Let us introduce open knot vectors:

$$E_l = \{\xi_{1,l}, \cdots, \xi_{n_l+p_l+2,l}\}$$

and the associated vectors without repetitions for each direction *l*

$$\zeta_l = \{\zeta_{1,l}, \cdots, \zeta_{m_l,l}\}$$

There is a parametric cartesian mesh Q_h associated with these knot vectors partitioning the parametric domain $\hat{\Omega}$ into a rectangular grid. So, we have:

$$Q_h = \{Q = \otimes_{l=1,2}(\zeta_{i_l,l}, \zeta_{i_{l+1},d}), 1 \le i_l \le m_l - 1\} \text{ [16]} \tag{7}$$

For each element $Q \in Q_h$, we associate a parametric mesh size $h_Q = h_{Q,\max}$ where $h_{Q,\max}$ denotes the length of the largest edge of *Q*. Also, for each element, we define a shape regularity constant as in [16]:

$$\lambda_Q = \frac{h_Q}{h_{Q,\min}} \tag{8}$$

where $h_{Q,\min}$ denotes the length of the smallest edge of *Q*.

We associate with each knot vector $E_l, (l = 1,2)$ univariate B-spline basis functions N_{i,p_l} of degree p_l for $i = 1, \cdots, n_l$.

On the mesh Q_h, we define the tensor-product B-spline basis functions as in [16] by:

$$N_{(i,j),(p_1,p_2)} = N_{i,p_1} \otimes N_{j,p_2}, i = 1, \cdots, n_1, j = 1, \cdots, n_2. \tag{9}$$

$$N_{(i,j),(p_1,p_2)} = N_{i,p_1} N_{j,p_2}, i = 1, \cdots, n_1, j = 1, \cdots, n_2. \tag{10}$$

The span of these functions form the space of two-dimensional splines over $\hat{\Omega}$, denoted by:

$$S_h = span\{N_{(i,j),(p_1,p_2)}\}_{i=1,j=1}^{n_1,n_2}$$

The physical domain Ω is defined through a geometrical mapping:

$$F = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} P_{ij} N_{i,n_1} N_{j,n_2} \text{ [16]}$$

where $P_{ij} \in \mathbb{R}^2$ are the so-called control points. *F* is a parametrization of the physical domain Ω, that is,

$$F:[0,1]^2 \to \Omega$$

For each element Q in the parametric domain $[0,1]^2$, there is a corresponding physical element $K = F(Q)$, as shown in **Figure 1**.

We assume throughout that F is invertible, with smooth inverse F^{-1}, on each element $Q \in Q_h$.

We define the physical mesh to be:

$$\tau_h = \{K : K = F(Q), Q \in Q_h\} = F(Q_h). \tag{11}$$

We assume (τ_h) is quasi-uniform:

$$\exists C > 0, h \le Ch_K. \tag{12}$$

with h_K the diameter of K and $h := \max_{K \in \tau_h} h_K$.

We introduce V_h, the space spanned by B-splines basis functions in Ω as the push-forward of the B-splines space S_h.

$$V_h := span\left\{N_{(i,j),(p_1,p_2)} \circ F^{-1}\right\}_{i=1,j=1}^{n_1,n_2}.$$

Given a function $\hat{v} \in L^2(\hat{\Omega})$, we define a projective operator over the B-splines space S_h as:

$$\pi_{S_h} : L^2(\hat{\Omega}) \to S_h, \pi_{S_h}\hat{v} := \sum_{i=1,j=1}^{n_1,n_2} \varphi(\hat{v}) N_{(i,j),(p_1,p_2)},$$

where the linear functionals $\varphi \in L^2(\hat{\Omega})'$ determine the dual basis for the set of B-splines.

The projective operator over the B-splines space V_h, is defined as the push-forward of the operator π_{S_h}.

$$\pi_h : L^2(\Omega) \to V_h, \pi_h v := \left(\pi_{S_h}(\hat{v})\right) \circ F^{-1}$$

4. Main Results

This section is devoted to our convergence results obtained for respectively a polynomial of degree $k \ge 2$ and a polynomial of degree $k = 1$. We present our main results whose proofs are given in the subsection 6.6.

Theorem 1. (*Convergence for RK2*, $k \ge 2$)

Assume the $\frac{4}{3}$ *CFL Condition*:

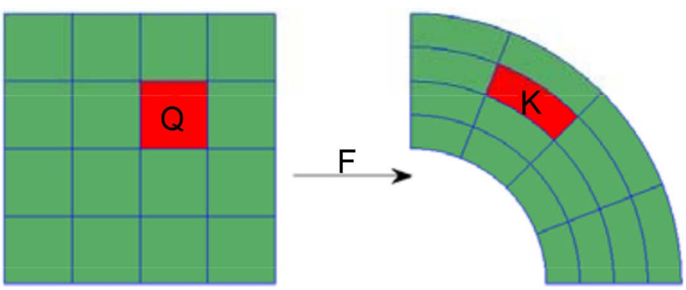

Figure 1. Definition of domains used in isogeometric analysis (Source [17]) (Page 181).

