Description of Incomplete Financial Markets for Time Evolution of Risk Assets

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Abstract

In the paper, a class of discrete evolutions of risk assets having the memory is considered. For such evolutions the description of all martingale measures is presented. It is proved that every martingale measure is an integral on the set of extreme points relative to some measure on it. For such a set of evolutions of risk assets, the contraction of the set of martingale measures on the filtration is described and the representation for it is found. The inequality for the integrals from a nonnegative random value relative to the contraction of the set of martingale measure on the filtration which is dominated by one is obtained. Using these inequalities a new proof of the optional decomposition theorem for super-martingales is presented. The description of all local regular super-martingales relative to the regular set of measures is presented. The applications of the results obtained to mathematical finance are presented. In the case, as evolution of a risk asset is given by the discrete geometric Brownian motion, the financial market is incomplete and a new formula for the fair price of super-hedge is founded.

1. Introduction

In the paper, the notion of the regular super-martingale relative to the set of equivalent measures is introduced. The necessary and sufficient conditions of the regular super-martingale relative to the set of equivalent measures are found. The notion of the family of equivalent measures consistent with filtration is introduced. Theorem giving the sufficient conditions of the existence of su-

per-martingale and martingale relative to the set of equivalent measures consistent with the filtration is proved. The sufficient conditions of the existence of the set of equivalent measures consistent with the filtration, satisfying the conditions: the mean value of the nonnegative random value relative to these set of measures equal one, are given. Further, we construct the set of equivalent measures consistent with the filtration satisfying the above conditions. First, we give the complete description of the set of equivalent measures satisfying the conditions: the mean value of the nonnegative random value relative to this set of measures equals one. Using the above result we construct the example of the set of equivalent measures consistent with the filtration satisfying the condition: the mean value of the nonnegative random value relative to every measure of this set of measures equals one. The above method we use for the construction of evolution of risk assets and we describe completely the set of equivalent martingale measures for this evolution. We prove that every martingale measure is an integral on the set of extreme points of the convex set of martingale measures relative to some measure on it. To give a new proof of the optional decomposition for super-martingale we describe the contraction of every martingale measure on the filtration and find the closure of integrals from the integrable random values over all martingale measures. To do this we introduce the notion of the exhaustive decomposition and prove that every separable metric space with the Borel σ-algebra has an exhaustive decomposition. For the integral from the nonnegative random value relative to all martingale measures which is dominated by one, the inequalities for this random value are obtained. This fact gives us the possibility to find a new proof of the optional decomposition for the nonnegative super-martingale. This proof does not use the no-arbitrage arguments and the measurable choice [1] [2] [3] [4]. This paper is a generalization of the results of the paper [5].

First, the optional decomposition for diffusion processes super-martingale was opened by El Karoui N. and Quenez M. C. [6]. After that, Kramkov D. O. and Follmer H. [1] [2] proved the optional decomposition for the nonnegative bounded super-martingales. Folmer H. and Kabanov Yu. M. [3] [4] proved analogous result for an arbitrary super-martingale. Recently, Bouchard B. and Nutz M. [7] considered a class of discrete models and proved the necessary and sufficient conditions for the validity of the optional decomposition.

The optional decomposition for super-martingales plays the fundamental role for the risk assessment in incomplete markets [1] [2] [6] [8] [9] [10] [11].

At last, we consider an application of the results obtained to find the new formula for the fair price of super-hedge in the case, as the risk asset evolves by the discrete geometric Brownian motion.

2. Local Regular Super-Martingales Relative to a Set of Equivalent Measures

We assume that on a measurable space $\{\Omega, \mathcal{F}\}$ a filtration $\mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \mathcal{F}$,

$m = \overline{0, \infty}$, and a set of equivalent measures M on $\mathcal{F}$ are given. Further, we assume that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and the σ-algebra $\mathcal{F} = \sigma\left(\bigvee_{n=1}^{\infty} \mathcal{F}_n\right)$ is a minimal σ-algebra generated by the algebra $\bigvee_{n=1}^{\infty} \mathcal{F}_n$. A random process $\psi = \{\psi_m\}_{m=0}^{\infty}$ is said to be adapted one relative to the filtration $\{\mathcal{F}_m\}_{m=0}^{\infty}$, if ψ_m is a $\mathcal{F}_m$ measurable random value, $m = \overline{0, \infty}$.

Definition 1. *An adapted random process* $f = \{f_m\}_{m=0}^{\infty}$ *is said to be a super-martingale relative to the filtration* $\mathcal{F}_m, m = \overline{0, \infty}$, *and the family of equivalent measures M, if* $E^P |f_m| < \infty, m = \overline{1, \infty}, P \in M$, *and the inequalities*

$$E^P\{f_m \mid \mathcal{F}_k\} \le f_k, \ \ 0 \le k \le m, \ m = \overline{1, \infty}, \ P \in M, \tag{1}$$

are valid.

Further, for an adapted process f we use both the denotation $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ and the denotation $\{f_m\}_{m=0}^{\infty}$.

Definition 2. *A super-martingale* $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ *relative to a set of equivalent measures M is a local regular one, if* $\sup_{P \in M} E^P |f_m| < \infty, m = \overline{1, \infty}$, *and there exists an adapted nonnegative increasing random process* $\{g_m, \mathcal{F}_m\}_{m=0}^{\infty}, g_0 = 0$, $\sup_{P \in M} E^P |g_m| < \infty, m = \overline{1, \infty}$, *such that* $\{f_m + g_m, \mathcal{F}_m\}_{m=0}^{\infty}$ *is a martingale relative to every measure from M.*

The next elementary Theorem 1 will be very useful later.

Theorem 1. *Let a super-martingale* $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$, *relative to a set of equivalent measures M be such that* $\sup_{P \in M} E^P |f_m| < \infty, m = \overline{1, \infty}$. *The necessary and sufficient condition for it to be a local regular one is the existence of an adapted nonnegative random process* $\{\bar{g}_m^0, \mathcal{F}_m\}_{m=0}^{\infty}$, $\sup_{P \in M} E^P |\bar{g}_m^0| < \infty, m = \overline{1, \infty}$, *such that*

$$f_{m-1} - E^P\{f_m \mid \mathcal{F}_{m-1}\} = E^P\{\bar{g}_m^0 \mid \mathcal{F}_{m-1}\}, \ m = \overline{1, \infty}, \ P \in M. \tag{2}$$

Proof. The necessity. If $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ is a local regular super-martingale, then there exist a martingale $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^{\infty}$ and a non-decreasing nonnegative random process $\{g_m, \mathcal{F}_m\}_{m=0}^{\infty}$, $g_0 = 0$, such that

$$f_m = \bar{M}_m - g_m, \ m = \overline{1, \infty}. \tag{3}$$

From here, we obtain the equalities

$$
\begin{aligned}
&E^P\{f_{m-1} - f_m \mid \mathcal{F}_{m-1}\} \\
&= E^P\{g_m - g_{m-1} \mid \mathcal{F}_{m-1}\} = E^P\{\bar{g}_m^0 \mid \mathcal{F}_{m-1}\}, \ m = \overline{1, \infty}, \ P \in M,
\end{aligned}
\tag{4}
$$

where we introduced the denotation $\bar{g}_m^0 = g_m - g_{m-1} \ge 0$. It is evident that $E^P \bar{g}_m^0 \le \sup_{P \in M} E^P g_m + \sup_{P \in M} E^P g_{m-1} < \infty$.

The sufficiency. Suppose that there exists an adapted nonnegative random process $\bar{g}^0 = \{\bar{g}_m^0\}_{m=0}^{\infty}, \bar{g}_0^0 = 0$, $E^P \bar{g}_m^0 < \infty, m = \overline{1, \infty}$, such that the equalities (2)

hold. Let us consider the random process $\left\{\bar{M}_m, \mathcal{F}_m\right\}_{m=0}^{\infty}$, where

$$\bar{M}_0 = f_0, \quad \bar{M}_m = f_m + \sum_{i=1}^{m} \bar{g}_m^0, \quad m = \overline{1,\infty}. \tag{5}$$

It is evident that $E^P \left|\bar{M}_m\right| < \infty$ and

$$E^P \left\{\bar{M}_{m-1} - \bar{M}_m \mid \mathcal{F}_{m-1}\right\} = E^P \left\{f_{m-1} - f_m - \bar{g}_m^0 \mid \mathcal{F}_{m-1}\right\} = 0. \tag{6}$$

Theorem 1 is proved. $\qquad\square$

Lemma 1. *Any super-martingale* $\left\{f_m, \mathcal{F}_m\right\}_{m=0}^{\infty}$ *relative to a family of measures M for which there hold equalities* $E^P f_m = f_0, m = \overline{1,\infty}, P \in M$, *is a martingale with respect to this family of measures and the filtration* $\mathcal{F}_m, m = \overline{1,\infty}$.

Proof. The proof of Lemma 1 see [12]. $\qquad\square$

In the next Lemma, we present the formula for calculation of the conditional expectation relative to another measure from *M*.

Lemma 2. *On the measurable space* $\left\{\Omega, \mathcal{F}\right\}$ *with the filtration* $\mathcal{F}_n$ *on it, let M be a set of equivalent measures and let* ξ *be an integrable random value. Then, the following formulas*

$$E^{P_1} \left\{\xi \mid \mathcal{F}_n\right\} = E^{P_2} \left\{\xi \varphi_n^{P_1} \mid F_n\right\}, \quad n = \overline{1,\infty}, \tag{7}$$

are valid, where

$$\varphi_n^{P_1} = \frac{dP_1}{dP_2} \left[E^{P_2} \left\{\frac{dP_1}{dP_2} \mid \mathcal{F}_n\right\} \right]^{-1}, \quad P_1, P_2 \in M. \tag{8}$$

Proof. The proof of Lemma 2 is evident. $\qquad\square$

3. Local Regular Super-Martingales Relative to a Set of Equivalent Measures Consistent with the Filtration

Definition 3. *On a measurable space* $\left\{\Omega, \mathcal{F}\right\}$ *with a filtration* $\mathcal{F}_n$ *on it, a set of equivalent measures M we call consistent with the filtration* $\mathcal{F}_n$, *if for every pair of measures* $\left(Q_1, Q_2\right) \in M^2$ *the set of measures*

$$R_s^k (A) = \int_A \frac{E^{Q_1} \left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_k\right\}}{E^{Q_1} \left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_s\right\}} dQ_1, \quad A \in \mathcal{F}, \quad k \geq s \geq n, \quad n = \overline{0,\infty}, \tag{9}$$

belongs to the set M, where M^2 *is a direct product of the set M by itself.*

Lemma 3. *On the measurable space* $\left\{\Omega, \mathcal{F}\right\}$ *with the filtration* $\mathcal{F}_n$ *on it, the set of measures*

$$M = \left\{Q, Q(A) = \int_A \alpha(\omega) dP, A \in \mathcal{F}, Q(\Omega) = 1\right\} \tag{10}$$

is a consistent one with the filtration $\mathcal{F}_n$, *if P is a measure on* $\left\{\Omega, \mathcal{F}\right\}$ *and a random value* $\alpha(\omega)$ *runs over all nonnegative random values, satisfying the condition* $P\left(\left\{\omega, \alpha(\omega) > 0\right\}\right) = 1$.

Proof. Suppose that $\left(Q_1, Q_2\right)$ belongs to M^2. Then, $\dfrac{dQ_2}{dQ_1} = \dfrac{\alpha_2(\omega)}{\alpha_1(\omega)}$ and

$$P\left(\left\{\omega, \frac{dQ_2}{dQ_1} > 0\right\}\right) = 1, \text{ since the equalities } P\left(\left\{\omega, 0 < \alpha_1(\omega) < \infty\right\}\right) = 1,$$

$P\left(\left\{\omega, 0 < \alpha_2(\omega) < \infty\right\}\right) = 1$ are true. It is evident that

$$R_s^k(A) = \int_A \frac{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_k\right\}}{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_s\right\}} dQ_1$$

$$= \int_A \frac{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_k\right\}}{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_s\right\}} \alpha_1(\omega) dP, \quad A \in \mathcal{F}, \ k \geq s \geq n, \ n = \overline{0, \infty}. \tag{11}$$

It is easy to see that

$$P\left(\left\{\omega, \frac{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_k\right\}}{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_s\right\}} \alpha_1(\omega) > 0\right\}\right) = 1, \ k \geq s, \tag{12}$$

since

$$P\left(\left\{\omega, E^{Q_1}\left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_k\right\} > 0\right\}\right) = 1, \ k \geq s, \tag{13}$$

$$P\left(\left\{\omega, 0 < E^{Q_1}\left\{\frac{dQ_2}{dQ_1} \mid \mathcal{F}_s\right\} < \infty\right\}\right) = 1, \ s \geq n, \ n = \overline{0, \infty}. \tag{14}$$

The last equality follows from the equivalence of the measures Q_1, Q_2 and P. Altogether, it means that the set of measures $R_s^k, k \geq s \geq n, n = \overline{0, \infty}$, belongs to the set M. The same is true for the pair $(Q_2, Q_1) \in M^2$. Lemma 3 is proved. $\square$

Theorem 2. *On the measurable space* $\{\Omega, \mathcal{F}\}$ *with the filtration* $\mathcal{F}_n$ *on it, let the set of equivalent measures M be consistent with the filtration* $\mathcal{F}_n$. *Then, for every nonnegative random value* ξ *such that* $\sup_{P \in M} E^P \xi < \infty$, *the random process* $\{f_n, \mathcal{F}_n\}_{n=0}^{\infty}$ *is a super-martingale relative to the set of measures M, where* $f_n = \operatorname*{ess\,sup}_{P \in M} E^P\{\xi \mid \mathcal{F}_n\}, n = \overline{0, \infty}$.

Proof. Let $Q \in M$, then, due to Lemma 2, for every $P \in M$

$$E^P\{\xi \mid \mathcal{F}_n\} = E^Q\left\{\xi \mid \frac{\frac{dP}{dQ}}{E^Q\left\{\frac{dP}{dQ} \mid \mathcal{F}_n\right\}} \mid \mathcal{F}_n\right\}. \tag{15}$$

If to put instead of the measure P the measure $R_s^k, k \geq s \geq n$, for the pair of measures (Q, P) we obtain

$$E^{R_s^k}\left\{\xi \mid F_n\right\} = E^Q\left\{\xi \mid \frac{\dfrac{\mathrm{d}R_s^k}{\mathrm{d}Q}}{E^Q\left\{\dfrac{\mathrm{d}R_s^k}{\mathrm{d}Q} \mid \mathcal{F}_n\right\}} \mid \mathcal{F}_n\right\}$$

$$= E^Q\left\{\xi \frac{E^Q\left\{\dfrac{\mathrm{d}P}{\mathrm{d}Q} \mid \mathcal{F}_k\right\}}{E^Q\left\{\dfrac{\mathrm{d}P}{\mathrm{d}Q} \mid \mathcal{F}_s\right\}} \mid \mathcal{F}_n\right\}, \tag{16}$$

where we took into account the equality

$$E^Q\left\{\frac{\mathrm{d}R_s^k}{\mathrm{d}Q} \mid \mathcal{F}_n\right\} = E^Q\left\{\frac{E^Q\left\{\dfrac{\mathrm{d}P}{\mathrm{d}Q} \mid \mathcal{F}_k\right\}}{E^Q\left\{\dfrac{\mathrm{d}P}{\mathrm{d}Q} \mid \mathcal{F}_s\right\}} \mid \mathcal{F}_n\right\} = 1, \ k \geq s \geq n. \tag{17}$$

From the formula (16), it follows the equality

$$\operatorname*{ess\,sup}_{P \in M} E^P\left\{\xi \mid \mathcal{F}_n\right\} = \operatorname*{ess\,sup}_{T \in R_n} E^P\left\{\xi T \mid \mathcal{F}_n\right\}, \tag{18}$$

where R_n is a set of martingales $T = \{T_m\}_{m=0}^{\infty}$ relative to the measure Q such

that $T_m = 1, m \leq n$, $T_m = \dfrac{E^Q\left\{\dfrac{\mathrm{d}P}{\mathrm{d}Q} \mid \mathcal{F}_m\right\}}{E^Q\left\{\dfrac{\mathrm{d}P}{\mathrm{d}Q} \mid \mathcal{F}_s\right\}}, m \geq s \geq n, P \in M$. The definition of

$\operatorname{ess\,sup}$ for the uncountable set of random values see [13]. It is evident that $T_n \subseteq T_{n-1}$. Let us consider

$$E^Q\left\{\operatorname*{ess\,sup}_{P \in M} E^P\left\{\xi \mid \mathcal{F}_n\right\} \mid \mathcal{F}_{n-1}\right\} = E^Q\left\{\operatorname*{ess\,sup}_{T \in R_n} E^P\left\{\xi T \mid \mathcal{F}_n\right\} \mid \mathcal{F}_{n-1}\right\}$$

$$= E^Q\left\{\sup_{i \geq 1} E^P\left\{\xi T_i \mid \mathcal{F}_n\right\} \mid \mathcal{F}_{n-1}\right\} = E^Q\left\{\lim_{k \to \infty} \max_{1 \leq i \leq k} E^P\left\{\xi T_i \mid \mathcal{F}_n\right\} \mid \mathcal{F}_{n-1}\right\}$$

$$= \lim_{k \to \infty} E^Q\left\{\max_{1 \leq i \leq k} E^P\left\{\xi T_i \mid \mathcal{F}_n\right\} \mid \mathcal{F}_{n-1}\right\} = \lim_{k \to \infty} E^P\left\{\xi T_{\tau_k} \mid \mathcal{F}_{n-1}\right\} \tag{19}$$

$$\leq \operatorname*{ess\,sup}_{T \in R_n} E^Q\left\{\xi T \mid \mathcal{F}_{n-1}\right\} \leq \operatorname*{ess\,sup}_{T \in R_{n-1}} E^Q\left\{\xi T \mid \mathcal{F}_{n-1}\right\}$$

$$= \operatorname*{ess\,sup}_{P \in M} E^Q\left\{\xi \mid \mathcal{F}_{n-1}\right\},$$

where

$$\tau_1 = 1, \tag{20}$$

$$\tau_i = \begin{cases} \tau_{i-1}, & E^P\left\{\xi T_{\tau_{i-1}} \mid \mathcal{F}_n\right\} > E^P\left\{\xi T_i \mid \mathcal{F}_n\right\}, \\ i, & E^P\left\{\xi T_{\tau_{i-1}} \mid \mathcal{F}_n\right\} \leq E^P\left\{\xi T_i \mid \mathcal{F}_n\right\}, \end{cases} \quad i = \overline{2, k}. \tag{21}$$

Lemma 2 is proved. $\qquad\square$

Theorem 3. *On the measurable space* $\{\Omega, \mathcal{F}\}$, $\mathcal{F} = \sigma\left(\bigvee_{i=1}^{\infty} \mathcal{F}_i\right)$, *let M be a set of equivalent measures being consistent with the filtration* $\mathcal{F}_n$. *If there exists a nonnegative random value* $\xi \neq 1$ *such that* $E^P \xi = 1, P \in M$, *then*

$E^P \{\xi \,|\, \mathcal{F}_n\}, P \in M$, *is a local regular martingale.*

Proof. Due to Lemma 2, the random process $\{f_n, \mathcal{F}_n\}_{n=0}^{\infty}$, where $f_n = \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\}$, $n = \overline{0, \infty}$, is a super-martingale relative to the set of measures *M*, that is,

$$E^Q \left\{ \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\} \,|\, \mathcal{F}_{n-1} \right\} \leq \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_{n-1}\}, \quad Q \in M, \quad n = \overline{0, \infty}. \quad (22)$$

From the inequality (22), it follows the inequality

$$E^Q \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\} \leq 1, \quad n = \overline{0, \infty}. \quad (23)$$

Since $E^Q \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\} \geq E^Q E^Q \{\xi \,|\, \mathcal{F}_n\} = 1$, we have

$$E^Q \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\} = 1, \quad Q \in M, \quad n = \overline{0, \infty}. \quad (24)$$

The inequalities (22) and the equalities (24) give the equalities

$$E^Q \left\{ \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\} \,|\, \mathcal{F}_{n-1} \right\} = \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_{n-1}\}, \quad Q \in M, \quad n = \overline{1, \infty}, \quad (25)$$

which are true with the probability 1. The last means that $\{f_n, \mathcal{F}_n\}_{n=0}^{\infty}$ is a martingale relative to the set of measures *M*, where $f_n = \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\}, n = \overline{0, \infty}$.