$$\delta t \leq \varrho' \tau_\star^{-\frac{1}{3}} \left(\frac{h}{\beta_c}\right)^{\frac{4}{3}} \quad \text{for some positive real number } \varrho' \tag{13}$$

and $d_t^s u \in C^0\left(H^{k+1-s}(\Omega)\right)$ for $s \in \{0,1\}$. Then,

$$\left\|u^N - u_h^N\right\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1} \delta t \left|u^m - u_h^m\right|_\beta^2\right)^{\frac{1}{2}} \lesssim \exp\left(\frac{4Cr^8}{\tau_\star} t_f\right)\left(\chi_1 \delta t^2 + \chi_2 h^{k+\frac{1}{2}}\right) \tag{14}$$

with

$$\chi_1 = r^4 t_f^{\frac{1}{2}} \tau_\star^{\frac{1}{2}} \left\|d_t^3 u\right\|_{C^0\left(L^2(\Omega)\right)} \tag{15}$$

$$\chi_2 = t_f^{\frac{1}{2}} r^5 \beta_c^{\frac{1}{2}} \|u\|_{C^0\left(H^{k+1}(\Omega)\right)} + t_f^{\frac{1}{2}} r^5 \beta_c^{-\frac{1}{2}} \left\|d_t u\right\|_{C^0\left(H^k(\Omega)\right)} \tag{16}$$

$$|v|_\beta^2 = \int_{\partial\Omega} \frac{1}{2}|\beta \cdot n| v^2 + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{1}{2}|\beta \cdot n_F| [v]^2, \tag{17}$$

where

$$r = \max\left(1, \sqrt{\max_{K \in \tau_h}\left(\lambda_Q \lambda_K\right)}\right) \tag{18}$$

and δt is the time step.

Theorem 2. (*Convergence for RK*2, $k = 1$)

Assume the $\frac{4}{3}$ *CFL Condition*, assume $d_t^s u \in C^0\left(H^{k+1-s}(\Omega)\right)$ for $s \in \{0,1\}$ and $4 - 3r^2 > 0$. Then,

$$\left\|u^N - u_h^N\right\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1} \delta t \left|u^m - u_h^m\right|_\beta^2\right)^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{4-3r^2}} \exp\left(\frac{8Cr^8}{\tau_\star} t_f\right)\left(\chi_1 \delta t^2 + \chi_2 h^{k+\frac{1}{2}}\right) \tag{19}$$

with

$$\chi_1 = r^4 t_f^{\frac{1}{2}} \tau_\star^{\frac{1}{2}} \left\|d_t^3 u\right\|_{C^0\left(L^2(\Omega)\right)} \tag{20}$$

and

$$\chi_2 = t_f^{\frac{1}{2}} r^5 \beta_c^{\frac{1}{2}} \|u\|_{C^0\left(H^{k+1}(\Omega)\right)} + t_f^{\frac{1}{2}} r^5 \beta_c^{-\frac{1}{2}} \left\|d_t u\right\|_{C^0\left(H^k(\Omega)\right)} \tag{21}$$

5. Inverse and Trace Inequalities

In this section, we present isogeometric inverse and trace inequalities, useful tools to analyze partial differential equations.

Let $K \in \tau_h$ and $Q = F^{-1}(K)$.

Theorem 3. (*see* [11] [18])

$$\forall h > 0, \forall K \in \tau_h \text{ and } \forall v_h \in \mathbb{P}_2^k(\tau_h), \left\|\nabla v_h\right\|_{\left[L^2(K)\right]^2} \leq C_\star h_K^{-1} \left\|v_h\right\|_{L^2(K)} \tag{22}$$

where $C_\star$ depends on k and on the parametrization F.

Theorem 4. (*see* [16])

$$\forall v_h \in \mathbb{P}_2^k(\tau_h), \|v_h\|^2_{L^2(\partial K)} \le C\lambda_Q\lambda_K h_K^{-1}\|v_h\|^2_{L^2(K)} \tag{23}$$

where C depends only on p_1 and p_2, λ_Q is the local shape regularity constant of Q, and λ_K is the shape regularity constant of K.

We set $p := \min\{p_1, p_2\}$ (see. [4] [19]).

Theorem 5. (*see* [20]) *Given the integers l and s such that* $0 \le l \le s \le p+1$ *and a function* $u \in H^s(\Omega)$, *then*:

$$\sum_{K\in\tau_h} |u - \pi_h u|^2_{H^l(K)} \le Ch^{2(s-l)}\|u\|^2_{H^s(\Omega)}, \tag{24}$$

where C is independent on h.

Theorem 6. *Given the integer s such that* $0 \le s \le p+1$ *and a function* $u \in H^s(\Omega)$, *then*:

$$\sum_{K\in\tau_h} \|u - \pi_h u\|^2_{L^2(\partial K)} \le C\max_{K\in\tau_h}(\lambda_Q\lambda_K)h^{(2s-1)}\|u\|^2_{H^s(\Omega)}, \tag{25}$$

where C is independent on h.

Proof 1. *Let* $u \in H^s(\Omega)$. *Using the inequality* (23), *we have*:

$$\sum_{K\in\tau_h} \|u - \pi_h u\|^2_{L^2(\partial K)} \le C\sum_{K\in\tau_h} \lambda_Q\lambda_K h_K^{-1}\|u - \pi_h u\|^2_{L^2(K)}, \tag{26}$$

$(\tau_h)_h$ being quasi-uniform, $h \le Ch_K$.

$$\begin{aligned} h \le Ch_K &\Rightarrow C^{-1}h_K^{-1} \le h^{-1} \\ &\Rightarrow h_K^{-1} \le Ch^{-1} \\ &\Rightarrow Ch_K^{-1} \le C'h^{-1} \end{aligned} \tag{27}$$

So

$$\sum_{K\in\tau_h} \|u - \pi_h u\|^2_{L^2(\partial K)} \le C\max_{K\in\tau_h}(\lambda_Q\lambda_K)h^{-1}\sum_{K\in\tau_h}\|u - \pi_h u\|^2_{L^2(K)} \tag{28}$$

Using the inequality (24), we have:

$$\sum_{K\in\tau_h} \|u - \pi_h u\|^2_{L^2(K)} \le Ch^{2s}\|u\|^2_{H^s(\Omega)} \tag{29}$$

Thus, we get:

$$\sum_{K\in\tau_h} \|u - \pi_h u\|^2_{L^2(\partial K)} \le C\max_{K\in\tau_h}(\lambda_Q\lambda_K)h^{(2s-1)}\|u\|^2_{H^s(\Omega)} \tag{30}$$

6. Functional Setting and Space Semidiscretization

6.1. Functional Setting

In this part, we introduce some basic notations for space-time functions and important theorems.

Theorem 7. (*see.* [11]) $C^l(V)$ *is a Banach space when equipped with the norm*:

$$\|\phi\|_{C^l(V)} = \max_{0\le m\le l}\|d_t^m\phi\|_{C^0(V)} \tag{31}$$

with

$$\|\phi\|_{C^0(V)} = \max_{t_0 \le t \le t_f} \|\phi(t)\|_V \quad [11] \text{ (page69)} \tag{32}$$

We want to specify mathematically the meaning of the boundary condition 1. Our aim is to give a meaning to such traces in the space. Thus, we need to investigate the trace on $\partial\Omega$ of functions in the space defined by:

$$L^2(|\beta\cdot n|,\partial\Omega) = \left\{ v \text{ is defined on } \partial\Omega, \int_{\partial\Omega} |\beta\cdot n| v^2 < \infty \right\} \quad [11] \tag{33}$$

6.2. Space Semidiscretization

Considering (τ_h), we present following notations:

- Interfaces are collected in the set $\mathcal{F}_h^i$ and boundary faces are collected in the set $\mathcal{F}_h^b$. We set $\mathcal{F}_h := \mathcal{F}_h^b \cup \mathcal{F}_h^i$. $\forall T \in \tau_h, \mathcal{F}_T := \{F \in \mathcal{F}_h, F \subset \partial T\}$.
- $\forall F \in \mathcal{F}_h^i$, the mean of v is denoted by $\{\{v\}\}$.
- The jump of v is denoted by $[v]$.
- Assume $P_\Omega = \{\Omega_i\}_{1\le i\le N_\Omega}$ is a partition of Ω such that, for the exact solution u,

$$u \in V_\star = V \cap H^{\frac{1}{2}+\varepsilon}(P_\Omega), \varepsilon > 0 \quad [11]$$

where

$$V = \left\{ u \in L^2(\Omega), \nabla\cdot(\beta u) \in L^2(\Omega) \right\} \tag{34}$$

We set

$$V_{\star h} := V_\star + V_h \quad \text{with} \quad V_h = \mathbb{P}_2^k(\tau_h) = \left\{ v \in L^2(\Omega); \forall T \in \tau_h, v_{/T} \in \mathbb{P}_2^k(T) \right\} \tag{35}$$

We define the discrete operator $A_h : V_{\star h} \to V_h$ such as $\forall (v, w_h) \in V_{\star h} \times V_h$,

$$\begin{aligned}(A_h v, w_h)_{L^2(\Omega)} &= \int_\Omega \nabla\cdot(\beta v) w_h + \int_{\partial\Omega} (\beta\cdot n)^{\ominus} v w_h \\ &- \sum_{F\in\mathcal{F}_h^i} \int_F (\beta n_F)[v]\{\{w_h\}\} + \sum_{F\in\mathcal{F}_h^i} \frac{1}{2}\int_F |\beta\cdot n_F| [v][w_h]\end{aligned} \quad [11] \tag{36}$$

6.3. Assumptions

For all $v \in V_{\star h}$, set:

$$\|v\|_{\star\star}^2 = \|v\|_{uwb,\star}^2 + \beta_c h \|\nabla v\|_{L^2(\Omega)}^2 \tag{37}$$

$$\|v\|_{uwb,\star}^2 = \|v\|_{uwb}^2 + \sum_{T\in\tau_h} \beta_c \|v\|_{L^2(\partial T)}^2 \tag{38}$$

$$\|v\|_{uwb}^2 = \frac{1}{\tau_c} \|v\|_{L^2(\Omega)}^2 + |v|_\beta^2 \tag{39}$$

$$E_h^n = \|\varepsilon_\pi^n\|_{\star\star} + \|\zeta_\pi^n\|_{\star\star} + \tau_\star^{\frac{1}{2}} \|d_t^3 u\|_{C^0(L^2(\Omega))} \delta t^2 + \tau_\star^{-\frac{1}{2}} \|\varepsilon_h^n\|_{L^2(\Omega)} \tag{40}$$

We abbreviate as $a \lesssim b$ the inequality $a \le Cb$ with positive C independent of $\beta, h, \delta t$. The value of C can change at each occurrence [11].

We now state some assumptions on the discrete operator A_h. The first one (41) is important to introduce the notion of numerical fluxes:

1) For all $(v, w_h) \in V_{\star h} \times V_h$,

$$\left(A_h v, w_h\right) = -\int_\Omega \left(\beta \cdot \nabla w_h\right) v + \int_{\partial\Omega} \left(\beta \cdot n\right)^{\oplus} v w_h + \sum_{F \in \mathcal{F}_h^i} \int_F \left(\beta \cdot n_F\right) \{\{v\}\} [w_h] + \sum_{F \in \mathcal{F}_h^i} \frac{1}{2} \int_F |\beta \cdot n_F| [v][w_h] \quad [11] \tag{41}$$

2) From equality (41), Cauchy-Schwarz inequality and inverse inequality (22), we can infer:

For all $(v, w_h) \in H^1(\Omega) \times V_h$,

$$\left(A_h \left(v - \pi_h v\right), w_h\right)_{L^2(\Omega)} \lesssim \|v - \pi_h v\|_{uwb,\star} \|w_h\|_{uwb} \tag{42}$$

3) The three next assumptions are useful to bound the operator A_h.