With the probability 1, $\lim_{n \to \infty} \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\} = f_\infty$, where the random value f_∞ is *F* measurable one. From the inequality (23) and Fatou Lemma [13] [14], we obtain

$$E^P f_\infty \leq 1, \quad P \in M. \quad (26)$$

Prove that $f_\infty = \xi$. Going to the limit in the inequality

$$\operatorname*{ess\,sup}_{P \in M} E^P \{\xi \,|\, \mathcal{F}_n\} \geq E^{P_1} \{\xi \,|\, \mathcal{F}_n\}, \quad (27)$$

as $n \to \infty$, we obtain the inequality

$$f_\infty \geq \xi. \quad (28)$$

From the inequality (26) and the inequality (28), we obtain the inequalities $1 \geq E^P f_\infty \geq E^P \xi = 1$. Or, $E^P f_\infty = 1$. The equalities $E^P f_\infty = 1, E^P \xi = 1$ and the inequality (28) give the equality $f_\infty = \xi$ with the probability 1. Lemma 3 is proved. $\square$

Lemma 4. *On the measurable space* $\{\Omega, \mathcal{F}\}$ *with the filtration* $\mathcal{F}_n$ *on it, let there exist k equivalent measures* $P_1, \cdots, P_k, k > 1$, *and a nonnegative random value* $\xi_0 \neq 1$ *be such that*

$$E^{P_i} \{\xi_0 \,|\, \mathcal{F}_n\} = E^{P_1} \{\xi_0 \,|\, \mathcal{F}_n\}, \quad E^{P_i} \xi_0 = 1, \quad i = \overline{2, k}, \quad n = \overline{0, \infty}. \quad (29)$$

Then, there exists the set of equivalent measures M consistent with the filtration $\mathcal{F}_n$, *satisfying the condition* $E^P \xi_0 = 1, P \in M$.

Proof. Let us consider the set of equivalent measures *M*, satisfying the condition

$$E^P \{\xi_0 \,|\, \mathcal{F}_n\} = E^{P_1} \{\xi_0 \,|\, \mathcal{F}_n\}, \quad n = \overline{0, \infty}, \quad P \in M. \quad (30)$$

Such a set of measures is a nonempty one. Suppose that $Q_1, Q_2 \in M$, then

$$E^{Q_1}\left\{\xi_0 \mid \mathcal{F}_n\right\} = E^{Q_2}\left\{\xi_0 \mid \mathcal{F}_n\right\}, \quad n = \overline{0, \infty}. \tag{31}$$

Let us prove that the formula

$$E^{Q_1}\left\{\xi_0 \frac{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_k\right\}}{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_s\right\}} \mid \mathcal{F}_n\right\} = E^{Q_1}\left\{\xi_0 \mid \mathcal{F}_n\right\}, \quad n \le s \le k, \quad n = \overline{0, \infty}, \tag{32}$$

is valid. Let $s \ge n$. Then, from the equalities (31), we have

$$E^{Q_1}\left\{E^{Q_2}\left\{\xi_0 \mid \mathcal{F}_s\right\} \mid \mathcal{F}_n\right\} = E^{Q_1}\left\{\xi_0 \mid \mathcal{F}_n\right\}.$$

Let $k \ge s$. Then,

$$E^{Q_1}\left\{E^{Q_2}\left\{\xi_0 \mid \mathcal{F}_s\right\} \mid \mathcal{F}_n\right\}$$

$$= E^{Q_1}\left\{E^{Q_2}\left\{E^{Q_2}\left\{\xi_0 \mid \mathcal{F}_k\right\} \mid \mathcal{F}_s\right\} \mid \mathcal{F}_n\right\}$$

$$= E^{Q_1}\left\{E^{Q_1}\left\{E^{Q_2}\left\{\xi_0 \mid \mathcal{F}_k\right\} \frac{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1}}{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_s\right\}} \mid \mathcal{F}_s\right\} \mid \mathcal{F}_n\right\}$$

$$= E^{Q_1}\left\{E^{Q_2}\left\{\xi_0 \mid \mathcal{F}_k\right\} \frac{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1}}{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_s\right\}} \mid \mathcal{F}_n\right\}$$

$$= E^{Q_1}\left\{E^{Q_1}\left\{\xi_0 \mid \mathcal{F}_k\right\} \frac{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1}}{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_s\right\}} \mid \mathcal{F}_n\right\}$$

$$= E^{Q_1}\left\{E^{Q_1}\left\{\xi_0 \mid \mathcal{F}_k\right\} \frac{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_k\right\}}{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_s\right\}} \mid \mathcal{F}_n\right\}$$

$$= E^{Q_1}\left\{\xi_0 \mid \frac{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_k\right\}}{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_s\right\}} \mid \mathcal{F}_n\right\}.$$

This proves the formula (32). To finish the proof of Lemma 4, it needs to prove that the set of measures

$$R_s^k(A) = \int_A \frac{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_k\right\}}{E^{Q_1}\left\{\dfrac{\mathrm{d}Q_2}{\mathrm{d}Q_1} \mid \mathcal{F}_s\right\}} \mathrm{d}Q_1, \quad A \in \mathcal{F}, \quad k \ge s \ge n, \quad n = \overline{0, \infty}, \tag{33}$$

belongs to the set M. Really,

$$E^{R_s^k}\left\{\xi_0\mid\mathcal{F}_n\right\}=E^{Q_1}\left\{\xi_0\mid\frac{\dfrac{dR_s^k}{dQ_1}}{E^{Q_1}\left\{\dfrac{dR_s^k}{dQ_1}\mid\mathcal{F}_n\right\}}\mid\mathcal{F}_n\right\}$$

$$=E^{Q_1}\left\{\xi_0\,\frac{E^{Q_1}\left\{\dfrac{dQ_2}{dQ_1}\mid\mathcal{F}_k\right\}}{E^{Q_1}\left\{\dfrac{dQ_2}{dQ_1}\mid\mathcal{F}_s\right\}}\mid\mathcal{F}_n\right\} \tag{34}$$

$$=E^{Q_1}\left\{\xi_0\mid\mathcal{F}_n\right\},$$

where we took into account the equality

$$E^{Q_1}\left\{\frac{dR_s^k}{dQ_1}\mid\mathcal{F}_n\right\}=E^{Q_1}\left\{\frac{E^{Q_1}\left\{\dfrac{dQ_2}{dQ_1}\mid\mathcal{F}_k\right\}}{E^{Q_1}\left\{\dfrac{dQ_2}{dQ_1}\mid\mathcal{F}_s\right\}}\mid\mathcal{F}_n\right\}=1,\ \ k\geq s\geq n. \tag{35}$$

From this, it follows that the set of measures $R_s^k\in M$. This proves the consistence with the filtration of the set of measures M. Lemma 4 is proved. $\square$

The next Lemma 5 is a key statement in the construction of the set of measures satisfying the conditions of Lemma 4.

On a probability space $\{\Omega,\mathcal{F},P\}$, let ξ be a random value, satisfying the conditions

$$0<P\left(\{\omega,\xi>0\}\right)<1,\ \ 0<P\left(\{\omega,\xi<0\}\right). \tag{36}$$

Denote $\Omega^+=\{\omega,\xi(\omega)>0\}$, $\Omega^-=\{\omega,\xi(\omega)\leq 0\}$ and let $\mathcal{F}^-,\mathcal{F}^+$ be the restrictions of the σ-algebra $\mathcal{F}$ on the sets Ω^- and Ω^+, correspondingly. Suppose that P^- and P^+ are the contractions of the measure P on the σ-algebras $\mathcal{F}^-,\mathcal{F}^+$, correspondingly. Consider the measurable space with measure $\{\Omega^-\times\Omega^+,\mathcal{F}^-\times\mathcal{F}^+,\mu\}$, which is a direct product of the measurable spaces with measures $\{\Omega^-,\mathcal{F}^-,P^-\}$ and $\{\Omega^+,\mathcal{F}^+,P^+\}$, where $\mu=P^-\times P^+$. Introduce the denotations

$$\xi^+(\omega)=\begin{cases}\xi(\omega), & \omega\in\{\xi(\omega)>0\},\\ 0, & \omega\in\{\xi(\omega)\leq 0\},\end{cases} \tag{37}$$

$$\xi^-(\omega)=\begin{cases}-\xi(\omega), & \omega\in\{\xi(\omega)\leq 0\},\\ 0, & \omega\in\{\xi(\omega)>0\}.\end{cases} \tag{38}$$

Then, $\xi(\omega)=\xi^+(\omega)-\xi^-(\omega)$.

On the measurable space $\{\Omega^-\times\Omega^+,\mathcal{F}^-\times\mathcal{F}^+,P^-\times P^+\}$, we assume that the set of nonnegative measurable functions $\alpha(\omega_1,\omega_2)$, satisfying the conditions

$$\mu\left(\{(\omega_1,\omega_2)\in\Omega^-\times\Omega^+,\alpha(\omega_1,\omega_2)>0\}\right)=P(\Omega^+)P(\Omega^-), \tag{39}$$

$$\int_{\Omega^-}\int_{\Omega^+}\alpha(\omega_1,\omega_2)\frac{\xi^-(\omega_1)\xi^+(\omega_2)}{\xi^-(\omega_1)+\xi^+(\omega_2)}d\mu(\omega_1,\omega_2)<\infty, \tag{40}$$

$$\int_{\Omega^-}\int_{\Omega^+}\alpha(\omega_1,\omega_2)d\mu(\omega_1,\omega_2)=1, \tag{41}$$

is a nonempty set. Such assumptions are true for the nonempty set of bounded random values $\alpha(\omega_1, \omega_2)$, for example, if the random value ξ is an integrable one relative to the measure P.

Lemma 5. *On the probability space* $\{\Omega, \mathcal{F}, P\}$ *, let a random value* ξ *satisfy the conditions* (36) *and let a measure Q be equivalent to the measure P and such that* $E^Q \xi = 0$ *. Then, for the measure Q the following representation*

$$
\begin{aligned}
Q(A) = & \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_1) \alpha(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) \\
& + \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_2) \alpha(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2), \quad A \in \mathcal{F},
\end{aligned}
\tag{42}
$$

is valid for those random value $\alpha(\omega_1, \omega_2)$ *that satisfy the conditions* (39)-(41).

Every measure Q, given by the formula (42), *with the random value* $\alpha(\omega_1, \omega_2)$, *satisfying the conditions* (39)-(41) *is equivalent to the measure P and is such that* $E^Q \xi = 0$ *. For the measure Q, the canonical representation*

$$
\begin{aligned}
Q(A) = & \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_1) \alpha_1(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) \\
& + \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_2) \alpha_1(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2), \quad A \in \mathcal{F},
\end{aligned}
\tag{43}
$$

is valid, where

$$
\alpha_1(\omega_1, \omega_2) = \frac{\psi_1(\omega_1)\psi_2(\omega_2)\left[\xi^-(\omega_1) + \xi^+(\omega_2)\right]}{d}, \quad (\omega_1, \omega_2) \in \Omega^- \times \Omega^+, \tag{44}
$$

$$
\psi_1(\omega_1) = \int_{\Omega^+} \alpha(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} dP(\omega_2), \quad \omega_1 \in \Omega^-, \tag{45}
$$

$$
\psi_2(\omega_2) = \int_{\Omega^-} \alpha(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} dP(\omega_1), \quad \omega_2 \in \Omega^+, \tag{46}
$$

$$
d = \int_{\Omega^-} \xi^-(\omega_1) \psi_1(\omega_1) dP(\omega_1) = \int_{\Omega^+} \xi^+(\omega_2) \psi_2(\omega_2) dP(\omega_2). \tag{47}
$$

Proof. From the Lemma 5 conditions,

$$
Q(A) = \int_A \psi(\omega) dP, \quad P(\{\omega, \psi(\omega) > 0\}) = 1, \tag{48}
$$

$$
\int_{\Omega} \psi(\omega) \xi(\omega) dP(\omega) = 0. \tag{49}
$$

The condition (49) means

$$
\int_{\Omega^+} \psi_2(\omega_2) \xi^+(\omega_2) dP(\omega_2) = \int_{\Omega^-} \psi_1(\omega_1) \xi^-(\omega_1) dP(\omega_1) = d > 0, \tag{50}
$$

where

$$
\psi_1(\omega) = \begin{cases} \psi(\omega), & \omega \in \Omega^-, \\ 0, & \omega \in \Omega^+, \end{cases} \tag{51}
$$

$$\psi_2(\omega) = \begin{cases} \psi(\omega), & \omega \in \Omega^+, \\ 0, & \omega \in \Omega^-. \end{cases} \tag{52}$$

Let us put

$$\alpha(\omega_1, \omega_2) = \frac{\psi_1(\omega_1)\psi_2(\omega_2)\left[\xi^-(\omega_1) + \xi^+(\omega_2)\right]}{d}, \quad (\omega_1, \omega_2) \in \Omega^- \times \Omega^+. \tag{53}$$

Then, for such $\alpha(\omega_1, \omega_2)$ the equality (39) is true. Moreover,

$$\int_{\Omega^-}\int_{\Omega^+} \alpha(\omega_1, \omega_2)\frac{\xi^-(\omega_1)\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)}d\mu(\omega_1, \omega_2) = d^2 < \infty, \tag{54}$$

$$\int_{\Omega^-}\int_{\Omega^+} \alpha(\omega_1, \omega_2)d\mu(\omega_1, \omega_2) = \int_{\Omega^-}\psi_1(\omega_1)dP(\omega_1) + \int_{\Omega^+}\psi_2(\omega_2)dP(\omega_2) = 1, \tag{55}$$

$$\begin{aligned} E^Q\xi = &\int_{\Omega^-}\int_{\Omega^+} \alpha(\omega_1, \omega_2)\xi(\omega_1)\frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)}d\mu(\omega_1, \omega_2) \\ &+ \int_{\Omega^-}\int_{\Omega^+} \alpha(\omega_1, \omega_2)\xi(\omega_2)\frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}d\mu(\omega_1, \omega_2) = 0, \end{aligned} \tag{56}$$

since $\xi(\omega_1) = -\xi^-(\omega_1), \omega_1 \in \Omega^-, \xi(\omega_2) = \xi^+(\omega_2), \omega_2 \in \Omega^+$.

Let us prove the last statement of Lemma 5. Suppose that the representation (42) for the measure Q, satisfying the conditions (39)-(41), is valid. Taking into account the denotations (45)-(47), we obtain

$$Q(A) = \int_{\Omega^-}\chi_A(\omega_1)\psi_1(\omega_1)dP(\omega_1) + \int_{\Omega^+}\chi_A(\omega_2)\psi_2(\omega_2)dP(\omega_2), \tag{57}$$

$$\begin{aligned} 0 = E^Q\xi &= \int_{\Omega^-}\xi(\omega_1)\psi_1(\omega_1)dP(\omega_1) + \int_{\Omega^+}\xi(\omega_2)\psi_2(\omega_2)dP(\omega_2) \\ &= -\int_{\Omega^-}\xi^-(\omega_1)\psi_1(\omega_1)dP(\omega_1) + \int_{\Omega^+}\xi^+(\omega_2)\psi_2(\omega_2)dP(\omega_2). \end{aligned} \tag{58}$$

If to introduce the denotation

$$\psi(\omega) = \begin{cases} \psi_1(\omega), & \omega \in \Omega^-, \\ \psi_2(\omega), & \omega \in \Omega^+, \end{cases} \tag{59}$$

then we obtain the representation

$$Q(A) = \int_A \psi(\omega)dP(\omega), \tag{60}$$

where $P(\psi_1(\omega) > 0) = P(\Omega^-), P(\psi_2(\omega) > 0) = P(\Omega^+)$.

The last formula proves the equivalence of the measures Q and P. At last, to prove the canonical representation (43) it is sufficient to substitute the expression (44) for $\alpha_1(\omega_1, \omega_2)$ into the expression (43) for $Q(A)$. We obtain the expression (57) for $Q(A)$. Then, if to substitute the expressions (45), (46) for $\psi_1(\omega_1), \psi_2(\omega_2)$ into the expression (57) for $Q(A)$, we obtain that the canonical representation for $Q(A)$ is true. This proves Lemma 5. □

For further investigations, the next Theorem 4 is very important [5].

Theorem 4. *The necessary and sufficient conditions of the local regularity of*

the nonnegative super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ relative to a set of equivalent measures M are the existence of $\mathcal{F}_m$-measurable random values $\xi_m^0 \in A_0, m = \overline{1, \infty}$, such that

$$\frac{f_m}{f_{m-1}} \le \xi_m^0, \quad E^P\left\{\xi_m^0 \mid \mathcal{F}_{m-1}\right\} = 1, \quad P \in M, \quad m = \overline{1, \infty}. \tag{61}$$

Proof. The necessity. Without loss of generality, we assume that $f_m \ge a$ for a certain real number $a > 0$. Really, if it is not so, then we can come to the consideration of the super-martingale $\{f_m + a, \mathcal{F}_m\}_{m=0}^{\infty}$. Thus, let $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ be a nonnegative local regular super-martingale. Then, there exists a nonnegative adapted random process $\{g_m\}_{m=0}^{\infty}$, $g_0 = 0$, such that $\sup_{P \in M} E^P g_m < \infty$,

$$f_{m-1} - E^P\left\{f_m \mid \mathcal{F}_{m-1}\right\} = E^P\left\{g_m \mid \mathcal{F}_{m-1}\right\}, \quad P \in M, \quad m = \overline{1, \infty}. \tag{62}$$

Let us put $\xi_m^0 = \dfrac{f_m + g_m}{f_{m-1}}, m = \overline{1, \infty}$. Then, $\xi_m^0 \in A_0$ and from the equalities (62) we obtain $E^P\left\{\xi_m^0 \mid \mathcal{F}_{m-1}\right\} = 1, P \in M, m = \overline{1, \infty}$. It is evident that the inequalities (61) are valid.

The sufficiency. Suppose that the conditions of Theorem 4 are valid. Then, $f_m \le f_{m-1} + f_{m-1}\left(\xi_m^0 - 1\right)$. Introduce the denotation $g_m = -f_m + f_{m-1}\xi_m^0$. Then, $g_m \ge 0$, $\sup_{P \in M} E^P g_m \le \sup_{P \in M} E^P f_m + \sup_{P \in M} E^P f_{m-1} < \infty, m = \overline{1, \infty}$. The last equality and the inequalities give

$$f_m = f_0 + \sum_{i=1}^{m} f_{i-1}\left(\xi_i^0 - 1\right) - \sum_{i=1}^{m} g_i, \quad m = \overline{1, \infty}. \tag{63}$$

Let us consider the random process $\{M_m, \mathcal{F}_m\}_{m=0}^{\infty}$, where $M_m = f_0 + \sum_{i=1}^{m} f_{i-1}\left(\xi_i^0 - 1\right)$. Then, $E^P\left\{M_m \mid \mathcal{F}_{m-1}\right\} = M_{m-1}, P \in M, m = \overline{1, \infty}$. Theorem 4 is proved. $\square$

4. Construction of the Regular Set of Measures

In the next two Lemmas, we investigate the closure of a convex set of equivalent measures presented in Lemma 5 by the formula (42). First, we consider the countable case.