For all

$$v \in V_{\star h}, \|A_h v\|_{L^2(\Omega)} \lesssim r \beta_c^{\frac{1}{2}} h^{-\frac{1}{2}} \|v\|_{\star\star} \quad \text{with} \quad r = \max\left(1, \sqrt{\max_{K \in \tau_h}\left(\lambda_Q \lambda_K\right)}\right) \tag{43}$$

For all

$$v_h \in V_h, \|v_h\|_{\star\star} \lesssim r \beta_c^{\frac{1}{2}} h^{-\frac{1}{2}} \|v_h\|_{L^2(\Omega)} \tag{44}$$

For all

$$v_h \in V_h, \|A_h v\|_{L^2(\Omega)} \lesssim r^2 \beta_c h^{-1} \|v_h\|_{L^2(\Omega)} \tag{45}$$

4) The two next inequalities are bounds of $\delta t\left(\alpha_h^n, \varepsilon_h^n\right)_{L^2(\Omega)} - \frac{\delta t}{2}\left|\varepsilon_h^n\right|_\beta^2$ and $\delta t\left(\beta_h^n, \zeta_h^n\right)_{L^2(\Omega)} - \frac{\delta t}{2}\left|\zeta_h^n\right|_\beta^2$.

$$\delta t\left(\alpha_h^n, \varepsilon_h^n\right)_{L^2(\Omega)} - \frac{\delta t}{2}\left|\varepsilon_h^n\right|_\beta^2 \lesssim \delta t\left(E_h^n\right)^2 \tag{46}$$

$$\delta t\left(\beta_h^n, \zeta_h^n\right)_{L^2(\Omega)} - \frac{\delta t}{2}\left|\zeta_h^n\right|_\beta^2 \lesssim r^4 \delta t\left(E_h^n\right)^2 \tag{47}$$

5) The two last inequalities are obtained thanks to CFL condition and isogeometric inverse and trace inequalities:

$$\left\|\varepsilon_\pi^m\right\|_{\star\star}^2 \lesssim r^2 \beta_c h^{2k+1} \left\|u^m\right\|_{H^{k+1}(\Omega)}^2 \tag{48}$$

$$\left\|\zeta_\pi^m\right\|_{\star\star}^2 \lesssim r^2 \beta_c h^{2k+1} \left(\left\|u^m\right\|_{H^{k+1}(\Omega)}^2 + \beta_c^{-2} \left\|d_t u^m\right\|_{H^k(\Omega)}^2\right) \tag{49}$$

For the time discretization, we are interested in an explicit scheme: the two stage Heun scheme.

7. The Explicit Two Stage Heun Scheme Analysis

In this section, we want to tackle the convergence analysis of the two stage Heun scheme.

7.1. The Explicit Two Stage Heun Scheme

Let δt be the time step such as $t_f = N\delta t$ where N is an integer. For

$n \in \{0, \cdots, N\}$, we define the discrete times $t^n := n\delta t$ and $u^n = u(t^n)$. Assume $\delta t \le \tau_\star$ with $\tau_\star = \min(t_f, \tau_c)$.

We consider the following explicit scheme:

$$\begin{cases} u_h^{n,1} = u_h^n - \delta t A_h u_h^n \\ u_h^{n+1} = \frac{1}{2}\left(u_h^n + u_h^{n,1}\right) - \frac{1}{2}\delta t A_h u_h^{n,1} \quad [11] \\ \text{with } u_h^0 = \pi_h u^0 \end{cases}$$

7.2. Error Equation

This step is to identify the error equation governing the time evolution of ε_h^n and ζ_h^n.

We set

$$\varepsilon_h^n = u_h^n - \pi_h u^n \tag{50}$$

$$\varepsilon_\pi^n = u^n - \pi_h u^n \tag{51}$$

$$\zeta_h^n = w_h^n - \pi_h w^n \tag{52}$$

$$\zeta_\pi^n = w^n - \pi_h w^n \tag{53}$$

with

$$w = u + \delta t d_t u \quad [11] \tag{54}$$

From (50) and (51), we have $u^n - u_h^n = \varepsilon_\pi^n - \varepsilon_h^n$.

From (52) and (53), we have $w^n - w_h^n = \zeta_\pi^n - \zeta_h^n$.

We get:

$$\zeta_h^n = \varepsilon_h^n - \delta t A_h \varepsilon_h^n + \delta t \alpha_h^n \quad [11] \tag{55}$$

$$\varepsilon_h^{n+1} = \frac{1}{2}\left(\varepsilon_h^n + \zeta_h^n\right) - \frac{1}{2}\delta t A_h \zeta_h^n + \frac{1}{2}\delta t \beta_h^n \quad [11] \tag{56}$$

where

$$\alpha_h^n = A_h \varepsilon_\pi^n \tag{57}$$

where

$$\beta_h^n = A_h \zeta_\pi^n - \pi_h \theta^n \tag{58}$$

and

$$\theta^n = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} \left(t^{n+1} - t\right)^2 u_t^3 \mathrm{d}t \tag{59}$$

7.3. Energy Identity

This step is to derive an energy identity for our scheme (6.1).

$$\begin{aligned} &\left\|\varepsilon_h^{n+1}\right\|_{L^2(\Omega)}^2 - \left\|\varepsilon_h^n\right\|_{L^2(\Omega)}^2 + \delta t \left|\varepsilon_h^n\right|_\beta^2 + \delta t \left|\zeta_h^n\right|_\beta^2 \\ &= \left\|\varepsilon_h^{n+1} - \zeta_h^n\right\|_{L^2(\Omega)}^2 + \delta t \left(\alpha_h^n, \varepsilon_h^n\right)_{L^2(\Omega)} + \delta t \left(\beta_h^n, \zeta_h^n\right)_{L^2(\Omega)} - \Lambda_h^n \end{aligned} \quad [11] \tag{60}$$

with

$$\Lambda_h^n = \delta t\left(\Lambda \varepsilon_h^n, \varepsilon_h^n\right)_{L^2(\Omega)} + \delta t\left(\Lambda \zeta_h^n, \zeta_h^n\right)_{L^2(\Omega)} \tag{61}$$

7.4. Stability Estimate

Our aim is to bound the right terms in the energy identity (60).