Suppose that Ω_1 contains the countable set of elementary events and let $\mathcal{F}_1$ be a σ-algebra of all subsets of the set Ω_1. Let P_1 be a measure on the σ-algebra $\mathcal{F}_1$. We assume that $P_1(\omega_i) = p_i > 0, i = \overline{1, \infty}$. On the probability space $\{\Omega_1, \mathcal{F}_1, P_1\}$, let us consider a nonnegative random value ξ_1, satisfying the conditions

$$0 < P_1\left(\left\{\omega \in \Omega_1, \eta_1(\omega) < 0\right\}\right) < 1, \quad 0 < P_1\left(\left\{\omega \in \Omega_1, \eta_1(\omega) > 0\right\}\right),$$

$$E^{P_1}\left|\eta_1(\omega)\right| < \infty, \tag{64}$$

where we introduced the denotation $\eta_1(\omega) = \xi_1(\omega) - 1$. On the measurable space $\{\Omega_1, \mathcal{F}_1\}$, let us consider the set of measures M_1, which are equivalent to the measure P_1 and are given by the formula

$$Q(A) = \sum_{\omega_1 \in \Omega_1^-} \sum_{\omega_2 \in \Omega_1^+} \chi_A(\omega_1) \alpha(\omega_1, \omega_2) \frac{\eta_1^+(\omega_2)}{\eta_1^-(\omega_1) + \eta_1^+(\omega_2)} P_1(\omega_1) P_1(\omega_2)$$

$$+ \sum_{\omega_1 \in \Omega_1^-} \sum_{\omega_2 \in \Omega_1^+} \chi_A(\omega_2) \alpha(\omega_1, \omega_2) \frac{\eta_1^-(\omega_1)}{\eta_1^-(\omega_1) + \eta_1^+(\omega_2)} P_1(\omega_1) P_1(\omega_2), \quad A \in \mathcal{F}_1, \tag{65}$$

where $\eta_1(\omega) = \eta_1^+(\omega) - \eta_1^-(\omega)$, $\Omega_1^+ = \{\omega, \eta_1(\omega) > 0\}$, $\Omega_1^- = \{\omega, \eta_1(\omega) \le 0\}$. Introduce the denotations $\mathcal{F}_1^+ = \Omega_1^+ \bigcap \mathcal{F}_1$, $\mathcal{F}_1^- = \Omega_1^- \bigcap \mathcal{F}_1$. Let P_1^- be a contraction of the measure P_1 on the σ-algebra $\mathcal{F}_1^-$ and let P_1^+ be a contraction of the measure P_1 on the σ-algebra $\mathcal{F}_1^+$. On the probability space $\{\Omega_1^- \times \Omega_1^+, \mathcal{F}_1^- \times \mathcal{F}_1^+, P_1^- \times P_1^+\}$, the set of random value $\alpha(\omega_1, \omega_2)$ satisfies the conditions

$$P_1^- \times P_1^+ \left(\{(\omega_1, \omega_2) \in \Omega_1^- \times \Omega_1^+, \alpha(\omega_1, \omega_2) > 0\} \right) = P_1(\Omega_1^+) P_1(\Omega_1^-), \tag{66}$$

$$\sum_{\omega_1 \in \Omega^-} \sum_{\omega_2 \in \Omega^+} \alpha(\omega_1, \omega_2) \frac{\eta_1^-(\omega_1) \eta_1^+(\omega_2)}{\eta_1^-(\omega_1) + \eta_1^+(\omega_2)} P_1(\omega_1) P_1(\omega_2) < \infty, \tag{67}$$

$$\sum_{\omega_1 \in \Omega_1^-} \sum_{\omega_2 \in \Omega_1^+} \alpha(\omega_1, \omega_2) P_1(\omega_1) P_1(\omega_2) = 1. \tag{68}$$

On the probability space $\{\Omega_1^- \times \Omega_1^+, \mathcal{F}_1^- \times \mathcal{F}_1^+, P_1^- \times P_1^+\}$, all the bounded strictly positive random values $\alpha(\omega_1, \omega_2)$ the above conditions satisfy. Introduce into the set of all measures on $\{\Omega_1, \mathcal{F}_1\}$ the metrics

$$\rho(Q_1, Q_2) = \sum_{i=1}^{\infty} |Q_1(\omega_i) - Q_2(\omega_i)|. \tag{69}$$

Lemma 6. *The closure of the set of measures M_1 in metrics (69) contains the set of measures*

$$v_{\omega_1, \omega_2}(A) = \chi_A(\omega_1) \frac{\eta_1^+(\omega_2)}{\eta_1^-(\omega_1) + \eta_1^+(\omega_2)} + \chi_A(\omega_2) \frac{\eta_1^-(\omega_1)}{\eta_1^-(\omega_1) + \eta_1^+(\omega_2)} \tag{70}$$

for $\omega_1 \in \Omega_1^-$, $\omega_2 \in \Omega_1^+$, $A \in \mathcal{F}_1$. For every bounded random value $f(\omega)$, the closure of the set of points $E^Q f, Q \in M_1$, in metrics $\rho(x, y) = |x - y|, x, y \in R^1$, contains the points $E^{v_{\omega_1, \omega_2}} f, (\omega_1, \omega_2) \in \Omega_1^- \times \Omega_1^+$.

Proof. Let us choose the set of equivalent measures Q^ε defined by $\alpha^\varepsilon(\omega_1, \omega_2), 0 < \varepsilon < 1$, and given by the law:

$$\alpha^\varepsilon(\omega_1^0, \omega_2^0) = \frac{1 - \varepsilon}{P_1(\omega_1^0) P_1(\omega_2^0)}, \quad \omega_1^0 \in \Omega_1^-, \ \omega_2^0 \in \Omega_1^+,$$

$$\alpha^\varepsilon(\omega_1, \omega_2) = \varepsilon \alpha_0^\varepsilon(\omega_1, \omega_2),$$

$$\alpha_0^\varepsilon(\omega_1, \omega_2) = \frac{1}{\displaystyle\sum_{\omega_1 \ne \omega_1^0} \sum_{\omega \ne \omega_2^0} P(\omega_1) P(\omega_2)}, \quad (\omega_1, \omega_2) \ne (\omega_1^0, \omega_2^0),$$

$$\omega_1 \in \Omega_1^-, \ \omega_2 \in \Omega_1^+.$$

It is evident that $\alpha^\varepsilon(\omega_1, \omega_2) > 0, (\omega_1, \omega_2) \in \Omega_1^- \times \Omega_1^+$, for every $1 > \varepsilon > 0$, and satisfy the equality

$$\sum_{(\omega_1, \omega_2) \in \Omega_1^- \times \Omega_1^+} \alpha^\varepsilon(\omega_1, \omega_2) P_1(\omega_1) P_1(\omega_2) = 1. \tag{71}$$

Then,

$$Q^{\varepsilon}\left(\omega_1^0\right) = \sum_{\omega_2 \in \Omega_1^+} \alpha^{\varepsilon}\left(\omega_1^0, \omega_2\right) \frac{\eta_1^+\left(\omega_2\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1^0\right) P_1\left(\omega_2\right), \qquad (72)$$

$$Q^{\varepsilon}\left(\omega_2^0\right) = \sum_{\omega_1 \in \Omega_1^-} \alpha^{\varepsilon}\left(\omega_1, \omega_2^0\right) \frac{\eta_1^-\left(\omega_1\right)}{\eta_1^-\left(\omega_1\right) + \eta_1^+\left(\omega_2^0\right)} P_1\left(\omega_1\right) P_1\left(\omega_2^0\right). \qquad (73)$$

$$Q^{\varepsilon}\left(\omega_1^0\right) = \left(1-\varepsilon\right) \frac{\eta_1^+\left(\omega_2^0\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2^0\right)}$$
$$+ \varepsilon \sum_{\omega_2 \in \Omega_1^+, \omega_2 \neq \omega_2^0} \alpha_0^{\varepsilon}\left(\omega_1^0, \omega_2\right) \frac{\eta_1^+\left(\omega_2\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1^0\right) P_1\left(\omega_2\right), \qquad (74)$$

$$Q^{\varepsilon}\left(\omega_2^0\right) = \left(1-\varepsilon\right) \frac{\eta_1^-\left(\omega_1^0\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2^0\right)}$$
$$+ \varepsilon \sum_{\omega_1 \in \Omega_1^-, \omega_1 \neq \omega_1^0} \alpha_0^{\varepsilon}\left(\omega_1, \omega_2^0\right) \frac{\eta_1^-\left(\omega_1\right)}{\eta_1^-\left(\omega_1\right) + \eta_1^+\left(\omega_2^0\right)} P_1\left(\omega_1\right) P_1\left(\omega_2^0\right). \qquad (75)$$

If $\omega_1 \neq \omega_1^0, \omega_2 \neq \omega_2^0$, then

$$Q^{\varepsilon}\left(\omega_1\right) = \varepsilon \sum_{\omega_2 \in \Omega_1^+} \alpha_0^{\varepsilon}\left(\omega_1, \omega_2\right) \frac{\eta_1^+\left(\omega_2\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1\right) P_1\left(\omega_2\right), \qquad (76)$$

$$Q^{\varepsilon}\left(\omega_2\right) = \varepsilon \sum_{\omega_1 \in \Omega_1^-} \alpha_0^{\varepsilon}\left(\omega_1, \omega_2\right) \frac{\eta_1^-\left(\omega_1\right)}{\eta_1^-\left(\omega_1\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1\right) P_1\left(\omega_2\right). \qquad (77)$$

The distance between the measures Q^{ε} and $\nu_{\omega_1^0, \omega_2^0}$ is given by the formula

$$\rho\left(Q^{\varepsilon}, \nu_{\omega_1^0, \omega_2^0}\right)$$
$$= \varepsilon + \varepsilon \sum_{\omega_2 \in \Omega_1^+, \omega_2 \neq \omega_2^0} \alpha_0^{\varepsilon}\left(\omega_1^0, \omega_2\right) \frac{\eta_1^+\left(\omega_2\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1^0\right) P_1\left(\omega_2\right)$$
$$+ \varepsilon \sum_{\omega_1 \in \Omega_1^-, \omega_1 \neq \omega_1^0} \alpha_0^{\varepsilon}\left(\omega_1, \omega_2^0\right) \frac{\eta_1^-\left(\omega_1\right)}{\eta_1^-\left(\omega_1\right) + \eta_1^+\left(\omega_2^0\right)} P_1\left(\omega_1\right) P_1\left(\omega_2^0\right) \qquad (78)$$
$$+ \varepsilon \sum_{\omega_1 \in \Omega_1^-, \omega_1 \neq \omega_1^0} \sum_{\omega_2 \in \Omega_1^+} \alpha_0^{\varepsilon}\left(\omega_1, \omega_2\right) \frac{\eta_1^+\left(\omega_2\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1\right) P_1\left(\omega_2\right)$$
$$+ \varepsilon \sum_{\omega_2 \in \Omega_1^+, \omega_2 \neq \omega_2^0} \sum_{\omega_1 \in \Omega_1^-} \alpha_0^{\varepsilon}\left(\omega_1, \omega_2\right) \frac{\eta_1^-\left(\omega_1\right)}{\eta_1^-\left(\omega_1\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1\right) P_1\left(\omega_2\right).$$

Since

$$\sum_{\omega_2 \in \Omega_1^+, \omega_2 \neq \omega_2^0} \alpha_0^{\varepsilon}\left(\omega_1^0, \omega_2\right) \frac{\eta_1^+\left(\omega_2\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1^0\right) P_1\left(\omega_2\right)$$
$$+ \sum_{\omega_1 \in \Omega_1^-, \omega_1 \neq \omega_1^0} \alpha_0^{\varepsilon}\left(\omega_1, \omega_2^0\right) \frac{\eta_1^-\left(\omega_1\right)}{\eta_1^-\left(\omega_1\right) + \eta_1^+\left(\omega_2^0\right)} P_1\left(\omega_1\right) P_1\left(\omega_2^0\right) \leq 1,$$

$$\sum_{\omega_1 \in \Omega_1^-, \omega_1 \neq \omega_1^0} \sum_{\omega_2 \in \Omega_1^+} \alpha_0^{\varepsilon}\left(\omega_1, \omega_2\right) \frac{\eta_1^+\left(\omega_2\right)}{\eta_1^-\left(\omega_1^0\right) + \eta_1^+\left(\omega_2\right)} P_1\left(\omega_1\right) P_1\left(\omega_2\right) \leq 1,$$

$$\sum_{\omega_2 \in \Omega_1^+, \omega_2 \neq \omega_2^0} \sum_{\omega_1 \in \Omega_1^-} \alpha_0^\varepsilon (\omega_1, \omega_2) \frac{\eta_1^- (\omega_1)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P_1(\omega_2) \leq 1,$$

we obtain

$$\rho \left(Q^\varepsilon, \nu_{\omega_1^0, \omega_2^0} \right) \leq 4\varepsilon.$$

Let us prove the second part of Lemma 6. It is evident that the inequality

$$\left| E^{Q^\varepsilon} f - E^{\nu_{\omega_1^0, \omega_2^0}} f \right| \leq 4\varepsilon \sup_{\omega \in \Omega_1} |f(\omega)| \tag{79}$$

is true. Due to arbitrariness of the small ε, Lemma 6 is proved. $\qquad\square$

Definition 4. *Let $\{\Omega_1, \mathcal{F}_1\}$ be a measurable space. The decomposition $A_{n,k}, n, k = \overline{1, \infty}$, of the space Ω_1 we call exhaustive one if the following conditions are valid:*

1) $A_{n,k} \in \mathcal{F}_1, A_{n,k} \bigcap A_{n,s} = \varnothing, k \neq s, \bigcup_{k=1}^{\infty} A_{n,k} = \Omega_1, n = \overline{1, \infty}$;

2) *the $(n+1)$-th decomposition is a sub-decomposition of the n-th one, that is, for every j, $A_{n+1, j} \subseteq A_{n,k}$ for a certain $k = k(j)$;*

3) *the minimal σ-algebra containing all $A_{n,k}, n, k = \overline{1, \infty}$, coincides with $\mathcal{F}_1$.*

The next Remark 1 is important for the construction of the filtration having the exhaustive decomposition.

Remark 1. *Suppose that the measurable spaces $\{\Omega_1, \mathcal{F}_1\}$ and $\{\Omega_2, \mathcal{F}_2\}$ have the exhaustive decompositions $A_{n,k}^1, n, k = \overline{1, \infty}$ and $A_{m,s}^2, m, s = \overline{1, \infty}$, respectively, then the measurable space $\{\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2\}$ also have the exhaustive decomposition $B_{n,ks}, n = \overline{1, \infty}, k, s = \overline{1, \infty}$, $B_{n,ks} = A_{n,k}^1 \times A_{n,s}^2, k, s = \overline{1, \infty}, n = \overline{1, \infty}$. Really,*

1) $A_{n,k}^1 \times A_{n,s}^2 \in \mathcal{F}_1 \times \mathcal{F}_2, A_{n,k}^1 \times A_{n,s}^2 \bigcap A_{n,t}^1 \times A_{n,r}^2 = \varnothing, (k,s) \neq (t,r)$,

$$\bigcup_{k,s=1}^{\infty} B_{n,ks} = \Omega_1 \times \Omega_2, n = \overline{1, \infty};$$

2) *the $(n+1)$-th decomposition is a sub-decomposition of the n-th one, that is, for every k, s $B_{n+1,ks} \subseteq B_{n,ij}$ for a certain $i = i(k), j = j(s)$;*

3) *the minimal σ-algebra containing all $B_{n,ks}, n, k, s = \overline{1, \infty}$, coincides with $F_1 \times F_2$.*

In the next Lemma we give the sufficient condition of the existence of exhaustive decomposition.

Lemma 7. *Let $\{\Omega_1, \mathcal{F}_1\}$ be a measurable space with a complete separable metric space Ω_1 and Borel σ-algebra $\mathcal{F}_1$ on it. Then $\{\Omega_1, \mathcal{F}_1\}$ has an exhaustive decomposition.*

Proof. If $\{\omega_1, \cdots, \omega_n, \cdots\}$ is a countable dense set in Ω_1, then we denote

$$B(\omega_n, \varepsilon_m) = \{\omega \in \Omega_1, \rho(\omega, \omega_n) < \varepsilon_m\}, \quad n, m = \overline{1, \infty}, \tag{80}$$

the countable set of open balls as ε_m runs all positive rational numbers, where $\rho(\omega_1, \omega_2), \omega_1, \omega_2 \in \Omega_1$ is a metric in Ω_1. Prove that $\mathcal{F}_1 = \sigma\left(B(\omega_n, \varepsilon_m), n, m = \overline{1, \infty}\right)$, where $\sigma\left(B(\omega_n, \varepsilon_m), n, m = \overline{1, \infty}\right)$ is a minimal σ-algebra generated by the sets (80). For this purpose let us prove that for every open set $A \in \Omega_1$ the representation

$$A = \bigcup_{n_k \in N_1, m_s \in Q_+^1} B\left(\omega_{n_k}, \varepsilon_{m_s}\right) \tag{81}$$

is true, where N_1 is a subset of positive integers, and Q_+^1 is a subset of positive rational numbers. Let us denote $\left\{\omega_1^A, \cdots, \omega_n^A, \cdots\right\} = A \bigcap \left\{\omega_1, \cdots, \omega_n, \cdots\right\}$. Suppose that $\omega_0 \in A$, then $d = \inf_{\omega \in \overline{A} \setminus A} \rho(\omega_0, \omega) > 0$, where $\overline{A}$ is a closure of the set A. Let the point $\omega_{k_0}^A$ belong to the ball $C\left(\omega_0, \dfrac{d}{8}\right) = \left\{\omega \in \Omega_1, \rho(\omega_0, \omega) < \dfrac{d}{8}\right\}$ and let us consider the ball

$C\left(\omega_{k_0}, \dfrac{d}{8} + \rho\left(\omega_0, \omega_{k_0}^A\right)\right) = \left\{\omega \in \Omega_1, \rho\left(\omega_{k_0}, \omega\right) < \dfrac{d}{8} + \rho\left(\omega_0, \omega_{k_0}^A\right)\right\}$. The point ω_0

belongs to this ball and for every $\omega \in C\left(\omega_{k_0}, \dfrac{d}{8} + \rho\left(\omega_0, \omega_{k_0}^A\right)\right)$ the inequality

$$\rho(\omega_0, \omega) \le \rho\left(\omega_0, \omega_{k_0}^A\right) + \rho\left(\omega_{k_0}^A, \omega\right) < \frac{d}{8} + 2\rho\left(\omega_0, \omega_{k_0}^A\right) < \frac{3d}{8} \tag{82}$$

is true. Therefore $C\left(\omega_{k_0}, \dfrac{d}{8} + \rho\left(\omega_0, \omega_{k_0}^A\right)\right) \subset C\left(\omega_0, \dfrac{3d}{8}\right)$. Let the rational number ε_{k_0} satisfies the inequalities

$$\frac{d}{8} + 2\rho\left(\omega_0, \omega_{k_0}^A\right) < \varepsilon_{k_0} < \frac{3d}{8}, \tag{83}$$

then $C\left(\omega_{k_0}^A, \varepsilon_{k_0}\right) \subseteq C\left(\omega_0, \dfrac{d}{2}\right)$, since for every $\omega \in C\left(\omega_{k_0}^A, \varepsilon_{k_0}\right)$,

$\rho(\omega_0, \omega) \le \rho\left(\omega_0, \omega_{k_0}\right) + \rho\left(\omega_{k_0}, \omega\right) < \dfrac{d}{8} + \varepsilon_{k_0} < \dfrac{d}{2}$. So, for $\omega_0 \in A$ we found $\omega_{k_0} \in \left\{\omega_1, \cdots, \omega_n, \cdots\right\}$ and the rational number ε_{k_0} such that

$\omega_0 \in C\left(\omega_{k_0}^A, \varepsilon_{k_0}\right) \subset C\left(\omega_0, \dfrac{d}{2}\right) \subset A$. The last prove the needed statement. To complete the proof of Lemma 7 let us construct the exhaustive decomposition. Let us renumber the sets $B(\omega_n, \varepsilon_m)$ putting by $D_1 = B(\omega_1, \varepsilon_1)$, $D_2 = B(\omega_1, \varepsilon_2)$, $D_3 = B(\omega_2, \varepsilon_1)$, and so on. We put that $\left\{A_{1k}\right\}_{k=1}^\infty$ consists of two sets D_1 and $\overline{D}_1 = \Omega_1 \setminus D_1$. If the set $\left\{A_{nk}\right\}_{k=1}^\infty$ is constructed, then the set $\left\{A_{n+1k}\right\}_{k=1}^\infty$ we construct from the various set of the kind $A_{nk} \bigcap D_{n+1}, A_{nk} \bigcap \overline{D}_{n+1}$. By construction the minimal σ-algebra $\sigma\left\{A_{nk}, n, k = \overline{1, \infty}\right\} = \sigma\left\{B(\omega_n, \varepsilon_m), n, m = \overline{1, \infty}\right\}$. Taking into account the previous part of the proof we have $\sigma\left\{A_{nk}, n, k = \overline{1, \infty}\right\} = \mathcal{F}_1$. Lemma 7 is proved. $\square$