We want now to establish a stability lemma for a polynomial of degree $k=1$ (68). To get it, we need the next lemma.

Lemma 1. *Let* π_h^0 *denote the* L^2*-orthogonal projection onto* $\mathbb{P}_2^0\left(\tau_h\right)$. $\mathbb{P}_2^0\left(\tau_h\right)$ is spanned by piecewise constant functions on τ_h.

Then, $\forall\left(v_h, w_h\right) \in V_h \times V_h$,

$$\left(A_h v_h, w_h - \pi_h^0 w_h\right) \le C_{\star\star} r \beta_c^{\frac{1}{2}} h^{-\frac{1}{2}} \left\|v_h\right\|_{uwb} \left\|w_h - \pi_h^0 w_h\right\|_{L^2(\Omega)} \tag{62}$$

where $C_{\star\star}$ is independent of $h, \delta t$ and of β.

Proof 2. *This result is obtained using Cauchy-Schwarz inequality and equality* (41).

7.5. Preliminary Results

This lemma is a preliminary stability bound.

Lemma 2. *Assume* $u \in C^3\left(L^2(\Omega)\right) \cap C^0\left(H^1(\Omega)\right)$.

Assume the CFL condition:

$$\delta t \le \varrho \frac{h}{\beta_c} \text{ for some positive real number } \varrho. \tag{63}$$

Thus,

$$\left\|\varepsilon_h^{n+1}\right\|_{L^2(\Omega)}^2 - \left\|\varepsilon_h^n\right\|_{L^2(\Omega)}^2 + \frac{\delta t}{2}\left|\varepsilon_h^n\right|_\beta^2 + \frac{\delta t}{2}\left|\zeta_h^n\right|_\beta^2 \le \left\|\varepsilon_h^{n+1} - \zeta_h^n\right\|_{L^2(\Omega)}^2 + C r^4 \delta t \left(E_h^n\right)^2 \tag{64}$$

where C is independent of $h, \delta t$ and β.

Proof 3. *Using CFL condition, energy identity* (60), *inequalities* (46) *and* (47), *we get* (64).

Lemma 3. (*Stability lemma*, $k \ge 2$) *Assume* $u \in C^3\left(L^2(\Omega)\right) \cap C^0\left(H^1(\Omega)\right)$.

Assume the ($\frac{4}{3}$ *CFL Condition*)

$$\delta t \le \varrho' \tau_\star^{-\frac{1}{3}} \left(\frac{h}{\beta_c}\right)^{\frac{4}{3}} \text{ for some positive real number } \varrho' \tag{65}$$

Then, we infer:

$$\left\|\varepsilon_h^{n+1}\right\|_{L^2(\Omega)}^2 - \left\|\varepsilon_h^n\right\|_{L^2(\Omega)}^2 + \frac{\delta t}{2}\left|\varepsilon_h^n\right|_\beta^2 + \frac{\delta t}{2}\left|\zeta_h^n\right|_\beta^2 \le C r^8 \delta t \left(E_h^n\right)^2 \tag{66}$$

Proof 4. *The stability lemma* (66) *for a polynomial of degree* $k \ge 2$ *is obtained by bounding the term* $\left\|\varepsilon_h^{n+1} - \zeta_h^n\right\|_{L^2(\Omega)}^2$ *in the energy identity* (60).

Lemma 4. (*Stability lemma*, $k=1$) *Assume* $u \in C^3\left(L^2(\Omega)\right) \cap C^0\left(H^1(\Omega)\right)$.

Assume the CFL condition:

$$\delta t \leq \varrho \frac{h}{\beta_c} \text{ for some positive real number } \varrho$$
$$\text{with } \varrho \leq \min\left\{\frac{1}{8}(C')^{-2}; \frac{1}{2}(C_\star C_{\star\star})^{-\frac{2}{3}}\right\} \tag{67}$$

Thus,

$$\left\|\varepsilon_h^{n+1}\right\|_{L^2(\Omega)}^2 - \left\|\varepsilon_h^n\right\|_{L^2(\Omega)}^2 + \delta t\left(\frac{1}{2} - \frac{3}{8}r^2\right)\left|\varepsilon_h^n\right|_\beta^2 + \delta t\left(\frac{1}{2} - \frac{3}{8}r^2\right)\left|\zeta_h^n\right|_\beta^2 \leq Cr^4 \delta t\left(E_h^n\right)^2 \tag{68}$$

where C is independent of $h, \delta t$ and β.

Proof 5. *This lemma is proven as in* [11] (*Page* 96).