Lemma 8. *Let a measurable space* $\{\Omega, \mathcal{F}\}$ *have an exhaustive decomposition and let* ξ *be an integrable random value relative to the measure P, satisfying the conditions* (36). *Then, the closure of the set of measure Q, given by the formula* (42), *relative to the pointwise convergence of measures contains the set of measures*

$$\begin{aligned} v_{\omega_1, \omega_2}(A) &= \chi_A(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \\ &\quad + \chi_A(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}, \quad A \in \mathcal{F}, \ (\omega_1, \omega_2) \in \Omega^- \times \Omega^+, \end{aligned} \tag{84}$$

for those $(\omega_1, \omega_2) \in \Omega^- \times \Omega^+$ *which have the full measure* $\mu = P^- \times P^+$ *For every integrable finite valued random value* $f(\omega)$ *relative to all measures* Q, *the closure in metrics* $\rho(x_1, x_2) = |x_1 - x_2|$, $x_1, x_2 \in R^1$, *of the set of real numbers*

$$
\begin{aligned}
E^Q f &= \int_{\Omega^-} \int_{\Omega^+} f(\omega_1) \alpha(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \, d\mu(\omega_1, \omega_2) \\
&+ \int_{\Omega^-} \int_{\Omega^+} f(\omega_2) \alpha(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \, d\mu(\omega_1, \omega_2),
\end{aligned}
\tag{85}
$$

when $\alpha(\omega_1, \omega_2)$ *runs over all random values satisfying the conditions* (39), (41), *contains the set of numbers*

$$
f(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} + f(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}, \quad (\omega_1, \omega_2) \in \Omega^- \times \Omega^+. \tag{86}
$$

Proof. On a probability space $\{\Omega, \mathcal{F}, P\}$, let ξ be an integrable random value, satisfying the conditions (36). As before, let $\Omega^+ = \{\omega, \xi(\omega) > 0\}$, $\Omega^- = \{\omega, \xi(\omega) \leq 0\}$ and let $\mathcal{F}^-, \mathcal{F}^+$ be the restrictions of the σ-algebra $\mathcal{F}$ on the sets Ω^- and Ω^+, correspondingly. Suppose that P^- and P^+ are the contractions of the measure P on the σ-algebras $\mathcal{F}^-, \mathcal{F}^+$, correspondingly. Consider the probability space $\{\Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+, P^- \times P^+\}$ which is a direct product of the probability spaces $\{\Omega^-, \mathcal{F}^-, P^-\}$ and $\{\Omega^+, \mathcal{F}^+, P^+\}$. Due to Lemma 8 conditions and Remark 1, the measurable space $\{\Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+\}$ has the exhaustive decomposition $B_{n,ks}, k, s = \overline{1, \infty}, n = \overline{1, \infty}$. Denote $\mathcal{F}_n$ the minimal σ-algebra generated by decomposition $B_{n,ks}, k, s = \overline{1, \infty}$. It is evident that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Moreover, $\sigma\left(\bigvee_{n=1}^{\infty} \mathcal{F}_n\right) = \mathcal{F}^- \times \mathcal{F}^+$. On the probability space $\{\Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+, P^- \times P^+\}$, for every integrable finite valued random value $f(\omega_1, \omega_2)$ the sequence $E^\mu\{f(\omega_1, \omega_2) | \mathcal{F}_n\}$ converges to $f(\omega_1, \omega_2)$ with probability one, as $n \to \infty$, since it is a regular martingale. It is evident that for those $B_{n,ks}$ for which $\mu(B_{n,ks}) \neq 0$

$$
E^\mu\{f(\omega_1, \omega_2) | \mathcal{F}_n\} = \frac{\int_{B_{n,ks}} f(\omega_1, \omega_2) \, d\mu}{\mu(B_{n,ks})}, \quad (\omega_1, \omega_2) \in B_{n,ks}. \tag{87}
$$

Denote $D_0 = \bigcup_{n,k,s, \mu(B_{n,ks})=0} B_{n,ks}$. It is evident that $\mu(D_0) = 0$. For every $(\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus D_0$, the formula (87) is well defined and is finite. Let D_1 be the subset of the set $\Omega^- \times \Omega^+ \setminus D_0$, where the limit of the left hand side of the formula (87) does not exist. Then, $\mu(D_1) = 0$. For every $(\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus (D_0 \bigcup D_1)$, the right hand side of the formula (87) converges to $f(\omega_1, \omega_2)$. For $(\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus (D_0 \bigcup D_1)$, denote $A_n = A_n(\omega_1, \omega_2)$ those set $B_{n,ks}$ for which $(\omega_1, \omega_2) \in B_{n,ks}$ for a certain k, s. Then, for every integrable finite valued $f(\omega_1, \omega_2)$

$$\lim_{n \to \infty} \frac{\int_{A_n} f(\omega_1, \omega_2) \mathrm{d}\mu}{\mu(A_n)} = f(\omega_1, \omega_2). \tag{88}$$

Let us consider the sequence

$$\alpha_n^{\varepsilon_n}(\omega_1, \omega_2) = (1 - \varepsilon_n) \frac{\chi_{A_n}(\omega_1, \omega_2)}{\mu(A_n)} + \varepsilon_n \frac{\chi_{\Omega^- \times \Omega^+ \setminus A_n}(\omega_1, \omega_2)}{\mu(\Omega^- \times \Omega^+ \setminus A_n)}, \tag{89}$$

where $0 < \varepsilon_n < 1$, $\lim_{n \to \infty} \varepsilon_n = 0$. Such a sequence $\alpha_n^{\varepsilon_n}(\omega_1, \omega_2)$ satisfies the conditions (39)-(41) and

$$Q_n^{\varepsilon_n}(A) = \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_1) \alpha_n^{\varepsilon_n}(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)$$

$$+ \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_2) \alpha_n^{\varepsilon_n}(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)$$

$$= (1 - \varepsilon_n) \frac{\int_{A_n} \chi_A(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)}{\mu(A_n)}$$

$$+ (1 - \varepsilon_n) \frac{\int_{A_n} \chi_A(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)}{\mu(A_n)}$$

$$+ \varepsilon_n \frac{\int_{\Omega^- \times \Omega^+ \setminus A_n} \chi_A(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)}{\mu(\Omega^- \times \Omega^+ \setminus A_n)}$$

$$+ \varepsilon_n \frac{\int_{\Omega^- \times \Omega^+ \setminus A_n} \chi_A(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)}{\mu(\Omega^- \times \Omega^+ \setminus A_n)}. \tag{90}$$

From the formula (90), we obtain

$$\lim_{n \to \infty} Q_n^{\varepsilon_n}(A) = \chi_A(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} + \chi_A(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}, \tag{91}$$

$A \in \mathcal{F}, (\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus (D_0 \bigcup D_1)$.

Further,

$$E^{Q_n^{\varepsilon_n}} f(\omega) = \int_{\Omega^-} \int_{\Omega^+} f(\omega_1) \alpha_n^{\varepsilon_n}(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)$$

$$+ \int_{\Omega^-} \int_{\Omega^+} f(\omega_2) \alpha_n^{\varepsilon_n}(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)$$

$$= (1 - \varepsilon_n) \frac{\int_{A_n} f(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)}{\mu(A_n)}$$

$$+ (1 - \varepsilon_n) \frac{\int_{A_n} f(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)}{\mu(A_n)}$$

$$+ \varepsilon_n \frac{\displaystyle\int_{\Omega^- \times \Omega^+ \setminus A_n} f(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)}{\mu(\Omega^- \times \Omega^+ \setminus A_n)}$$

$$+ \varepsilon_n \frac{\displaystyle\int_{\Omega^- \times \Omega^+ \setminus A_n} f(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)}{\mu(\Omega^- \times \Omega^+ \setminus A_n)}. \tag{92}$$

From the formula (92), we obtain

$$\lim_{n \to \infty} E^{Q_n^{\varepsilon_n}} f(\omega) = f(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} + f(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}, \tag{93}$$

$$(\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus (D_0 \cup D_1).$$

Lemma 8 is proved. □

The next Theorem 5 is a consequence of Lemma 5.

Theorem 5. *On the probability space* $\{\Omega, \mathcal{F}, P\}$, *for the nonnegative random value* $\xi \neq 1$ *the set of measures* M_0 *on the measurable space* $\{\Omega, \mathcal{F}\}$, *being equivalent to the measure P, satisfies the condition*

$$E^Q \xi = 1, \ Q \in M_0, \tag{94}$$

if and only if as for $Q \in M_0$ *the representation*

$$Q(A) = \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_1) \alpha(\omega_1, \omega_2) \frac{(\xi - 1)^+(\omega_2)}{(\xi - 1)^-(\omega_1) + (\xi - 1)^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2)$$

$$+ \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_2) \alpha(\omega_1, \omega_2) \frac{(\xi - 1)^-(\omega_1)}{(\xi - 1)^-(\omega_1) + (\xi - 1)^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2), \ A \in \mathcal{F}, \tag{95}$$

is true, where on the measurable space $\{\Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+, P^- \times P^+\}$, *the random value* $\alpha(\omega_1, \omega_2)$ *satisfies the conditions*

$$\mu(\{(\omega_1, \omega_2) \in \Omega^- \times \Omega^+, \alpha(\omega_1, \omega_2) > 0\}) = P(\Omega^+) P(\Omega^-), \tag{96}$$

$$\int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) \frac{(\xi - 1)^-(\omega_1)(\xi - 1)^+(\omega_2)}{(\xi - 1)^-(\omega_1) + (\xi - 1)^+(\omega_2)} \mathrm{d}\mu(\omega_1, \omega_2) < \infty, \tag{97}$$

$$\int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) \mathrm{d}\mu(\omega_1, \omega_2) = 1. \tag{98}$$

We introduced above the following denotations: $\mu = P^- \times P^+$, P^- is a contraction of the measure P on the set $\Omega^- = \{\omega \in \Omega, \xi - 1 \le 0\}$, P^+ is a contraction of the measure P on the set $\Omega^+ = \{\omega \in \Omega, \xi - 1 > 0\}$, $\mathcal{F}^- = \Omega^- \cap \mathcal{F}$, $\mathcal{F}^+ = \Omega^+ \cap \mathcal{F}$.

It is evident that the set of measure M_0 is a nonempty one, since it contains those measures Q, for which the random value $\alpha(\omega_1, \omega_2)$ is bounded, since $E^Q |\xi - 1| < \infty$. The set of measure M_0 is consistent with the filtration $\{\mathcal{F}_0, \mathcal{F}\}$ on the measurable space $\{\Omega, \mathcal{F}\}$, where $\mathcal{F}_0 = \{\varnothing, \Omega\}$.

Theorem 6. *On the probability space* $\{\Omega, \mathcal{F}, P\}$ *with the filtration* $\mathcal{F}_n$ *on it, the set of measures* M_0, *given by the formula (95), is consistent with the filtration* $\mathcal{F}_n$, *if and only if, as* $E^Q \{\xi \mid \mathcal{F}_n\}, Q \in M_0$, *is a local regular martingale.*

Proof. The necessity. Let the set of measures M_0 be consistent with the filtration. Then, due to Theorem 3, $E^Q\{\xi\,|\,\mathcal{F}_n\}, Q\in M_0$, is a local regular martingale.

The sufficiency. Suppose that $E^Q\{\xi\,|\,\mathcal{F}_n\}, Q\in M_0$, is a local regular martingale. Let us prove that, if $Q_1, Q_2 \in M_0$, then the set of measures

$$R_s^k(A) = \int_A \frac{E^{Q_2}\left\{\dfrac{dQ_1}{dQ_2}\,|\,\mathcal{F}_k\right\}}{E^{Q_2}\left\{\dfrac{dQ_1}{dQ_2}\,|\,\mathcal{F}_s\right\}}\,dQ_2, \quad A\in\mathcal{F},\ k\geq s\geq n,\ n=\overline{0,\infty}, \tag{99}$$

belongs to the set M_0. For this, it is to prove that $E^{R_s^k}(\xi-1) = 0$, or $E^{R_s^k}\xi = 1$. Really, if $E^{Q_1}\xi = 1, E^{Q_2}\xi = 1$, then

$$E^{R_s^k}\xi = E^{Q_2}\xi\,\frac{E^{Q_2}\left\{\dfrac{dQ_1}{dQ_2}\,|\,\mathcal{F}_k\right\}}{E^{Q_2}\left\{\dfrac{dQ_1}{dQ_2}\,|\,\mathcal{F}_s\right\}} = E^{Q_2}E^{Q_2}\{\xi\,|\,\mathcal{F}_k\}\,\frac{\dfrac{dQ_1}{dQ_2}}{E^{Q_2}\left\{\dfrac{dQ_1}{dQ_2}\,|\,\mathcal{F}_s\right\}}$$

$$= E^{Q_2}E^{Q_2}\left\{E^{Q_2}\{\xi\,|\,\mathcal{F}_k\}\,\frac{\dfrac{dQ_1}{dQ_2}}{E^{Q_2}\left\{\dfrac{dQ_1}{dQ_2}\,|\,\mathcal{F}_s\right\}}\,|\,\mathcal{F}_s\right\} \tag{100}$$

$$= E^{Q_2}E^{Q_1}\{E^{Q_2}\{\xi\,|\,\mathcal{F}_k\}\,|\,\mathcal{F}_s\} = E^{Q_2}E^{Q_1}\{E^{Q_1}\{\xi\,|\,\mathcal{F}_k\}\,|\,\mathcal{F}_s\}$$

$$= E^{Q_2}E^{Q_1}\{\xi\,|\,\mathcal{F}_s\} = E^{Q_2}E^{Q_2}\{\xi\,|\,\mathcal{F}_s\} = E^{Q_2}\xi = 1.$$

Theorem 6 is proved. $\qquad\square$

Theorem 7. *On the probability space* $\{\Omega,\mathcal{F},P\}$ *with the filtration* $\mathcal{F}_n$ *on it, the set of measures* M_0*, given by the formula* (95)*, is consistent with the filtration* $\mathcal{F}_n$*, if and only if there exists not depending on* $(\omega_1,\omega_2)\in\Omega^-\times\Omega^+$ *the random process* $\{m_n,\mathcal{F}_n\}_{n=0}^{\infty}$ *such that*

$$E^{\nu_{\omega_1,\omega_2}}\{\xi\,|\,\mathcal{F}_n\} = m_n, \quad n=\overline{1,\infty}, \tag{101}$$

for those $(\omega_1,\omega_2)\in\Omega^-\times\Omega^+$ *that have the full measure* $\mu = P^-\times P^+$*, where*

$$\nu_{\omega_1,\omega_2}(A) = \chi_A(\omega_1)\frac{(\xi-1)^+(\omega_2)}{(\xi-1)^-(\omega_1)+(\xi-1)^+(\omega_2)}$$

$$+ \chi_A(\omega_2)\frac{(\xi-1)^-(\omega_1)}{(\xi-1)^-(\omega_1)+(\xi-1)^+(\omega_2)}, \quad A\in\mathcal{F},\ (\omega_1,\omega_2)\in\Omega^-\times\Omega^+. \tag{102}$$

Proof. The necessity. Suppose that the set of measures M_0, given by the formula (95), is consistent with the filtration $\mathcal{F}_n$. Due to Theorem 6, $E^Q\{\xi\,|\,\mathcal{F}_n\}, Q\in M_0$, is a local regular martingale. Then, $E^Q\{\xi\,|\,\mathcal{F}_n\} = m_n$. Using Lemma 8, we obtain $E^{\nu_{\omega_1,\omega_2}}\{\xi\,|\,\mathcal{F}_n\} = m_n$ for those $(\omega_1,\omega_2)\in\Omega^-\times\Omega^+$ that have the full measure μ.

The sufficiency. If the formula (101) is true, then $E^Q\{\xi\,|\,\mathcal{F}_n\} = m_n, Q\in M_0$. From this, it follows that $E^Q\{\xi\,|\,\mathcal{F}_n\}, Q\in M_0$, is a local regular martingale. Theorem 7 is proved. $\qquad\square$

Definition 5. *On the probability space* $\{\Omega, \mathcal{F}, P\}$ *with the filtration* $\mathcal{F}_n$ *on it, the consistent with the filtration* $\mathcal{F}_n$ *subset of the measures M of the set of the measures* M_0 *generating by the nonnegative random value* $\xi \neq 1$, $E^Q \xi = 1, Q \in M_0$, *we call the regular set of measures.*

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space. On the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it, let $M \subseteq M_0$ be a set of regular measures, where the set M_0 is generated by the nonnegative random value $\xi \neq 1$. Denote by $\{m_n, \mathcal{F}_n\}_{n=0}^{\infty}$ the regular martingale, where $m_n = E^Q \{\xi \mid \mathcal{F}_n\}, Q \in M, n = \overline{1, \infty}$. Assume that M_n is a contraction of the set of regular measures M onto the σ-algebra $\mathcal{F}_n$. Every $Q^n \in M_n$ is equivalent to P_n, where P_n is a contraction of the measure P on the σ-algebra $\mathcal{F}_n$. For every $Q^n \in M_n$, we have $E^{Q^n} [m_n - m_{n-1}] = 0$. Therefore, for the measure $Q^n \in M_n$ the representation

$$
\begin{aligned}
Q^n(A) = &\int_{\Omega_n^- \times \Omega_n^+} \chi_A(\omega_1) \frac{\alpha_n(\omega_1, \omega_2)[m_n - m_{n-1}]^+(\omega_2)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} d\mu_n(\omega_1, \omega_2) \\
&+ \int_{\Omega_n^- \times \Omega_n^+} \chi_A(\omega_2) \frac{\alpha_n(\omega_1, \omega_2)[m_n - m_{n-1}]^-(\omega_1)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} d\mu_n(\omega_1, \omega_2), \quad A \in \mathcal{F}_n,
\end{aligned}
\tag{103}
$$

$$
\Omega_n^- = \left\{ \omega_1 \in \Omega, [m_n - m_{n-1}](\omega_1) \le 0 \right\},
$$

$$
\Omega_n^+ = \left\{ \omega_2 \in \Omega, [m_n - m_{n-1}](\omega_2) > 0 \right\},
$$

is true, where, on the measurable space $\{\Omega_n^- \times \Omega_n^+, \mathcal{F}_n^- \times \mathcal{F}_n^+\}$, the random value $\alpha_n(\omega_1, \omega_2)$ satisfies the conditions

$$
\mu_n\left(\left\{(\omega_1, \omega_2) \in \Omega_n^- \times \Omega_n^+, \alpha_n(\omega_1, \omega_2) > 0\right\}\right) = P_n(\Omega^+) P_n(\Omega^-),
\tag{104}
$$

$$
\int_{\Omega_n^-} \int_{\Omega_n^+} \alpha_n(\omega_1, \omega_2) \frac{[m_n - m_{n-1}]^-(\omega_1)[m_n - m_{n-1}]^+(\omega_2)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} d\mu_n(\omega_1, \omega_2) < \infty,
\tag{105}
$$

$$
\int_{\Omega_n^-} \int_{\Omega_n^+} \alpha_n(\omega_1, \omega_2) d\mu_n(\omega_1, \omega_2) = 1.
\tag{106}
$$

Here, the measure $\mu_n = P_n^- \times P_n^+$ is given on the measurable space $\{\Omega_n^- \times \Omega_n^+, \mathcal{F}_n^- \times \mathcal{F}_n^+\}$ and it is a direct product of the measures P_n^- and P_n^+, where the measure P_n^+ is a contraction of the measure P_n on the σ-algebra $\mathcal{F}_n^+ = \Omega_n^+ \bigcap \mathcal{F}_n$ and P_n^- is a contraction of the measure P_n on the σ-algebra $\mathcal{F}_n^- = \Omega_n^- \bigcap \mathcal{F}_n$. It is evident that the regular set of measures M is a convex set of measure.

Definition 6. *On the probability space* $\{\Omega, \mathcal{F}, P\}$ *with the filtration* $\mathcal{F}_n$ *on it, denote by* A_0 *the set of all nonnegative integrable random values* ζ *relative to the set of regular measures M, satisfying the conditions:*

$$
E^P \zeta = 1, \quad P \in M.
\tag{107}
$$

Due to Theorem 3, $\left\{ E^P \{\zeta \mid \mathcal{F}_n\}, \mathcal{F}_n \right\}_{n=0}^{\infty}$ is a regular martingale relative to the set of measures M.

It is evident that the set A_0 is a nonempty one, since it contains the random value $\zeta = 1$. The more interesting case is as A_0 contains more than one ele-

ment. So, further we consider the regular set of measure M with the set A_0, containing more than one element.

In the next Lemma 9, using Lemma 5, we construct a set of measures consistent with the filtration. On the probability space $\left\{\Omega_1^0, \mathcal{F}_1^0, P_1^0\right\}$, let us consider a nonnegative random value ξ_1, satisfying the conditions

$$0 < P_1^0\left(\left\{\omega_1 \in \Omega_1^0, \eta_1(\omega_1) < 0\right\}\right) < 1,$$

$$0 < P_1^0\left(\left\{\omega_1 \in \Omega_1^0, \eta_1(\omega_1) > 0\right\}\right), \tag{108}$$

where we introduced the denotation $\eta_1(\omega_1) = \xi_1(\omega_1) - 1$. Described in Lemma 5 the set of equivalent measures to the measure P_1^0 and such that $E^Q \eta_1(\omega_1) = 0$, we denote by M_1. Let us construct the infinite direct product of the measurable spaces $\left\{\Omega_i^0, \mathcal{F}_i^0\right\}, i = \overline{1, \infty}$, where $\Omega_i^0 = \Omega_1^0, \mathcal{F}_i^0 = \mathcal{F}_1^0$. Denote $\Omega = \prod_{i=1}^{\infty} \Omega_i^0$. On the space Ω, under the σ-algebra $\mathcal{F}$ we understand the minimal σ-algebra, generated by the sets $\prod_{i=1}^{\infty} G_i, G_i \in F_i^0$, where in the last product only the finite set of G_i do not equal Ω_i^0. On the measurable space $\left\{\Omega, \mathcal{F}\right\}$, under the filtration $\mathcal{F}_n$ we understand the minimal σ-algebra generated by the sets $\prod_{i=1}^{\infty} G_i, G_i \in \mathcal{F}_i^0$, where $G_i = \Omega_i^0$ for $i > n$. We consider the probability space $\left\{\Omega, \mathcal{F}, P\right\}$, where $P = \prod_{i=1}^{\infty} P_i^0, P_i^0 = P_1^0, i = \overline{1, \infty}$.

On the measurable space $\left\{\Omega, \mathcal{F}\right\}$, we introduce into consideration the set of measures M, where Q belongs to M, if $Q = \prod_{i=1}^{\infty} Q_i, Q_i \in M_1$. We denote by M^{Q_0} a subset of the set M of those measures $Q = \prod_{i=1}^{\infty} Q_i, Q_i \in M_1$, for which only the finite set of Q_i does not coincide with the measure $Q_0 \in M_1$.

Lemma 9. *On the measurable space* $\left\{\Omega, \mathcal{F}\right\}$ *with the filtration* $\mathcal{F}_n$ *on it, there exists consistent with the filtration* $\mathcal{F}_n$ *the set of measures* M_0 *and the nonnegative random variable* ξ_0 *such that* $E^Q \xi_0 = 1, Q \in M_0$, *if the random value* ξ_1, *satisfying the conditions* (108), *is bounded.*

Proof. To prove Lemma 9, we need to construct a nonnegative bounded random value ξ_0 on the measurable space $\left\{\Omega, \mathcal{F}\right\}$ and a set of equivalent measures M_0 on it, such that $E^Q \xi_0 = 1, Q \in M_0$, and to prove that the set of measures M_0 is consistent with the filtration $\mathcal{F}_n$. From the Lemma 9 conditions, the random value $\eta_1(\omega_1) = \xi_1(\omega_1) - 1$ is also bounded. Let us put

$$\xi_0 = \prod_{i=1}^{\infty}\left[1 + a_i(\omega_1, \cdots, \omega_{i-1})\eta_i(\omega_i)\right], \tag{109}$$

where the random values $a_i(\omega_1, \cdots, \omega_{i-1})$ are $\mathcal{F}_{i-1}$-measurable, $i = \overline{1, \infty}$, they satisfy the conditions $0 < a_i(\omega_1, \cdots, \omega_{i-1}) \le b_i < 1$. The constants b_i are such that $\sum_{i=1}^{\infty} b_i < \infty$, the random value $\eta_i(\omega_i)$ is given on $\left\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\right\}$ and is

distributed as $\eta_1(\omega_1)$ on $\{\Omega_1^0, \mathcal{F}_1^0, P_1^0\}$. From this, it follows that the random value ξ_0 is bounded by the constant $\prod_{i=1}^{\infty}[1+Cb_i]$, where $C>0$ and it is such that $|\eta_i(\omega_i)| < C, i = \overline{1,\infty}$. It is evident that $E^Q \xi_0 = 1, Q \in M^{Q_0}$. Really,

$$E^Q \prod_{i=1}^{n}\left[1+a_i(\omega_1,\cdots,\omega_{i-1})\eta_i(\omega_i)\right]$$
$$= E^{Q_0^{n-1}} \prod_{i=1}^{n-1}\left[1+a_i(\omega_1,\cdots,\omega_{i-1})\eta_i(\omega_i)\right] \tag{110}$$
$$\times E^{Q_n}\left[1+a_{n-1}(\omega_1,\cdots,\omega_{i-1})\eta_n(\omega_n)\right],$$

where $Q = \prod_{i=1}^{\infty}Q_i, Q_0^{n-1} = \prod_{i=1}^{n-1}Q_i$,

$$E^{Q_n}\left[1+a_{n-1}(\omega_1,\cdots,\omega_{n-1})\eta_i(\omega_n)\right]$$
$$= \left[1+a_{n-1}(\omega_1,\cdots,\omega_{n-1})E^{Q_n}\eta_n(\omega_n)\right]=1. \tag{111}$$

From the last equality, we have

$$E^Q \prod_{i=1}^{n}\left[1+a_i(\omega_1,\cdots,\omega_{i-1})\eta_i(\omega_i)\right]=1. \tag{112}$$

Since $\xi_0 = \lim_{n\to\infty}\prod_{i=1}^{n}\left[1+a_i(\omega_1,\cdots,\omega_{i-1})\eta_i(\omega_i)\right]$, from the equality (112) and the possibility to go to the limit under the mathematical expectation, we prove the needed statement. Let us prove the existence of the set of measures M_0 consistent with the filtration $\mathcal{F}_n$. If $Q \in M^{Q_0}$, then

$$E^Q\{\xi_0 \mid \mathcal{F}_n\} = \prod_{i=1}^{n}\left[1+a_i(\omega_1,\cdots,\omega_{i-1})\eta_i(\omega_i)\right], \quad Q \in M^{Q_0}. \tag{113}$$

Due to Lemma 4, there exists a set of measures M_0 such that it is consistent with the filtration and $M_0 \supseteq M^{Q_0}$, $E^Q \xi_0 = 1, Q \in M_0$. The set M_0 is a linear convex span of the set M^{Q_0}. It means that the set of measures M_0 is consistent with the filtration. Lemma 9 is proved. $\quad\square$

Remark 2 *The boundedness of the random value ξ_1 is not essential. For the applications, the case, as $a_i(\omega_1,\cdots,\omega_{i-1})=0, i \geq n+1$, is very important (see Section 8). In this case, Lemma 9 is true as the random value η_1 is an integrable one. The random value ξ_0 is also integrable one relative to every measures from the set M_0 and it is $\mathcal{F}_n$-measurable one.*

Below, we describe one class of evolutions of risk assets satisfying no arbitrage condition [15]-[20] and give the complete description of the set of equivalent martingale measures.

On the introduced measurable space $\{\Omega, \mathcal{F}, P\}$ we consider the evolution of the risk asset given by the law

$$S_n = S_{n-1}\left(1+a_n(\omega_1,\cdots,\omega_{n-1})\right)\eta_n(\omega_n), \quad n = \overline{1,N}, \tag{114}$$

where the random values $a_i(\omega_1,\cdots,\omega_{i-1})$ are $\mathcal{F}_{i-1}$-measurable, $i = \overline{1,N}$, they satisfy the conditions $0 < a_i(\omega_1,\cdots,\omega_{i-1}) \leq 1$, $\eta_1(\omega_1) = \xi_1(\omega_1)-1$, the random

value $\eta_i(\omega_i)$ is given on $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$ and is distributed as $\eta_1(\omega_1)$ on $\{\Omega_1^0, \mathcal{F}_1^0, P_1^0\}$. The main aim is to describe the set of martingale measures for the evolution of risk asset given by the formula (114). This problem we solve in Theorem 8.

Below, we describe completely the regular set of measures in the case as

$$\xi_0 = \prod_{i=1}^{N}\left[1 + a_i(\omega_1, \cdots, \omega_{i-1})\eta_i(\omega_i)\right], N < \infty, \quad 0 < a_i(\omega_1, \cdots, \omega_{i-1}) \le 1, i = \overline{1, N}, \text{ and}$$

the random value ξ_1 is an integrable one relative to the measure P_1^0. For this purpose, we introduce the denotations: $\Omega_1^- = \{\omega_1 \in \Omega_1^0, \eta_1(\omega_1) \le 0\}$, $\Omega_1^+ = \{\omega_1 \in \Omega_1^0, \eta_1(\omega_1) > 0\}$, P_1^- is a contraction of the measure P_1^0 on the σ-algebra $\mathcal{F}_1^-$, P_1^+ is a contraction of the measure P_1^0 on the σ-algebra $\mathcal{F}_1^+$, $\mathcal{F}_1^- = \Omega_1^- \cap \mathcal{F}_1^0$, $\mathcal{F}_1^+ = \Omega_1^+ \cap \mathcal{F}_1^0$.

Denote $U_1 = \Omega_1^- \times \Omega_1^+$ and introduce the measure $\mu_1 = P_1^- \times P_1^+$ on the σ-algebra $G_1 = \mathcal{F}_1^- \times \mathcal{F}_1^+$. Let us introduce the measurable space $\{\mathcal{V}, \mathcal{L}, \mu\}$, where $\mathcal{V} = \prod_{i=1}^{N} U_i, U_i = U_1, i = \overline{1, N}$, is a direct product of the spaces

$$U_i = \Omega_i^- \times \Omega_i^+, \quad \Omega_i^- = \Omega_1^-, \quad \Omega_i^+ = \Omega_1^+, \quad \mathcal{L} = \prod_{i=1}^{N} G_i \text{ is a direct product of the}$$

σ-algebras $G_i = G_1, i = \overline{1, N}$. At last, let $\mu = \prod_{i=1}^{N} \mu_i$ be a direct product of the

measures $\mu_i = \mu_1, i = \overline{1, N}$, and let $\nu_v = \prod_{i=1}^{N} \nu_{\omega_i^1, \omega_i^2}, v = \left\{\left(\omega_1^1, \omega_1^2\right), \cdots, \left(\omega_N^1, \omega_N^2\right)\right\}$,

be a direct product of the measures $\nu_{\omega_i^1, \omega_i^2}, i = \overline{1, N}$, which is a countable additive function on the σ-algebra $\mathcal{F}_N$ for every $v \in \mathcal{V}$, where

$$\nu_{\omega_i^1, \omega_i^2}(A_i) = \chi_{A_i}(\omega_i^1)\frac{\eta_i^+(\omega_i^2)}{\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2)} + \chi_{A_i}(\omega_i^2)\frac{\eta_i^-(\omega_i^1)}{\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2)} \quad (115)$$

for $\omega_i^1 \in \Omega_i^-, \omega_i^2 \in \Omega_i^+, A_i \in \mathcal{F}_i^0$.

In the next Theorem 8, we assume that the random value $\eta_1(\omega_1)$ is an integrable one.

Theorem 8. *On the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it, every measure Q of the regular set of measures M for the random value*

$$\xi_0 = \prod_{i=1}^{N}\left[1 + a_i(\omega_1, \cdots, \omega_{i-1})\eta_i(\omega_i)\right], N < \infty, \quad 0 < a_i(\omega_1, \cdots, \omega_{i-1}) \le 1, i = \overline{1, N}, \text{ has}$$

the representation

$$Q(A) = \int_V \alpha(v)\nu_v(A)\,\mathrm{d}\mu(v), \quad (116)$$

where the random value $\alpha(v)$ satisfies the conditions

$$\mu\left(\{v \in \mathcal{V}, \alpha(v) > 0\}\right) = \left[P_1^0(\Omega_1^-)P_1^0(\Omega_1^+)\right]^N, \quad (117)$$

$$\int_V \alpha(v)\prod_{i=1}^{N}\frac{\eta_i^-(\omega_i^1)\eta_i^+(\omega_i^2)}{\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2)}\,\mathrm{d}\mu(v) < \infty, \quad (118)$$

$$\int_V \alpha(v)\,\mathrm{d}\mu(v) = 1. \quad (119)$$

Proof. To prove Theorem, it needs to prove that the countable additive measure $\nu_v(A)$ at every fixed $v \in \mathcal{V}$ is a measurable map from the measurable space $\{\mathcal{V}, \mathcal{L}\}$ into the measurable space $\{[0,1], B([0,1])\}$ for every fixed $A \in \mathcal{F}_N$. For $A = \prod_{i=1}^{N} A_i$, $A_i \in \mathcal{F}_i^0$, $\nu_v(A)$ is a measurable map from the measurable space $\{\mathcal{V}, \mathcal{L}\}$ into the measurable space $\{[0,1], B([0,1])\}$. The family of sets of the kind $\bigcup_{i \in I} E_i$, $E_i = \prod_{s=1}^{N} A_s^i$, $A_s^i \in F_s^0$, where $E_i \bigcap E_j = \varnothing$, the set I is an arbitrary finite set, forms the algebra of the sets that we denote by U_0. From the countable additivity of $\nu_v(A)$, $\nu_v\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} \nu_v(E_i)$ is a measurable map from the measurable space $\{\mathcal{V}, \mathcal{L}\}$ into the measurable space $\{[0,1], B([0,1])\}$. Let T be a class of the sets from the minimal σ-algebra Σ generated by U_0 for every subset E of that $\nu_v(E)$ is a measurable map from the measurable space $\{\mathcal{V}, \mathcal{L}\}$ into the measurable space $\{[0,1], B([0,1])\}$. Let us prove that T is a monotonic class. Suppose that $E_i \subset E_{i+1}, i = \overline{1, \infty}, E_i \in T$. Then, $\nu_v(E_i) \le \nu_v(E_{i+1})$. From this, it follows that $\lim_{i \to \infty} \nu_v(E_i)$ is a measurable map from the measurable space $\{\mathcal{V}, \mathcal{L}\}$ into the measurable space $\{[0,1], B([0,1])\}$. But, $\nu_v(E_{i+1} \setminus E_i) = \nu_v(E_{i+1}) - \nu_v(E_i)$ is a measurable map from $\{\mathcal{V}, \mathcal{L}\}$ into $\{[0,1], B([0,1])\}$. From this equality, it follows that the set $E_{i+1} \setminus E_i$ belongs to the class T. Since $\bigcup_{i=1}^{\infty} E_i = E_1 \bigcup \bigcup_{i=1}^{\infty}[E_{i+1} \setminus E_i]$, we have

$$\lim_{n \to \infty} \nu_v(E_n) = \nu_v(E_1) + \lim_{n \to \infty} \sum_{i=1}^{n} \nu_v(E_{i+1} \setminus E_i)$$

$$= \nu_v(E_1) + \sum_{i=1}^{\infty} \nu_v(E_{i+1} \setminus E_i) \tag{120}$$

$$= \nu_v\left(E_1 \bigcup \bigcup_{i=1}^{\infty}[E_{i+1} \setminus E_i]\right) = \nu_v\left(\bigcup_{i=1}^{\infty} E_i\right).$$

The equalities (120) mean that $\bigcup_{i=1}^{\infty} E_i$ belongs to T, since $\nu_v\left(\bigcup_{i=1}^{\infty} E_i\right)$ is a measurable map of $\{\mathcal{V}, \mathcal{L}\}$ into $\{[0,1], B([0,1])\}$. Suppose that $E_i \supset E_{i+1}, E_i \in T, i = \overline{1, \infty}$. Then, this case is reduced to the previous one by the note that the sequence $\bar{E}_i = \prod_{i=1}^{N} \Omega_i^0 \setminus E_i, i = \overline{1, \infty}$ is monotonically increasing. From this, it follows that $\bar{E} = \bigcup_{i=1}^{\infty} \bar{E}_i \in T$. Therefore, $\bigcap_{i=1}^{\infty} E_i = \prod_{i=1}^{N} \Omega_i^0 \setminus \bigcup_{i=1}^{\infty} \bar{E}_i \in T$. Thus, T is a monotone class. But, $U_0 \subset T$. Hence, T contains the minimal monotone class generated by the algebra U_0, that is, $m(U_0) = \Sigma$, therefore, $\Sigma \subset T$. Thus, $\nu_v(E)$ is a measurable map of $\{\mathcal{V}, \mathcal{L}\}$ into $\{[0,1], B([0,1])\}$ for $A \in \Sigma$. The fact that the random value $\alpha(v)$ satisfies the conditions (117)-(119) means that Q, given by the formula (116), is a countable additive function of sets and $E^Q \xi_0 < \infty$. Moreover, $E^Q \xi_0 = 1$. It is evident that

$E^{Q}\left\{\xi_{0} \mid \mathcal{F}_{n}\right\}=\prod_{i=1}^{n}\left[1+a_{i}\left(\omega_{1}, \cdots, \omega_{i-1}\right) \eta_{i}\left(\omega_{i}\right)\right], Q \in M$. Due to Lemma 4, this proves that the set M is a regular set of measure. Theorem 8 is proved. $\qquad \square$

Remark 3. *The representation* (116) *for the regular set of measures M means that M is a convex set of equivalent measures. Since the random value* $\alpha(v)$ *runs all bounded random values, satisfying the conditions* (117 - 119), *it is easy to show that the set of measures* $v_{v}(A), v \in \mathcal{V}, A \in \mathcal{F}_{N}$, *is the set of extreme points for the set M.*

Let us introduce the denotations

$$m_{n}(\{\omega\})=\prod_{i=1}^{n}\left[1+a_{i}\left(\omega_{1}, \cdots, \omega_{i-1}\right) \eta_{i}\left(\omega_{i}\right)\right], 1 \leq n \leq N<\infty, \qquad (121)$$

$$0<a_{i}\left(\omega_{1}, \cdots, \omega_{i-1}\right) \leq 1, \ i=\overline{1, N}.$$

$$\{\omega\}=\left\{\omega_{1}, \cdots, \omega_{N}\right\}, \ \{\omega\}^{i}=\left\{\omega_{1}^{i}, \cdots, \omega_{N}^{i}\right\}, \ i=1,2, \ \Omega_{n}=\prod_{i=1}^{n} \Omega_{i}^{0}, \ n=\overline{1, N},$$

$$\{\omega\}_{n}^{1}=\left\{\omega_{1}^{1}, \cdots, \omega_{n}^{1}\right\}, \ \{\omega\}_{n}^{i}=\left\{\omega_{1}^{i}, \cdots, \omega_{n}^{i}\right\}, \ i=1,2,$$

$$\Omega_{N-n}=\prod_{i=n+1}^{N} \Omega_{i}^{0}, \ n=\overline{1, N-1},$$

$$\bar{\Omega}_{n}^{-}=\left\{\{\omega\}_{n} \in \Omega_{n},\left[m_{n}-m_{n-1}\right]\left(\{\omega\}_{n}\right) \leq 0\right\}$$
$$=\Omega_{n-1} \times\left\{\omega_{n} \in \Omega_{n}^{0}, \eta_{n}\left(\omega_{n}\right) \leq 0\right\},$$

$$\bar{\Omega}_{n}^{+}=\left\{\{\omega\}_{n} \in \Omega_{n},\left[m_{n}-m_{n-1}\right]\left(\{\omega\}_{n}\right)>0\right\}$$
$$=\Omega_{n-1} \times\left\{\omega_{n} \in \Omega_{n}^{0}, \eta_{n}\left(\omega_{n}\right)>0\right\},$$

$$\Omega_{n}^{-}=\left\{\{\omega\} \in \Omega_{N},\left[m_{n}-m_{n-1}\right]\left(\{\omega\}\right) \leq 0\right\}$$
$$=\Omega_{n-1} \times\left\{\omega_{n} \in \Omega_{n}^{0}, \eta_{n}\left(\omega_{n}\right) \leq 0\right\} \times \Omega_{N-n},$$

$$\Omega_{n}^{+}=\left\{\{\omega\} \in \Omega_{N},\left[m_{n}-m_{n-1}\right]\left(\{\omega\}\right)>0\right\}$$
$$=\Omega_{n-1} \times\left\{\omega_{n} \in \Omega_{n}^{0}, \eta_{n}\left(\omega_{n}\right)>0\right\} \times \Omega_{N-n}. \qquad (122)$$