7.6. Proofs of Our Main Results

Proof of theorem 1

$$\left\|u^N - u_h^N\right\|_{L^2(\Omega)} = \left\|u^N - \pi_h u^n + \pi_h u^n - u_h^N\right\|_{L^2(\Omega)}$$
$$\leq \left\|u^N - \pi_h u^n\right\|_{L^2(\Omega)} + \left\|\pi_h u^n - u_h^N\right\|_{L^2(\Omega)}$$
$$\left\|u^N - u_h^N\right\|_{L^2(\Omega)} \leq \left\|\varepsilon_\pi^N\right\|_{L^2(\Omega)} + \left\|\varepsilon_h^N\right\|_{L^2(\Omega)}$$

Using the triangle and Young inequalities, we deduce:

$$\left(\sum_{m=0}^{N-1} \delta t \left|u^m - u_h^m\right|_\beta^2\right)^{\frac{1}{2}} \lesssim \sum_{m=0}^{N-1} \delta t^{\frac{1}{2}} \left\|\varepsilon_\pi^m\right\|_{\star\star} + \sum_{m=0}^{N-1} \delta t^{\frac{1}{2}} \left|\varepsilon_h^m\right|_\beta$$

Thus, we obtain:

$$\left\|u^N - u_h^N\right\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1} \delta t \left|u^m - u_h^m\right|_\beta^2\right)^{\frac{1}{2}} \lesssim \left\|\varepsilon_\pi^N\right\|_{L^2(\Omega)} + \left\|\varepsilon_h^N\right\|_{L^2(\Omega)} + \sum_{m=0}^{N-1} \delta t^{\frac{1}{2}} \left\|\varepsilon_\pi^m\right\|_{\star\star} + \sum_{m=0}^{N-1} \delta t^{\frac{1}{2}} \left|\varepsilon_h^m\right|_\beta \tag{69}$$

Let $n \in \{0, \cdots, N\}$

$$\text{Set } b^n = \frac{\delta t}{2}\left|\varepsilon_h^n\right|_\beta^2 + \frac{\delta t}{2}\left|\zeta_h^n\right|_\beta^2$$

$$\text{Given } (a+b+c+d)^2 \leq 4a^2 + 4b^2 + 4c^2 + 4d^2$$

From the relation (66), we deduce that:

$$\left\|\varepsilon_h^{n+1}\right\|_{L^2(\Omega)}^2 + b^n \leq \left\|\varepsilon_h^n\right\|_{L^2(\Omega)}^2 + Cr^8 \delta t\Big(4\left\|\varepsilon_\pi^n\right\|_{\star\star}^2 + 4\left\|\zeta_\pi^n\right\|_{\star\star}^2 + 4\tau_\star \left\|d_t^3 u\right\|_{C^0(L^2(\Omega))}^2 \delta t^4 + 4\delta t \tau_\star^{-1} \left\|\varepsilon_h^n\right\|_{L^2(\Omega)}^2\Big)$$

$$\text{Set } d^n = 4Cr^8 \delta t\left(\left\|\varepsilon_\pi^n\right\|_{\star\star}^2 + \left\|\zeta_\pi^n\right\|_{\star\star}^2 + \tau_\star \left\|d_t^3 u\right\|_{C^0(L^2(\Omega))}^2 \delta t^4\right) \tag{70}$$

whence, we have:

$$\left\|\varepsilon_h^{n+1}\right\|_{L^2(\Omega)}^2 \leq \left(1 + 4Cr^8 \delta t \tau_\star^{-1}\right)\left\|\varepsilon_h^n\right\|_{L^2(\Omega)}^2 + d^n - b^n \tag{71}$$

Applying the Gronwall lemma, we get for $n = N-1$:

$$\left\|\varepsilon_h^N\right\|_{L^2(\Omega)}^2 \le \exp\left(\frac{4Cr^8}{\tau_\star}\left(t_f - t_0\right)\right)\left\|\varepsilon_h^0\right\|_{L^2(\Omega)}^2 + \sum_{i=0}^{N-1}\exp\left(\frac{4Cr^8}{\tau_\star}\left(t_f - t_{i+1}\right)\right)\left(d^i - b^i\right) \quad (72)$$

So

$$\left\|\varepsilon_h^N\right\|_{L^2(\Omega)}^2 \le \sum_{i=0}^{N-1}\exp\left(\frac{4Cr^8}{\tau_\star}\left(t_f - t_{i+1}\right)\right)\left(d^i - b^i\right) \quad \text{for} \quad u_h^0 = \pi_h u^0 \Rightarrow \varepsilon_h^0 = 0 \quad (73)$$

So

$$\left\|\varepsilon_h^N\right\|_{L^2(\Omega)} + \left(\sum_{i=0}^{N-1}\delta t^{\frac{1}{2}}\left|\varepsilon_h^i\right|_\beta\right) \lesssim \left(\sum_{i=0}^{N-1}\exp\left(\frac{4Cr^8}{\tau_\star}t_f\right)d^i\right)^{\frac{1}{2}} \quad (74)$$

where

$$d^i = 4Cr^8\delta t\left(\left\|\varepsilon_\pi^i\right\|_{\star\star}^2 + \left\|\zeta_\pi^i\right\|_{\star\star}^2 + \tau_\star\left\|d_t^3 u\right\|_{C^0\left(L^2(\Omega)\right)}^2\delta t^4\right)$$

Therefore, we have:

$$\left(\sum_{i=0}^{N-1}\delta t^{\frac{1}{2}}\left\|\varepsilon_\pi^i\right\|_{\star\star}\right) \lesssim \left(\sum_{i=0}^{N-1}\exp\left(\frac{4Cr^8}{\tau_\star}t_f\right)d^i\right)^{\frac{1}{2}} \quad (75)$$

From inequalities (69), (74) and (75), we get:

$$\left\|u^N - u_h^N\right\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1}\delta t\left|u^m - u_h^m\right|_\beta^2\right)^{\frac{1}{2}} \lesssim \left\|\varepsilon_\pi^N\right\|_{L^2(\Omega)} + \exp\left(\frac{4Cr^8}{\tau_\star}t_f\right)\left(\sum_{i=0}^{N-1}d^i\right)^{\frac{1}{2}} \quad (76)$$