Note that the σ-algebra $\mathcal{F}_{n}$ is generated by sets of the kind $G=\prod_{i=1}^{N} G_{i}$, where $G_{i} \in \mathcal{F}_{i}^{0}, i=\overline{1, n}, G_{i}=\Omega_{i}^{0}, i=\overline{n+1, N}$. Denote $P_{n}=\prod_{i=1}^{n} P_{i}^{0}$ the contraction of the measure $P_{N}=\prod_{i=1}^{N} P_{i}^{0}$ onto the σ-algebra $\mathcal{F}_{n}$. Further we use the denotations P_{n}^{-} and P_{n}^{+} which are the contractions the measure P_{n} onto the σ-algebras $\mathcal{F}_{n} \bigcap \Omega_{n}^{-}$ and $\mathcal{F}_{n} \bigcap \Omega_{n}^{+}$, correspondingly. If the measure Q belongs to the set of martingale measures (116), then $E^{Q}\left\{m_{n} \mid \mathcal{F}_{n-1}\right\}=m_{n-1}$, or $E^{Q}\left[m_{n}-m_{n-1}\right]=0$. From this, for the measure Q the representation

$$Q(A_1) = \int\limits_{\Omega_n^- \times \Omega_n^+} \frac{\chi_{A_1}\left(\{\omega\}^1\right)\alpha\left(\{\omega\}^1;\{\omega\}^2\right)\left[m_n - m_{n-1}\right]^+\left(\{\omega\}^2\right)}{\left[m_n - m_{n-1}\right]^-\left(\{\omega\}^1\right) + \left[m_n - m_{n-1}\right]^+\left(\{\omega\}^2\right)} \mathrm{d}\left[P_N^- \times P_N^+\right]$$

$$+ \int\limits_{\Omega_n^- \times \Omega_n^+} \frac{\chi_{A_1}\left(\{\omega\}^2\right)\alpha\left(\{\omega\}^1;\{\omega\}^2\right)\left[m_n - m_{n-1}\right]^-\left(\{\omega\}^1\right)}{\left[m_n - m_{n-1}\right]^-\left(\{\omega\}^1\right) + \left[m_n - m_{n-1}\right]^+\left(\{\omega\}^2\right)} \mathrm{d}\left[P_N^- \times P_N^+\right], \quad A_1 \in \mathcal{F}_n, \tag{123}$$

is true if the random value $\alpha\left(\{\omega\}^1;\{\omega\}^2\right) > 0$ satisfies the condition

$$\int\limits_{\Omega_n^- \times \Omega_n^+} \alpha\left(\{\omega\}^1;\{\omega\}^2\right)\mathrm{d}\left[P_N^- \times P_N^+\right] = 1. \tag{124}$$

Since for the set A_1 the representation $A_1 = A \times \prod\limits_{i=n+1}^{N} \Omega_i^0$, is true, where $A \in \bar{\mathcal{F}}_n = \prod\limits_{i=1}^{n} \mathcal{F}_i^0$, then for the contraction Q_n of the measure Q onto the σ-algebra $\bar{\mathcal{F}}_n$ the representation

$$Q_n(A) = \int\limits_{\bar{\Omega}_n^- \times \bar{\Omega}_n^+} \frac{\chi_A\left(\{\omega\}_n^1\right)\alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\left[m_n - m_{n-1}\right]^+\left(\{\omega\}_n^2\right)}{\left[m_n - m_{n-1}\right]^-\left(\{\omega\}_n^1\right) + \left[m_n - m_{n-1}\right]^+\left(\{\omega\}_n^2\right)} \mathrm{d}\left[\bar{P}_n^- \times \bar{P}_n^+\right]$$

$$+ \int\limits_{\bar{\Omega}_n^- \times \bar{\Omega}_n^+} \frac{\chi_A\left(\{\omega\}_n^2\right)\alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\left[m_n - m_{n-1}\right]^-\left(\{\omega\}_n^1\right)}{\left[m_n - m_{n-1}\right]^-\left(\{\omega\}_n^1\right) + \left[m_n - m_{n-1}\right]^+\left(\{\omega\}_n^2\right)} \mathrm{d}\left[\bar{P}_n^- \times \bar{P}_n^+\right], \quad A \in \bar{\mathcal{F}}_n, \tag{125}$$

is true, where we introduced the denotations $\bar{P}_n^-$ and $\bar{P}_n^+$ which are the contractions of the measure P_n onto the σ-algebras $\bar{\mathcal{F}}_n \bigcap \bar{\Omega}_n^-$ and $\bar{\mathcal{F}}_n \bigcap \bar{\Omega}_n^+$, correspondingly,

$$\alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right) = \int\limits_{\Omega_{N-n} \times \Omega_{N-n}} \alpha\left(\{\omega\}^1;\{\omega\}^2\right)\mathrm{d}\left[P_{N-n} \times P_{N-n}\right],$$

$$P_{N-n} = \prod\limits_{i=n+1}^{N} P_i^0, \quad \int\limits_{\bar{\Omega}_n^- \times \bar{\Omega}_n^+} \alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\left[\bar{P}_n^- \times \bar{P}_n^+\right] = 1. \tag{126}$$

In the set $\bar{\Omega}_n^- \times \bar{\Omega}_n^+$ let us introduce the transformation

$$T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right) = \left(T_n^1\left(\{\omega\}_n^1\right); T_n^2\left(\{\omega\}_n^2\right)\right),$$

$$T_n^1\left(\{\omega\}_n^1\right) = \left\{\{\omega\}_{n-1}^2, \omega_n^1\right\}, \quad T_n^2\left(\{\omega\}_n^2\right) = \left\{\{\omega\}_{n-1}^1, \omega_n^2\right\}, \quad n = \overline{1, N}. \tag{127}$$

By the definition we put that for $n = 1$ the transformation T_1 is identical one. Introduce the denotations

$$v_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right) = \frac{\left[m_n - m_{n-1}\right]^+\left(\{\omega\}_n^2\right)}{\varphi_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}, \tag{128}$$

$$v_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right) = \frac{\left[m_n - m_{n-1}\right]^-\left(\{\omega\}_n^1\right)}{\varphi_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}. \tag{129}$$

$$\varphi_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right) = \left[m_n - m_{n-1}\right]^-\left(\{\omega\}_n^1\right) + \left[m_n - m_{n-1}\right]^+\left(\{\omega\}_n^2\right), \tag{130}$$

$$\varphi_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)=\varphi_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)+\varphi_n^1\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right). \tag{131}$$

Theorem 9. *Let* Ω_1^0 *be a complete separable metric space and* $\mathcal{F}_1^0$ *be a Borel* σ-*algebra on it. If the condition*

$$\int_{\Omega_n} f\left(\{\omega\}_n\right)\mathrm{d}P_n<\infty, \tag{132}$$

is true for $\mathcal{F}_n$ *-measurable nonnegative random value* $f\left(\{\omega\}_n\right)$, *then the closure of the set of points* $E^{Q_n}f\left(\{\omega\}_n\right), Q_n\in M_n$, *in metrics* $\rho(x,y)=|x-y|$ *on the real line contains the set of points*

$$f\left(\{\omega\}_n^1\right)v_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)+f\left(\{\omega\}_n^2\right)v_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$$
$$+f\left(T_n^1\left(\{\omega\}_n^1\right)\right)v_n^1\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right) \tag{133}$$
$$+f\left(T_n^2\left(\{\omega\}_n^2\right)\right)v_n^2\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right),\ n=\overline{1,N}.$$

Proof. Let us find the conditions for the measurable functions $\alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$ under which $E^{Q_n}\left\{m_n\mid\overline{\mathcal{F}}_{n-1}\right\}=m_{n-1}$. Introduce the denotation

$$\alpha_n^0\left(\{\omega\}_n^1;\{\omega\}_n^2\right)=\frac{\alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}{\left[m_n-m_{n-1}\right]^-\left(\{\omega\}_n^1\right)+\left[m_n-m_{n-1}\right]^+\left(\{\omega\}_n^2\right)}. \tag{134}$$

Let the set B belongs to $\overline{\mathcal{F}}_{n-1}$, then

$$E^{Q_n}\chi_B\left(\{\omega\}_{n-1}\right)\left[m_n-m_{n-1}\right]\left(\{\omega\}_n\right)$$
$$=\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+}\chi_B\left(\{\omega\}_{n-1}^1\right)\alpha_n^0\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\left[m_n-m_{n-1}\right]\left(\{\omega\}_n^1\right)$$
$$\times\left[m_n-m_{n-1}\right]^+\left(\{\omega\}_n^2\right)\mathrm{d}\left[\overline{P}_n^-\times\overline{P}_n^+\right] \tag{135}$$
$$+\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+}\chi_B\left(\{\omega\}_{n-1}^2\right)\alpha_n^0\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\left[m_n-m_{n-1}\right]\left(\{\omega\}_n^2\right)$$
$$\times\left[m_n-m_{n-1}\right]^-\left(\{\omega\}_n^1\right)\mathrm{d}\left[\overline{P}_n^-\times\overline{P}_n^+\right].$$

If to take into account the relations

$$\left[m_n-m_{n-1}\right]\left(\{\omega\}_n\right)=m_{n-1}\left(\{\omega\}_{n-1}\right)a_n\left(\{\omega\}_{n-1}\right)\eta_n\left(\omega_n\right),$$

$$\left[m_n-m_{n-1}\right]^+\left(\{\omega\}_n\right)=m_{n-1}\left(\{\omega\}_{n-1}\right)a_n\left(\{\omega\}_{n-1}\right)\eta_n^+\left(\omega_n\right),$$

$$\left[m_n-m_{n-1}\right]^-\left(\{\omega\}_n\right)=m_{n-1}\left(\{\omega\}_{n-1}\right)a_n\left(\{\omega\}_{n-1}\right)\eta_n^-\left(\omega_n\right), \tag{136}$$

and introduce the denotations

$$\theta_1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$$
$$=m_{n-1}\left(\{\omega\}_{n-1}^1\right)a_n\left(\{\omega\}_{n-1}^1\right)\times m_{n-1}\left(\{\omega\}_{n-1}^2\right)a_n\left(\{\omega\}_{n-1}^2\right)\alpha_n^0\left(\{\omega\}_n^1;\{\omega\}_n^2\right), \tag{137}$$

$$\theta_2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$$
$$=m_{n-1}\left(\{\omega\}_{n-1}^1\right)a_n\left(\{\omega\}_{n-1}^1\right)\times m_{n-1}\left(\{\omega\}_{n-1}^2\right)a_n\left(\{\omega\}_{n-1}^2\right)\alpha_n^0\left(\{\omega\}_n^1;\{\omega\}_n^2\right), \tag{138}$$

we obtain

$$E^{Q_n} \chi_B \left(\{\omega\}_{n-1} \right) \left[m_n - m_{n-1} \right] \left(\{\omega\}_n \right)$$

$$= - \int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \chi_B \left(\{\omega\}_{n-1}^1 \right) \theta_1 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) \eta_n^- \left(\omega_n^1 \right) \eta_n^+ \left(\omega_n^2 \right) \mathrm{d} \left[\overline{P}_n^- \times \overline{P}_n^+ \right]$$

$$+ \int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \chi_B \left(\{\omega\}_{n-1}^2 \right) \theta_2 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) \eta_n^- \left(\omega_n^1 \right) \eta_n^+ \left(\omega_n^2 \right) \mathrm{d} \left[\overline{P}_n^- \times \overline{P}_n^+ \right]$$

$$= \int_{\left\{ \eta_n \left(\omega_n^1 \right) \leq 0 \right\} \times \left\{ \eta_n \left(\omega_n^2 \right) > 0 \right\}} \mathrm{d} \left[P_1 \left(\omega_n^1 \right) \times P_1 \left(\omega_n^2 \right) \right] \eta_n^- \left(\omega_n^1 \right) \eta_n^+ \left(\omega_n^2 \right)$$

$$\times \int_{\Omega_{n-1} \times \Omega_{n-1}} \chi_B \left(\{\omega\}_{n-1}^1 \right) \left[\theta_1 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) - \theta_1 \left(\{\omega\}_{n-1}^2 , \omega_n^1 ; \{\omega\}_{n-1}^1 , \omega_n^2 \right) \right] \quad (139)$$

$$\times \mathrm{d} \left[P_{n-1} \left(\{\omega\}_{n-1}^1 \right) \times P_{n-1} \left(\{\omega\}_{n-1}^2 \right) \right].$$

It is evident that the expression (139) equals zero for every $B \in \overline{\mathcal{F}}_{n-1}$ if and only if as

$$\theta_1 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) - \theta_1 \left(\{\omega\}_{n-1}^2 , \omega_n^1 ; \{\omega\}_{n-1}^1 , \omega_n^2 \right) = 0. \quad (140)$$

The last equality (140) is valid if the equality

$$\alpha_n^0 \left(\{\omega\}_{n-1}^1 , \omega_n^1 ; \{\omega\}_{n-1}^2 , \omega_n^2 \right) = \alpha_n^0 \left(\{\omega\}_{n-1}^2 , \omega_n^1 ; \{\omega\}_{n-1}^1 , \omega_n^2 \right) \quad (141)$$

is true.

Now if for $\alpha_n^2 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) > 0$ satisfying the condition

$$\int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \alpha_n^2 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) \mathrm{d} \left[\overline{P}_n^- \times \overline{P}_n^+ \right] = 1 \quad (142)$$

to put

$$\alpha_n^0 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) = \frac{\alpha_n^2 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) + \alpha_n^2 \left(T_n \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) \right)}{\varphi_n \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right)}, \quad (143)$$

then

$$Q_n (A) = \int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \chi_A \left(\{\omega\}_n^1 \right) \alpha_n^0 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) \left[m_n - m_{n-1} \right]^+ \left(\{\omega\}_n^2 \right) \mathrm{d} \left[\overline{P}_n^- \times \overline{P}_n^+ \right]$$

$$+ \int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \chi_A \left(\{\omega\}_n^2 \right) \alpha_n^0 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) \left[m_n - m_{n-1} \right]^- \left(\{\omega\}_n^1 \right) \mathrm{d} \left[\overline{P}_n^- \times \overline{P}_n^+ \right] \quad (144)$$

is a probability measure on the σ-algebra $\overline{\mathcal{F}}_n$.

Taking into account the denotation (134) and the formula (143) we obtain that the measure

$$Q_n (A) = \int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \chi_A \left(\{\omega\}_n^1 \right) \alpha_n^1 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) \frac{\left[m_n - m_{n-1} \right]^+ \left(\{\omega\}_n^2 \right)}{\varphi_n^1 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right)} \mathrm{d} \left[\overline{P}_n^- \times \overline{P}_n^+ \right]$$

$$+ \int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \chi_A \left(\{\omega\}_n^2 \right) \alpha_n^1 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right) \frac{\left[m_n - m_{n-1} \right]^- \left(\{\omega\}_n^1 \right)}{\varphi_n^1 \left(\{\omega\}_n^1 ; \{\omega\}_n^2 \right)} \mathrm{d} \left[\overline{P}_n^- \times \overline{P}_n^+ \right] \quad (145)$$

is a probability measure on the σ-algebra $\overline{\mathcal{F}}_n$, where

$$\alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right) = \alpha_n^0\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\varphi_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right). \tag{146}$$

satisfy the condition

$$\int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \mathrm{d}\left[\overline{P}_n^- \times \overline{P}_n^+\right] = 1, \tag{147}$$

due to the condition

$$\int_{\overline{\Omega}_n^- \times \overline{\Omega}_n^+} \alpha_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \mathrm{d}\left[\overline{P}_n^- \times \overline{P}_n^+\right] = 1. \tag{148}$$

So, we described the contraction of the martingale measure Q on the σ-algebra $\mathcal{F}_n$ for which $E^{Q_n}\{m_n \mid \mathcal{F}_{n-1}\} = m_{n-1}$. It has the representation (145) with the strictly positive random values $\alpha_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right), \alpha_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$ satisfying conditions (147), (148).

Since Ω_1^0 is a separable metric space, then it has an exhaustive decomposition. This is true for Ω_n which is also separable metric space for every $n = \overline{2,N}$. On the probability space $\left\{\overline{\Omega}_n^- \times \overline{\Omega}_n^+, \overline{\mathcal{F}}_n^- \times \overline{\mathcal{F}}_n^+, \overline{P}_n^- \times \overline{P}_n^+\right\}$, for every integrable finite valued random value $f\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$ the sequence

$E^{\overline{\mu}_n}\left\{f\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \mid \overline{\mathcal{F}}_m\right\}$ converges to $f\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$ with probability one,

as $m \to \infty$, since it is a regular martingale. It is evident that for those $B_{m,ks}$ for which $\overline{\mu}_n\left(B_{m,ks}\right) \neq 0, \overline{\mu}_n = \overline{P}_n^- \times \overline{P}_n^+$,

$$E^{\overline{\mu}_n}\left\{f\left(\{\omega\}_n^1,\{\omega\}_n^2\right) \mid \overline{\mathcal{F}}_m\right\}$$
$$= \frac{\int_{B_{m,ks}} f\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(B_{m,ks}\right)}, \quad \left(\{\omega\}_n^1;\{\omega\}_n^2\right) \in B_{m,ks}. \tag{149}$$

Denote $D_0 = \bigcup\limits_{m,k,s,\mu\left(B_{m,ks}\right)=0} B_{m,ks}$. It is evident that $\overline{\mu}_n\left(D_0\right) = 0$. For every $\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \in \overline{\Omega}_n^- \times \overline{\Omega}_n^+ \setminus D_0$, the formula (149) is well defined and is finite. Let D_1 be the subset of the set $\overline{\Omega}_n^- \times \overline{\Omega}_n^+ \setminus D_0$, where the limit of the left hand side of the formula (149) does not exists. Then, $\overline{\mu}_n\left(D_1\right) = 0$. For every $\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \in \overline{\Omega}_n^- \times \overline{\Omega}_n^+ \setminus \left(D_0 \bigcup D_1\right)$, the right hand side of the formula (149) converges to $f\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$. For $\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \in \overline{\Omega}_n^- \times \overline{\Omega}_n^+ \setminus \left(D_0 \bigcup D_1\right)$, denote $A_m = A_m\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$ those set $B_{m,ks}$ for which $\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \in B_{m,ks}$ for a certain k,s. Then, for every integrable finite valued $f\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$

$$\lim_{m \to \infty} \frac{\int_{A_m} f\left(\{\omega\}_n^1;\{\omega\}_n^2\right) \mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(A_m\right)} = f\left(\{\omega\}_n^1;\{\omega\}_n^2\right). \tag{150}$$

Choose the sequence

$$\alpha_n^{2,\varepsilon_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$$

$$=\left(1-\varepsilon_m\right)\frac{\chi_{A_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}{\overline{\mu}_n\left(A_m\right)}+\varepsilon_m\frac{\chi_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}{\overline{\mu}_n\left(\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m\right)}, \tag{151}$$

where $0<\varepsilon_m<1$, $\lim_{m\to\infty}\varepsilon_m=0$. Then the sequence $\alpha_n^{2,\varepsilon_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$ satisfy the condition (148). Let us consider the sequence

$$\alpha_n^{0,\varepsilon_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$$

$$=\frac{1}{\varphi_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}\left[\left(1-\varepsilon_m\right)\frac{\chi_{A_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}{\overline{\mu}_n\left(A_m\right)}\right.$$

$$\left.+\varepsilon_m\frac{\chi_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}{\overline{\mu}_n\left(\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m\right)}\right] \tag{152}$$

$$+\frac{1}{\varphi_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)}\left[\left(1-\varepsilon_m\right)\frac{\chi_{A_m}\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)}{\overline{\mu}_n\left(A_m\right)}\right.$$

$$\left.+\varepsilon_m\frac{\chi_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m}\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)}{\overline{\mu}_n\left(\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m\right)}\right].$$

Then the contraction of the sequence of martingale measures $Q_n^{\varepsilon_m}$ generated by sequence (152) on the σ-algebra $\mathcal{F}_n$ is given by the formula

$$E^{Q_n^{\varepsilon_m}}f(\omega)$$

$$=\int_{\overline{\Omega}_n^-}\int_{\overline{\Omega}_n^+}f\left(\{\omega\}_n^1\right)\alpha_n^{0,\varepsilon_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\left[m_n-m_{n-1}\right]^+\left(\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n$$

$$+\int_{\overline{\Omega}_n^-}\int_{\overline{\Omega}_n^+}f\left(\{\omega\}_n^2\right)\alpha_n^{0,\varepsilon_m}\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\left[m_n-m_{n-1}\right]^-\left(\{\omega\}_n^1\right)\mathrm{d}\overline{\mu}_n$$

$$=\left(1-\varepsilon_m\right)\frac{\int_{A_m}f\left(\{\omega\}_n^1\right)v_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(A_m\right)}$$

$$+\left(1-\varepsilon_m\right)\frac{\int_{A_m}f\left(\{\omega\}_n^2\right)v_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(A_m\right)}$$

$$+\varepsilon_m\frac{\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m}f\left(\{\omega\}_n^1\right)v_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m\right)}$$

$$+\varepsilon_m\frac{\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m}f\left(\{\omega\}_n^2\right)v_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(\overline{\Omega}_n^-\times\overline{\Omega}_n^+\setminus A_m\right)}$$

$$+\left(1-\varepsilon_m\right)\frac{\displaystyle\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+}\chi_{A_m}\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)f\left(\{\omega\}_n^1\right)v_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(A_m\right)}$$

$$+\left(1-\varepsilon_m\right)\frac{\displaystyle\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+}\chi_{A_m}\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)f\left(\{\omega\}_n^2\right)v_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(A_m\right)}$$

$$+\varepsilon_m\frac{\displaystyle\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\backslash A_m}\chi_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\backslash A_m}\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)f\left(\{\omega\}_n^1\right)v_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(\overline{\Omega}_n^-\times\overline{\Omega}_n^+\backslash A_m\right)}$$

$$+\varepsilon_m\frac{\displaystyle\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\backslash A_m}\chi_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+\backslash A_m}\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)f\left(\{\omega\}_n^2\right)v_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(\overline{\Omega}_n^-\times\overline{\Omega}_n^+\backslash A_m\right)}. \tag{153}$$

Due to the invariance of the measure $\overline{\mu}_n$ relative to the transformation T_n we have

$$\left(1-\varepsilon_m\right)\frac{\displaystyle\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+}\chi_{A_m}\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)f\left(\{\omega\}_n^1\right)v_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(A_m\right)}$$

$$+\left(1-\varepsilon_m\right)\frac{\displaystyle\int_{\overline{\Omega}_n^-\times\overline{\Omega}_n^+}\chi_{A_m}\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)f\left(\{\omega\}_n^2\right)v_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(A_m\right)}$$

$$=\left(1-\varepsilon_m\right)\frac{\displaystyle\int_{A_m}f\left(T_n^1\left(\{\omega\}_n^1\right)\right)v_n^1\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}_n\left(A_m\right)}$$

$$+\left(1-\varepsilon_m\right)\frac{\displaystyle\int_{A_m}f\left(T_n^2\left(\{\omega\}_n^2\right)\right)v_n^2\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)\mathrm{d}\overline{\mu}_n}{\overline{\mu}\left(A_m\right)}. \tag{154}$$

From the equalities (153), (154) it follows that

$$\lim_{m\to\infty}E^{Q_n^{\varepsilon_m}}f_n\left(\{\omega\}_n\right)$$
$$=f\left(\{\omega\}_n^1\right)v_n^1\left(\{\omega\}_n^1;\{\omega\}_n^2\right)+f\left(\{\omega\}_n^2\right)v_n^2\left(\{\omega\}_n^1;\{\omega\}_n^2\right)$$
$$+f\left(T_n^1\left(\{\omega\}_n^1\right)\right)v_n^1\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right)$$
$$+f\left(T_n^2\left(\{\omega\}_n^2\right)\right)v_n^2\left(T_n\left(\{\omega\}_n^1;\{\omega\}_n^2\right)\right),\ n=\overline{1,N}. \tag{155}$$

Theorem 9 is proved. □

Theorem 10. *On the probability space* $\{\Omega,\mathcal{F},P\}$ *with the filtration* $\mathcal{F}_n$ *on it, let* Ω_1^0 *be a complete separable metric space. Suppose that* $f_n\left(\{\omega\}_n\right)$ *is a nonnegative integrable* $\mathcal{F}_n$ *-measurable random value, satisfying the condition* $E^{Q^n}f_n\left(\{\omega\}_n\right)\le 1,Q^n\in M_n$ *. Then, there exists a* F_{n-1} *-measurable random value* α_n *, depending on* $f_n\left(\{\omega\}_n\right)$ *, such that*

$$f_n\left(\{\omega\}_n\right)\le 1+\alpha_n\left(\{\omega\}_{n-1}\right)\left[m_n-m_{n-1}\right]\left(\{\omega\}_n\right),\ \{\omega\}_n\in\Omega_n. \tag{156}$$

Proof. First, let us consider the case $n = 1$. From Theorem 9, we have the inequality

$$f_1(\omega_1) \frac{[m_1 - 1]^+(\omega_2)}{[m_1 - 1]^-(\omega_1) + [m_1 - 1]^+(\omega_2)}$$

$$+ f_1(\omega_2) \frac{[m_1 - 1]^-(\omega_1)}{[m_1 - 1]^-(\omega_1) + [m_1 - 1]^+(\omega_2)} \le 1, \ (\omega_1, \omega_2) \in \Omega_1^- \times \Omega_1^+, \tag{157}$$

where $\Omega_1^- = \{\omega_1 \in \Omega_1^0, [m_1 - 1](\omega_1) \le 0\}$, $\Omega_1^+ = \{\omega_2 \in \Omega_1^0, [m_1 - 1](\omega_2) > 0\}$.

Let us put $\eta_1(\omega) = [m_1 - 1](\omega)$. Then, the formula (157) is written in the form

$$f_1(\omega_1) \frac{\eta_1^+(\omega_2)}{\eta_1^-(\omega_1) + \eta_1^+(\omega_2)} + \frac{\eta_1^-(\omega_1)}{\eta_1^-(\omega_1) + \eta_1^+(\omega_2)} f_1(\omega_2) \le 1, \ \omega_1 \in \Omega_1^-, \ \omega_2 \in \Omega_1^+. \tag{158}$$

From the inequalities (158), we obtain the inequalities

$$f_1(\omega_2) \le 1 + \frac{1 - f_1(\omega_1)}{\eta_1^-(\omega_1)} \eta_1^+(\omega_2), \tag{159}$$

$$\eta_1^-(\omega_1) > 0, \ \eta_1^+(\omega_2) > 0, \ \omega_1 \in \Omega_1^-, \ \omega_2 \in \Omega_1^+. \tag{160}$$

Two cases are possible: a) for all $\omega_1 \in \Omega_n^-$, $f_1(\omega_1) \le 1$; b) there exists $\omega_1 \in \Omega_1^-$ such that $f_1(\omega_1) > 1$. First, let us consider the case a).

Since the inequalities (159) are valid for every value $\dfrac{1 - f_1(\omega_1)}{\eta_1^-(\omega_1)}$, as

$\eta_1^-(\omega_1) > 0$, and $f_1(\omega_1) \le 1, \omega_1 \in \Omega_1^-$, then, if to denote

$$\alpha_1 = \inf_{\{\omega_1, \eta_1^-(\omega_1) > 0\}} \frac{1 - f_1(\omega_1)}{\eta_1^-(\omega_1)}, \tag{161}$$

we have $0 \le \alpha_1 < \infty$ and

$$f_1(\omega_2) \le 1 + \alpha_1 \eta_1^+(\omega_2), \ \eta_1^+(\omega_2) > 0, \ \omega_2 \in \Omega_1^+. \tag{162}$$

From the definition of α_1, we obtain the inequalities

$$f_1(\omega_1) \le 1 - \alpha_1 \eta_1^-(\omega_1), \ \eta_1^-(\omega_1) > 0, \ \omega_1 \in \Omega_1^-. \tag{163}$$

Now, if $\eta_1^-(\omega_1) = 0$ for some $\omega_1 \in \Omega_1^-$, then in this case $f_1(\omega_1) \le 1$. All these inequalities give the inequalities

$$f_1(\omega) \le 1 + \alpha_1 \eta_1(\omega), \ \omega \in \Omega_1^- \cup \Omega_1^+. \tag{164}$$

Consider the case b). From the inequality (159), we obtain the inequalities

$$f_1(\omega_2) \le 1 - \frac{1 - f_1(\omega_1)}{-\eta_1^-(\omega_1)} \eta_1^+(\omega_2), \tag{165}$$

$$\eta_1^-(\omega_1) > 0, \ \eta_1^+(\omega_2) > 0, \ \omega_1 \in \Omega_1^-, \ \omega_2 \in \Omega_1^+. \tag{166}$$

The inequalities (165) give the inequalities

$$\frac{1 - f_1(\omega_1)}{-\eta_1^-(\omega_1)} \le \inf_{\{\omega_2, \eta_1^+(\omega_2) > 0\}} \frac{1}{\eta_1^+(\omega_2)} < \infty, \ \eta_1^-(\omega_1) > 0, \ \omega_1 \in \Omega_1^-. \tag{167}$$

Let us define $\alpha_1 = \sup\limits_{\{\omega_1, \eta_1^-(\omega_1) > 0\}} \dfrac{1 - f_1(\omega_1)}{-\eta_1^-(\omega_1)} < \infty$. Then, from (165) we obtain the inequalities

$$f_1(\omega_2) \le 1 - \alpha_1 \eta_1^+(\omega_2), \ \eta_1^+(\omega_2) > 0, \ \omega_2 \in \Omega_1^+. \tag{168}$$

From the definition of α_1, we have the inequalities

$$f_1(\omega_1) \le 1 + \alpha_1 \eta_1^-(\omega_1), \ \eta_1^-(\omega_1) > 0, \ \omega_1 \in \Omega_1^-. \tag{169}$$

The inequalities (168), (169) give the inequalities

$$f_1(\omega) \le 1 - \alpha_1 \eta_1(\omega), \ \omega \in \Omega_1^- \bigcup \Omega_1^+. \tag{170}$$

Theorem 10 in the case $n = 1$ is proved, since the set $\Omega_1^- \bigcup \Omega_1^+$ has the probability one.

Now let us consider the case of arbitrary $2 \le n \le N$. In this case we have the inequality

$$f\left(\{\omega\}_n^1\right) v_n^1\left(\{\omega\}_n^1 ; \{\omega\}_n^2\right) + f\left(\{\omega\}_n^2\right) v_n^2\left(\{\omega\}_n^1 ; \{\omega\}_n^2\right)$$
$$+ f\left(T_n^1\left(\{\omega\}_n^1\right)\right) v_n^1\left(T_n\left(\{\omega\}_n^1 ; \{\omega\}_n^2\right)\right) + f\left(T_n^2\left(\{\omega\}_n^2\right)\right) v_n^2\left(T_n\left(\{\omega\}_n^1 ; \{\omega\}_n^2\right)\right) \le 1. \tag{171}$$

Let us put in this inequality $\{\omega\}_{n-1}^1 = \{\omega\}_{n-1}^2 = \{\omega\}_{n-1}$, then the inequality (171) is transformed into the inequality

$$f_n\left(\{\omega\}_{n-1}, \omega_n^1\right) \frac{\eta_n^+(\omega_n^2)}{\eta_n^-(\omega_n^1) + \eta_n^+(\omega_n^2)} + \frac{\eta_n^-(\omega_n^1)}{\eta_n^-(\omega_n^1) + \eta_n^+(\omega_n^2)} f_n\left(\{\omega\}_{n-1}, \omega_n^2\right) \le 1, \tag{172}$$
$$\left(\{\omega\}_{n-1}, \omega_n^1\right) \in \Omega_n^-, \ \left(\{\omega\}_{n-1}, \omega_n^2\right) \in \Omega_n^+.$$

Taking into account the first part of the proof of Theorem 10 from the inequality (172) we obtain

$$f\left(\{\omega\}_n\right) \le 1 + \alpha_1 \eta_n(\omega_n) = 1 + \frac{\alpha_1 [m_n - m_{n-1}]\left(\{\omega\}_n\right)}{m_{n-1}\left(\{\omega\}_{n-1}\right) a_n\left(\{\omega\}_{n-1}\right)}, \tag{173}$$

where the constant α_1 is the same as in the first part of the proof of Theorem 10. Theorem 10 is completely proved. $\square$

Theorem 11. *On the probability space $\{\Omega, \mathcal{F}, P\}$ with the filtration $\mathcal{F}_n$ on it, let Ω_1^0 be a complete separable metric space. Then, every nonnegative super-martingale $\{f_n, \mathcal{F}_n\}_{n=0}^\infty$ is a local regular one.*

Proof. Without loss of generality, we assume that $f_n \ge d_0 > 0$. From the last fact, we obtain

$$E^{Q^n} \frac{f_n}{f_{n-1}} \le 1, \ Q^n \in M_n, \ n = \overline{1, \infty}. \tag{174}$$

The inequalities (174) and Theorems 4, 10 prove Theorem 11. $\square$

Theorem 12. *On the probability space $\{\Omega, \mathcal{F}, P\}$ with the filtration $\mathcal{F}_n$ on it, let Ω_1^0 be a complete separable metric space. Then, every bounded from below super-martingale $\{f_n, \mathcal{F}_n\}_{n=0}^\infty$ is a local regular one.*

Proof. Since the super-martingale $\{f_n, \mathcal{F}_n\}_{n=0}^\infty$ is bounded from below, then

there exists a real number C_0 such that $f_n + C_0 > 0$. If to consider the super-martingale $\{f_n + C_0, \mathcal{F}_n\}_{n=0}^{\infty}$, then all conditions of Theorem 11 are true. Theorem 12 is proved. □

Theorem 13. *On the probability space $\{\Omega, \mathcal{F}, P\}$ with the filtration $\mathcal{F}_n$ on it, let Ω_1^0 be a complete separable metric space. Suppose that evolution of the risk asset is defined by the formula* (114) *and non risk asset evolve by the law $B_n = 1, n = \overline{0, N}$. If the nonnegative payment function f_N is $\mathcal{F}_N$ measurable integrable random value relative to every martingale measure and satisfying conditions Theorem* 16 *from* [5], *then the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} \int_{\Omega} f_N \mathrm{d}Q = \sup_{v \in \mathcal{V}} \int_{\Omega} f_N \mathrm{d}\nu_v. \tag{175}$$

Proof. The left part of the formula (175) for the super-hedge is true (see: [5]), since for the super-martingale $\left\{ \operatorname*{ess\,sup}_{P \in M} E^P \{\xi \mid \mathcal{F}_n\}, F_n \right\}_{n=0}^{N}$ the optional decomposition is true due to Theorem 11. The left hand side of the formula (175) equals right hand side one due to Theorem 8. Theorem 13 is proved. □

5. Description of Local Regular Super-Martingales Relative to a Regular Set of Measures

In this section, we give the description of local regular super-martingales.

Theorem 14. *On the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it, let M be a regular set of measures. If $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ is an adapted random process, satisfying the conditions*

$$f_m \leq f_{m-1}, \ E^P \xi |f_m| < \infty, \ P \in M, \ m = \overline{1, \infty}, \ \xi \in A_0, \tag{176}$$

then the random process

$$\left\{ f_m E^P \{\xi \mid \mathcal{F}_m\}, \mathcal{F}_m \right\}_{m=0}^{\infty}, \ P \in M, \tag{177}$$

is a local regular super-martingale relative to the regular set of measures M.

Proof. Due to Theorem 3, the random process $\left\{ E^P \{\xi \mid \mathcal{F}_m\}, \mathcal{F}_m \right\}_{m=0}^{\infty}$ is a martingale relative to the regular set of measures *M*. Therefore,

$$\begin{aligned} &f_{m-1} E^P \{\xi \mid \mathcal{F}_{m-1}\} - E^P \{f_m E^P \{\xi \mid \mathcal{F}_m\} \mid \mathcal{F}_{m-1}\} \\ &= E^P \{(f_{m-1} - f_m) E^P \{\xi \mid \mathcal{F}_m\} \mid \mathcal{F}_{m-1}\}, \ m = \overline{1, \infty}. \end{aligned} \tag{178}$$

So, if to put $\bar{g}_m^0 = (f_{m-1} - f_m) E^P \{\xi \mid \mathcal{F}_m\}, m = \overline{1, \infty}$, then $\bar{g}_m^0 \geq 0$, it is $\mathcal{F}_m$-measurable and $E^P \bar{g}_m^0 \leq E^P \xi (|f_{m-1}| + |f_m|) < \infty$. Due to Theorem 1, we obtain the proof of Theorem 14. □

Corollary 1. *If $f_m = \alpha, m = \overline{1, \infty}, \alpha \in R^1, \xi \in A_0$, then $\left\{ \alpha E^P \{\xi \mid \mathcal{F}_m\}, \mathcal{F}_m \right\}_{m=0}^{\infty}$ is a local regular martingale. Assume that $\xi = 1$, then $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ is a local regular super-martingale relative to the regular set of measures M.*

Denote F_0 the set of adapted processes

$$F_0 = \left\{ f = \{f_m\}_{m=0}^{\infty}, P(|f_m| < \infty) = 1, P \in M, f_m \leq f_{m-1} \right\}. \tag{179}$$

For every $\xi \in A_0$, let us introduce the set of adapted processes

$$L_\xi = \left\{ \overline{f} = \left\{ f_m E^P \left\{ \xi \mid \mathcal{F}_m \right\} \right\}_{m=0}^\infty, \left\{ f_m \right\}_{m=0}^\infty \in F_0, E^P \xi \left| f_m \right| < \infty, P \in M \right\}, \quad (180)$$

and

$$V = \bigcup_{\xi \in A_0} L_\xi. \quad (181)$$

Corollary 2. *Every random process from the set K, where*

$$K = \left\{ \sum_{i=1}^m C_i \overline{f}_i, \overline{f}_i \in V, C_i \geq 0, i = \overline{1, m}, m = \overline{1, \infty} \right\}, \quad (182)$$

is a local regular super-martingale relative to the regular set of measures M on the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it.

Proof. The proof is evident. $\qquad\square$

Theorem 15. *On the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it, let M be a regular set of measures. Suppose that $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a nonnegative uniformly integrable super-martingale relative to the set of measures M, then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set K.*

Proof. The necessity. It is evident that if $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ belongs to K, then it is a local regular super-martingale.