From inequalities (48), (49) and CFL condition, we obtain:

$$\left(\sum_{i=0}^{N-1}d^i\right)^{\frac{1}{2}} \lesssim t_f^{\frac{1}{2}}r^5\beta_c^{\frac{1}{2}}h^{k+\frac{1}{2}}\left\|u^m\right\|_{H^{k+1}(\Omega)} + t_f^{\frac{1}{2}}r^5\beta_c^{-\frac{1}{2}}h^{k+\frac{1}{2}}\left\|d_t u^m\right\|_{H^k(\Omega)} + r^4 t_f^{\frac{1}{2}}\tau_\star^{\frac{1}{2}}\left\|d_t^3 u\right\|_{C^0\left(L^2(\Omega)\right)}\delta t^2$$

Using inverse inequality (24),

$$\left\|\varepsilon_\pi^N\right\|_{L^2(\Omega)}^2 \lesssim h^{2k+2}\left\|u^m\right\|_{H^{k+1}(\Omega)}^2$$

So

$$\left\|\varepsilon_\pi^N\right\|_{L^2(\Omega)} \lesssim h^{k+1}\left\|u\right\|_{C^0\left(H^{k+1}(\Omega)\right)}$$

Therefore, inequality (76) becomes:

$$\begin{aligned}&\left\|u^N - u_h^N\right\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1}\delta t\left|u^m - u_h^m\right|_\beta^2\right)^{\frac{1}{2}} \\ &\lesssim h^{k+1}\left\|u\right\|_{C^0\left(H^{k+1}(\Omega)\right)} + \exp\left(\frac{4Cr^8}{\tau_\star}t_f\right)\left(t_f^{\frac{1}{2}}r^5\beta_c^{\frac{1}{2}}h^{k+\frac{1}{2}}\left\|u\right\|_{C^0\left(H^{k+1}(\Omega)\right)}\right. \\ &\left.+ t_f^{\frac{1}{2}}r^5\beta_c^{-\frac{1}{2}}h^{k+\frac{1}{2}}\left\|d_t u\right\|_{C^0\left(H^k(\Omega)\right)}\right) + \exp\left(\frac{4Cr^8}{\tau_\star}t_f\right)r^4 t_f^{\frac{1}{2}}\tau_\star^{\frac{1}{2}}\left\|d_t^3 u\right\|_{C^0\left(L^2(\Omega)\right)}\delta t^2\end{aligned} \quad (77)$$

Set

$$\chi_1 = r^4 t_f^{\frac{1}{2}} \tau_\star^{\frac{1}{2}} \left\| d_t^3 u \right\|_{C^0\left(L^2(\Omega)\right)}$$

and

$$\chi_2 = t_f^{\frac{1}{2}} r^5 \beta_c^{\frac{1}{2}} \left\| u \right\|_{C^0\left(H^{k+1}(\Omega)\right)} + t_f^{\frac{1}{2}} r^5 \beta_c^{-\frac{1}{2}} \left\| d_t u \right\|_{C^0\left(H^k(\Omega)\right)} \tag{78}$$

$$h^{k+1} \left\| u \right\|_{C^0\left(H^{k+1}(\Omega)\right)} = \sqrt{h} h^{k+\frac{1}{2}} \left\| u \right\|_{C^0\left(H^{k+1}(\Omega)\right)}$$

$$\lesssim t_f^{\frac{1}{2}} \beta_c^{\frac{1}{2}} h^{k+\frac{1}{2}} \left\| u \right\|_{C^0\left(H^{k+1}(\Omega)\right)} \quad \text{for } h \lesssim t_f \beta_c$$

$$h^{k+1} \left\| u \right\|_{C^0\left(H^{k+1}(\Omega)\right)} \lesssim \chi_2 h^{k+\frac{1}{2}}$$

We obtain thus:

$$\left\| u^N - u_h^N \right\|_{L^2(\Omega)} + \left(\sum_{m=0}^{N-1} \delta t \left| u^m - u_h^m \right|_\beta^2 \right)^{\frac{1}{2}} \lesssim \exp\left(\frac{4Cr^8}{\tau_\star} t_f \right) \left(\chi_1 \delta t^2 + \chi_2 h^{k+\frac{1}{2}} \right)$$

Proof of theorem 2

Using inequality (68), we get inequality:

$$\left\| \varepsilon_h^N \right\|_{L^2(\Omega)}^2 + \sum_{i=0}^{N-1} b^i \le \sum_{i=0}^{N-1} \exp\left(\frac{8Cr^8}{\tau_\star} t_f \right) d^i \tag{79}$$

with

$$b^i = \delta t \left(\frac{1}{2} - \frac{3}{8} r^2 \right) \left| \varepsilon_h^i \right|_\beta^2 + \delta t \left(\frac{1}{2} - \frac{3}{8} r^2 \right) \left| \zeta_h^i \right|_\beta^2$$

$$b^i = \delta t \left(\frac{4 - 3r^2}{2} \right) \left| \varepsilon_h^i \right|_\beta^2 + \delta t \left(\frac{4 - 3r^2}{8} \right) \left| \zeta_h^i \right|_\beta^2$$

$$\left(\left\| \varepsilon_h^N \right\|_{L^2(\Omega)} + \sum_{i=0}^{N-1} \frac{\sqrt{4-3r^2}}{2\sqrt{2}} \delta t^{\frac{1}{2}} \left| \varepsilon_h^i \right|_\beta + \sum_{i=0}^{N-1} \frac{\sqrt{4-3r^2}}{2\sqrt{2}} \delta t^{\frac{1}{2}} \left| \zeta_h^i \right|_\beta \right)^2$$

$$\lesssim \left\| \varepsilon_h^N \right\|_{L^2(\Omega)}^2 + \sum_{i=0}^{N-1} \delta t \frac{4-3r^2}{8} \left| \varepsilon_h^i \right|_\beta^2 + \sum_{i=0}^{N-1} \delta t \frac{4-3r^2}{8} \left| \zeta_h^i \right|_\beta^2$$

Thus

$$\left\| \varepsilon_h^N \right\|_{L^2(\Omega)} + \sum_{i=0}^{N-1} \frac{\sqrt{4-3r^2}}{2\sqrt{2}} \delta t^{\frac{1}{2}} \left| \varepsilon_h^i \right|_\beta + \sum_{i=0}^{N-1} \frac{\sqrt{4-3r^2}}{2\sqrt{2}} \delta t^{\frac{1}{2}} \left| \zeta_h^i \right|_\beta$$

$$\lesssim \left(\sum_{i=0}^{N-1} \exp\left(\frac{8Cr^8}{\tau_\star} t_f \right) d^i \right)^{\frac{1}{2}}$$

So

$$\left\| \varepsilon_h^N \right\|_{L^2(\Omega)} + \sum_{i=0}^{N-1} \sqrt{4-3r^2} \delta t^{\frac{1}{2}} \left| \varepsilon_h^i \right|_\beta \lesssim \left(\sum_{i=0}^{N-1} \exp\left(\frac{8Cr^8}{\tau_\star} t_f \right) d^i \right)^{\frac{1}{2}} \tag{80}$$

whence

$$\left\| \varepsilon_h^N \right\|_{L^2(\Omega)} + \sum_{i=0}^{N-1} \delta t^{\frac{1}{2}} \left| \varepsilon_h^i \right|_\beta \lesssim \frac{1}{\sqrt{4-3r^2}} \left(\sum_{i=0}^{N-1} \exp\left(\frac{8Cr^8}{\tau_\star} t_f \right) d^i \right)^{\frac{1}{2}} \text{ for } \sqrt{4-3r^2} \le 1 \tag{81}$$

Therefore,

$$\begin{aligned}&\left\|u^N-u_h^N\right\|_{L^2(\Omega)}+\left(\sum_{m=0}^{N-1}\delta t\left|u^m-u_h^m\right|_\beta^2\right)^{\frac{1}{2}}\\&\lesssim h^{k+1}\|u\|_{C^0\left(H^{k+1}(\Omega)\right)}+\frac{1}{\sqrt{4-3r^2}}\exp\left(\frac{8Cr^8}{\tau_\star}t_f\right)\\&\times\left(t_f^{\frac{1}{2}}r^5\beta_c^{\frac{1}{2}}h^{k+\frac{1}{2}}\|u\|_{C^0\left(H^{k+1}(\Omega)\right)}+t_f^{\frac{1}{2}}r^5\beta_c^{-\frac{1}{2}}h^{k+\frac{1}{2}}\|d_tu\|_{C^0\left(H^k(\Omega)\right)}\right)\\&+\frac{1}{\sqrt{4-3r^2}}\exp\left(\frac{8Cr^8}{\tau_\star}t_f\right)r^4t_f^{\frac{1}{2}}\tau_\star^{\frac{1}{2}}\left\|d_t^3u\right\|_{C^0\left(L^2(\Omega)\right)}\delta t^2\end{aligned}\tag{82}$$

$$\begin{aligned}&\left\|u^N-u_h^N\right\|_{L^2(\Omega)}+\left(\sum_{m=0}^{N-1}\delta t\left|u^m-u_h^m\right|_\beta^2\right)^{\frac{1}{2}}\\&\lesssim\frac{1}{\sqrt{4-3r^2}}h^{k+1}\|u\|_{C^0\left(H^{k+1}(\Omega)\right)}+\frac{1}{\sqrt{4-3r^2}}\exp\left(\frac{8Cr^8}{\tau_\star}t_f\right)\\&\times\left(t_f^{\frac{1}{2}}r^5\beta_c^{\frac{1}{2}}h^{k+\frac{1}{2}}\|u\|_{C^0\left(H^{k+1}(\Omega)\right)}+t_f^{\frac{1}{2}}r^5\beta_c^{-\frac{1}{2}}h^{k+\frac{1}{2}}\|d_tu\|_{C^0\left(H^k(\Omega)\right)}\right)\\&+\frac{1}{\sqrt{4-3r^2}}\exp\left(\frac{8Cr^8}{\tau_\star}t_f\right)r^4t_f^{\frac{1}{2}}\tau_\star^{\frac{1}{2}}\left\|d_t^3u\right\|_{C^0\left(L^2(\Omega)\right)}\delta t^2\quad\text{for }1\le\frac{1}{\sqrt{4-3r^2}}\end{aligned}\tag{83}$$

$$\begin{aligned}&\left\|u^N-u_h^N\right\|_{L^2(\Omega)}+\left(\sum_{m=0}^{N-1}\delta t\left|u^m-u_h^m\right|_\beta^2\right)^{\frac{1}{2}}\\&\lesssim\frac{1}{\sqrt{4-3r^2}}\exp\left(\frac{8Cr^8}{\tau_\star}t_f\right)\left(\chi_1\delta t^2+\chi_2h^{k+\frac{1}{2}}\right)\end{aligned}\tag{84}$$

Remark 1. *When* F *is the identity mapping,* $K=Q$ *so* $\lambda_K=\lambda_Q=1$ [16] (*page* 10). *Therefore* $r=1$. *We get same results as Alexandre Ern. Thus, Our results are a generalization of Ern results because in finite elements method, Ern obtained his error estimate, working on a polygon* [11]. *In the framework of our work, we got the same order of precision in time and space like Alexandre Ern. But our result is obtained for anygeometry.*

8. Conclusion

The isogeometric method has been used to establish an error estimate for transport equation in 2D using the explicit two stage Heun scheme, for smooth solutions, in the energy norm comprising the L^2-norm and the jumps. These results generalize Ern results. An extension of this present paper is to tackle Burgers equation to get an isogeometric error estimate.

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Conflicts of Interest

The authors declare that there is no conflict of interest as far as the publication of this paper is concerned.

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