The sufficiency. Suppose that $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a nonnegative uniformly integrable local regular super-martingale. Then, there exists a nonnegative adapted process $\{\overline{g}_m^0\}_{m=1}^\infty, E^P \overline{g}_m^0 < \infty, m = \overline{1, \infty}$, and a martingale $\{M_m, \mathcal{F}_m\}_{m=0}^\infty$, such that

$$f_m = M_m - \sum_{i=1}^m \overline{g}_i^0, \quad m = \overline{0, \infty}. \quad (183)$$

Then, $M_m \geq 0, m = \overline{0, \infty}, E^P M_m < \infty, P \in M$. Since $0 < E^P M_m = f_0 < \infty$, we have $E^P \sum_{i=1}^m \overline{g}_i^0 < f_0$. Let us put $g_\infty = \lim_{m \to \infty} \sum_{i=1}^m \overline{g}_i^0$. Using the uniform integrability of $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$, we can pass to the limit in the equality

$$E^P \left(f_m + \sum_{i=1}^m \overline{g}_i^0 \right) = f_0, \quad P \in M, \quad (184)$$

as $m \to \infty$. Passing to the limit in the last equality, as $m \to \infty$, we obtain

$$E^P \left(f_\infty + g_\infty \right) = f_0, \quad P \in M. \quad (185)$$

Introduce into consideration a random value $\xi = \dfrac{f_\infty + g_\infty}{f_0}$. Then,

$E^P \xi = 1, P \in M$. From here, we obtain that $\xi \in A_0$ and

$$M_m = f_0 E^P \left\{ \xi \mid \mathcal{F}_m \right\}, m = \overline{0, \infty}. \quad (186)$$

Let us put $\overline{f}_m^2 = -\sum_{i=1}^m \overline{g}_i^0$. It is easy to see that the adapted random process $\overline{f}_2 = \left\{ \overline{f}_m^2, \mathcal{F}_m \right\}_{m=0}^\infty$ belongs to $\mathcal{F}_0$. Therefore, for the super-martingale

$f = \left\{ f_m, \mathcal{F}_m \right\}_{m=0}^{\infty}$ the representation

$$f = \overline{f}_1 + \overline{f}_2,$$

is valid, where $\overline{f}_1 = \left\{ f_0 E^P \left\{ \xi \mid \mathcal{F}_m \right\}, \mathcal{F}_m \right\}_{m=0}^{\infty}$ belongs to L_ξ with $\xi = \dfrac{f_\infty + g_\infty}{f_0}$

and $f_m^1 = f_0, m = \overline{0, \infty}$. The same is valid for $\overline{f}_2$ with $\xi = 1$. This implies that f belongs to the set K. Theorem 15 is proved. $\qquad\square$

Theorem 16. *On the measurable space* $\left\{ \Omega, \mathcal{F} \right\}$ *with the filtration* $\mathcal{F}_n$ *on it, let M be a regular set of measures. Suppose that the super-martingale* $\left\{ f_m, \mathcal{F}_m \right\}_{m=0}^{\infty}$ *relative to the set of measures M satisfy the conditions*

$$\left| f_m \right| \leq C\xi_0, \ m = \overline{1, \infty}, \ \xi_0 \in A_0, \ 0 < C < \infty, \tag{187}$$

then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set K.

Proof. The necessity is evident.

The sufficiency. Suppose that $\left\{ f_m, \mathcal{F}_m \right\}_{m=0}^{\infty}$ is a local regular super-martingale. Then, there exists a nonnegative adapted random process $\left\{ \overline{g}_m^0 \right\}_{m=1}^{\infty}, E^P \overline{g}_m^0 < \infty, m = \overline{1, \infty}$, and a martingale $\left\{ M_m \right\}_{m=0}^{\infty}, E^P \left| M_m \right| < \infty, m = \overline{1, \infty}, P \in M$, such that

$$f_m = M_m - \sum_{i=1}^{m} \overline{g}_i^0, \ m = \overline{0, \infty}. \tag{188}$$

The inequalities $f_m + C\xi_0 \geq 0, m = \overline{1, \infty}$, give the inequalities

$$f_m + CE^P \left\{ \xi_0 \mid \mathcal{F}_m \right\} \geq 0, \ m = \overline{0, \infty}. \tag{189}$$

From the inequalities (187), it follows that the super-martingale $\left\{ f_m, \mathcal{F}_m \right\}_{m=0}^{\infty}$ is a uniformly integrable one relative to the regular set of measures *M*. The martingale $\left\{ E^P \left\{ \xi_0 \mid \mathcal{F}_m \right\}, \mathcal{F}_m \right\}_{m=0}^{\infty}$ relative to the regular set of measures *M* is also uniformly integrable one.

Then, $M_m + CE^P \left\{ \xi_0 \mid \mathcal{F}_m \right\} \geq 0, m = \overline{0, \infty}$. Since

$0 < E^P \left[M_m + CE^P \left\{ \xi_0 \mid \mathcal{F}_m \right\} \right] = f_0 + C < \infty$, we have $E^P \sum_{i=1}^{m} \overline{g}_i^0 < f_0 + C$. Let us

put $g_\infty = \lim_{m \to \infty} \sum_{i=1}^{m} \overline{g}_i^0$. Using the uniform integrability of f_m and $\sum_{i=1}^{m} \overline{g}_i^0$, we can

pass to the limit in the equality

$$E^P \left(f_m + \sum_{i=1}^{m} \overline{g}_i^0 \right) = f_0, \ P \in M, \tag{190}$$

as $m \to \infty$. Passing to the limit in the last equality, as $m \to \infty$, we obtain

$$E^P \left(f_\infty + g_\infty \right) = f_0, \ P \in M. \tag{191}$$

Introduce into consideration a random value $\xi_1 = \dfrac{f_\infty + C\xi_0 + g_\infty}{f_0 + C} \geq 0$. Then,

$E^P \xi_1 = 1, P \in M$. From here, we obtain that $\xi_1 \in A_0$ and for the super-martingale $f = \left\{ f_m, \mathcal{F}_m \right\}_{m=0}^{\infty}$ the representation

$$f_m = f_m^0 E^P \{\xi_0 \mid \mathcal{F}_m\} + f_m^1 E^P \{\xi_1 \mid \mathcal{F}_m\} + f_m^2 E^P \{\xi_2 \mid \mathcal{F}_m\}, \quad m = \overline{0, \infty}, \qquad (192)$$

is valid, where $f_m^0 = -C, f_m^1 = f_0 + C, f_m^2 = -\sum_{i=1}^{m} \overline{g}_i^0, m = \overline{0, \infty}, \xi_2 = 1$. From the last representation, it follows that the super-martingale $f = \{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ belongs to the set K. Theorem 16 is proved. $\qquad \square$

Corollary 3. *Let* $f_N, N < \infty$, *be a* $\mathcal{F}_N$ *-measurable integrable random value,* $\sup_{P \in M} E^P |f_N| < \infty$, *and let there exist* $\alpha_0 \in R^1$ *such that*

$$-\alpha_0 M_N + f_N \leq 0, \quad \omega \in \Omega,$$

where $\{M_m, \mathcal{F}_m\}_{m=0}^{\infty} = \{E^P \{\xi \mid \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^{\infty}, \xi \in A_0$. *Then, a super-martingale* $\{f_m^0 + \overline{f}_m\}_{m=0}^{\infty}$ *is a local regular one relative to the regular set of measures* M, *where*

$$f_m^0 = \alpha_0 M_m, \qquad (193)$$

$$\overline{f}_m = \begin{cases} 0, & m < N, \\ f_N - \alpha_0 M_N, & m \geq N. \end{cases} \qquad (194)$$

Proof. It is evident that $\overline{f}_{m-1} - \overline{f}_m \geq 0, m = \overline{0, \infty}$. Therefore, the super-martingale

$$f_m^0 + \overline{f}_m = \begin{cases} \alpha_0 M_m, & m < N, \\ f_N, & m = N, \\ f_N - \alpha_0 M_N + \alpha_0 M_m, & m > N \end{cases} \qquad (195)$$

is a local regular one relative to the regular set of measures M. Corollary 3 is proved. $\qquad \square$

6. Discrete Geometric Brownian Motion

In this section, we construct for the discrete evolution of risk assets the set of equivalent martingale measures and give a new formula for the fair price of super-hedge. Let $\Omega_1^0 = R^1, F_1^0 = B(R^1)$, where R^1 is a real axis, $B(R^1)$ is a Borel σ-algebra of R^1. Let us put $\Omega_i = \Omega_1, F_i^0 = F_1^0, i = \overline{1, \infty}$, and let us construct the infinite direct product of the measurable spaces $\{\Omega_i^0, \mathcal{F}_i^0\}, i = \overline{1, \infty}$. Denote $\Omega = \prod_{i=1}^{\infty} \Omega_i^0$. Under the σ-algebra $\mathcal{F}$ on Ω, we understand the minimal σ-algebra generated by sets $\prod_{i=1}^{\infty} G_i, G_i \in \mathcal{F}_i^0$, where in the last product only the finite set of G_i do not equal Ω_i^0. On the measurable space $\{\Omega, \mathcal{F}\}$, under the filtration $\mathcal{F}_n$ we understand the minimal σ-algebra, generated by sets $\prod_{i=1}^{\infty} G_i, G_i \in F_i^0$, where $G_i = \Omega_i^0$ for $i > n$. Suppose that the points $t_0 = 0, t_1, t_2, \cdots, t_n, \cdots$, belongs to R_1^+ with $\Delta t = t_i - t_{i-1}$ not depending on the index i. Let us consider the probability space $\{\Omega, \mathcal{F}, P\}$, where

$$P = \prod_{i=1}^{\infty} P_i^0, P_i^0 = P_1^0, i = \overline{1, \infty},$$

$$P_1^0(A) = \frac{1}{[2\pi\Delta t]^{1/2}} \int_A e^{-\frac{y^2}{2\Delta t}} dy, \quad A \in \mathcal{F}_1^0. \tag{196}$$

Define on the set $t_0 = 0, t_1, t_2, \cdots, t_n, \cdots$, the discrete Brownian motion. We say that the random process $w(t_i), i = \overline{0, \infty}$, is a discrete Brownian motion, if on $\{\Omega, \mathcal{F}\}$ the joint distribution function is given by the formula

$$P_0\left(w(t_{i_1}) \in A_{i_1}, \cdots, w(t_{i_k}) \in A\right)_{i_k}$$

$$= \frac{1}{D} \int_{A_{i_1} \times \cdots \times A_{i_k}} e^{-\frac{y_{i_1}^2}{2\Delta t_{i_1}}} \times \cdots \times e^{-\frac{\left[y_{i_k} - y_{i_{k-1}}\right]^2}{2\Delta t_{i_k}}} dy_{i_1} \cdots dy_{i_k}, \quad A_{i_s} \in \mathcal{F}_{i_s}^0, \tag{197}$$

$$D = [2\pi]^{k/2} \left[\Delta t_{i_1} \times \cdots \times \Delta t_{i_k}\right]^{1/2}, \Delta t_{i_s} = t_{i_s} - t_{i_{s-1}}.$$

So defined above the random process $w(t_i)$ on the set $t_0, t_1, t_2, \cdots, t_n, \cdots$, with $w(0) = 0$, is a homogeneous one relative to the displacement on $k\Delta t$, where $k \geq 1$, and is a natural number, with the independent increments, the zero expectation and the correlation function $E^{P_0} w(t_s) w(t_k) = \min\{t_s, t_k\}$.

We assume that the evolution of non risk asset is given by the formula $B_n = e^{rt_n}, n = \overline{0, \infty}$, where r is an interest rate. Let us consider on $\{\Omega, \mathcal{F}, P\}$ the evolution of risk asset given by the law

$$\overline{S}_n = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t_n + \sigma w(t_n)}. \tag{198}$$

Further, we consider the discount evolution of the risk asset

$$S_n = \frac{\overline{S}_n}{B_n} = S_0 e^{\left(\mu - \frac{\sigma^2}{2} - r\right)t_n + \sigma w(t_n)}. \tag{199}$$

It is convenient to present these evolutions in the form

$$S_n = (1 + \rho_n) S_{n-1}, \quad n = \overline{1, \infty}, \tag{200}$$

with $\rho_n = e^{\left(\mu - \frac{\sigma^2}{2} - r\right)\Delta t + \sigma(w(t_n) - w(t_{n-1}))} - 1$.

On the probability space $\{\Omega, \mathcal{F}, P\}$ with the filtration $\mathcal{F}_n$ on it, for further investigations it is convenient to present the Brownian motion in the equivalent form. We present the Brownian motion by the sequence of random values $\zeta_n = \sum_{i=1}^{n} y_i, y_i \in \Omega_i^0, n = \overline{1, \infty}$, with the joint distribution functions

$$P\left(\zeta_{i_1} \in A_{i_1}, \cdots, \zeta_{i_k} \in A_{i_k}\right)$$

$$= \frac{1}{D} \int_{A_{i_1} \times \cdots \times A_{i_k}} e^{-\frac{y_{i_1}^2}{2i_1\Delta t}} \times \cdots \times e^{-\frac{\left[y_{i_k} - y_{i_{k-1}}\right]^2}{2(i_k - i_{k-1})\Delta t}} dy_{i_1} \cdots dy_{i_k}, \quad A_{i_s} \in \mathcal{F}_{i_s}^0, \tag{201}$$

$$D = [2\pi]^{i_k/2} [\Delta t]^{i_k/2} \left[i_1 (i_2 - i_1) \times \cdots \times (i_k - i_{k-1})\right]^{1/2}.$$

Then, the discount evolution of the risk asset we can rewrite in the form

$$S_n = S_0 e^{\left(\mu - \frac{\sigma^2}{2} - r\right)n\Delta t + \sigma\zeta_n}. \tag{202}$$

It is convenient to present discount evolution in the form

$$S_n = \left(1 + \rho_n\left(y_n\right)\right)S_{n-1}, \quad n = \overline{1, \infty},\tag{203}$$

with $\rho_n\left(y_n\right) = e^{\left(\mu - \frac{\sigma^2}{2} - r\right)\Delta t + \sigma y_n} - 1 = \rho_1\left(y_n\right)$.

On the measurable space $\left\{\Omega^N, \mathcal{F}^N\right\}$ with the filtration $\mathcal{F}_n, n = \overline{1, N}$, on it, where $\Omega^N = \prod\limits_{i=1}^{N} \Omega_i^0$, $\mathcal{F}^N = \prod\limits_{i=1}^{N} \mathcal{F}_i^0$, we introduce into consideration the set of measures M^N. A measure Q belongs to M^N, if $Q = \prod\limits_{i=1}^{N} Q_i, Q_i \in M_1^0$. For every $\overline{Q} \in M_1^0$ the representation

$$\overline{Q}(A) = \int\limits_{\Omega_1^-}\int\limits_{\Omega_1^+} \chi_A\left(y_1\right)\alpha\left(y_1, y_2\right)\frac{\rho_1^+\left(y_2\right)}{\rho_1^-\left(y_1\right) + \rho_1^+\left(y_2\right)}d\mu\left(y_1, y_2\right)$$

$$+ \int\limits_{\Omega_1^-}\int\limits_{\Omega_1^+} \chi_A\left(y_2\right)\alpha\left(y_1, y_2\right)\frac{\rho_1^-\left(y_1\right)}{\rho_1^-\left(y_1\right) + \rho_1^+\left(y_2\right)}d\mu\left(y_1, y_2\right), \quad A \in \mathcal{F}_1^0,\tag{204}$$

$$\Omega_1^- = \left\{y \in R^1, \rho_1\left(y\right) \le 0\right\} = \left\{y \in R^1, y \le -\frac{\left(\mu - \frac{\sigma^2}{2} - r\right)\Delta t}{\sigma}\right\},$$

$$\Omega_1^+ = \left\{y \in R^1, \rho_1\left(y\right) > 0\right\} = \left\{y \in R^1, y > -\frac{\left(\mu - \frac{\sigma^2}{2} - r\right)\Delta t}{\sigma}\right\},$$

is valid, where $\rho_1\left(y\right) = \rho_1^+\left(y\right) - \rho_1^-\left(y\right)$, $\rho_1\left(y\right) = e^{\left(\mu - \frac{\sigma^2}{2} - r\right)\Delta t + \sigma y} - 1$, $\mu = P^- \times P^+$,

$$P^-\left(A\right) = \frac{1}{\left[2\pi\Delta t\right]^{1/2}} \int\limits_A e^{-\frac{y^2}{2\Delta t}}dy, \quad A \in B\left(\Omega_1^-\right),$$

$$P^+\left(A\right) = \frac{1}{\left[2\pi\Delta t\right]^{1/2}} \int\limits_A e^{-\frac{y^2}{2\Delta t}}dy, \quad A \in B\left(\Omega_1^+\right).$$

On the measurable space $\left\{\Omega_1^- \times \Omega_1^+, B\left(\Omega_1^-\right) \times B\left(\Omega_1^+\right)\right\}$, the random value $\alpha\left(y_1, y_2\right)$ satisfy the conditions

$$\mu\left(\left\{\left(y_1, y_2\right) \in \Omega_1^- \times \Omega_1^+, \alpha\left(y_1, y_2\right) > 0\right\}\right) = P\left(\Omega_1^+\right)P\left(\Omega_1^-\right),\tag{205}$$

$$\int\limits_{\Omega_1^-}\int\limits_{\Omega_1^+} \alpha\left(y_1, y_2\right)\frac{\rho_1^-\left(y_1\right)\rho_1^+\left(y_2\right)}{\rho_1^-\left(y_1\right) + \rho_1^+\left(y_2\right)}d\mu\left(y_1, y_2\right) < \infty,\tag{206}$$

$$\int\limits_{\Omega_1^-}\int\limits_{\Omega_1^+} \alpha\left(y_1, y_2\right)d\mu\left(y_1, y_2\right) = 1,\tag{207}$$

for every bounded $\alpha\left(y_1, y_2\right) > 0$, since $E^{P_1^0}\left|\rho_1\left(y\right)\right| < \infty$. Denote $M_0^N = M_c^N$

the convex linear span of the set of measures M^N. On the measurable space $\{\Omega^N, \mathcal{F}^N\}$ with the filtration $\mathcal{F}_n, n = \overline{0, N}$, on it, in correspondence with Theorem 8, the set of measures M_0^N is a regular set of measures with the random variable $\xi_0 = \prod_{i=1}^{N}(1 + \rho_i(y_i))$, since the random value $\eta_1 = \rho_1(y_1)$, figuring in Theorem 8, is an integrable one relative to the measure P_1^0 and, therefore, $E^Q \xi_0 = 1, Q \in M_0^N$. It means that the set of equivalent martingale measures M_0^N for the discount evolution $S_n = S_0 e^{\left(\mu - \frac{\sigma^2}{2} - r\right) n \Delta t + \sigma \zeta_n}$ of the risk asset contains more than one martingale measure. In this case, the financial market is an incomplete one.

Theorem 17. *On the measurable space $\{\Omega^N, \mathcal{F}^N\}$ with the filtration $\mathcal{F}_n, n = \overline{0, N}$, on it, let the discount risk asset evolution is given by the formula $S_n = S_0 e^{\left(\mu - \frac{\sigma^2}{2} - r\right) n \Delta t + \sigma \zeta_n}$. For the payment function $f(S_N)$, satisfying the condition* $\sup_{Q \in M_0^N} E^Q f(S_N) < \infty$, *the fair price of super-hedge is giving by the formula*

$$\sup_{Q \in M_0^N} E^Q f(S_N)$$

$$= \sup_{y_i^1 \le -d, y_i^2 > -d, i = \overline{1, N}} \sum_{i_1, \cdots, i_N = 1}^{2} f\left(S_0 \prod_{s=1}^{N}(1 + \rho(y_s^{i_s}))\right) \prod_{s=1}^{N} \frac{\left|e^{\sigma\left(d + y_s^{i_s+1}\right)} - 1\right|}{\left|e^{\sigma\left(d + y_s^{i_s+1}\right)} - e^{\sigma\left(d + y_s^{i_s}\right)}\right|}, \quad (208)$$

where we put $d = \dfrac{\left(\mu - \dfrac{\sigma^2}{2} - r\right)\Delta t}{\sigma}$, $y_s^3 = y_s^1$.

Proof. Since R^1 is a separable metric space then due to Lemma 7 the Borel σ-algebra $B(R^1)$ has the exhaustive decomposition. Therefore, the filtration $\mathcal{F}_n, n = \overline{0, N}$, has the exhaustive decomposition, due to Remark 1. Theorem 11 guarantee the formula for the fair price of super-hedge [5]. Due to Remark 3 after Theorem 8, the set of measures $\prod_{i=1}^{n} \mu_{\{y_i^1, y_i^2\}}$, where

$$\mu_{\{y_i^1, y_i^2\}}(A) = \chi_A(y_i^1) \frac{\rho_i^+(y_i^2)}{\rho_i^-(y_i^1) + \rho_i^+(y_i^2)} + \chi_A(y_i^2) \frac{\rho_i^-(y_i^1)}{\rho_i^-(y_i^1) + \rho_i^+(y_i^2)}, \quad (209)$$

$$(y_i^1, y_i^2) \in \Omega_i^- \times \Omega_i^+, \Omega_i^- = \Omega_1^-, \Omega_i^+ = \Omega_1^+, i = \overline{1, N},$$

forms the extreme points of the convex set of measures M_0^N. The formula (208) is obtained by integration relative to the measure $\prod_{i=1}^{n} \mu_{\{y_i^1, y_i^2\}}$ of the random value $f(S_N)$ and taking the sup on the set of all extreme points. Theorem 17 is proved. □

7. Conclusions

In the paper, we generalize the results of the paper [5]. Section 2 contains the de-

finition of local regular super-martingales relative to the set of equivalent measures. Theorem 1 gives the necessary and sufficient conditions of the local regularity of a super-martingale. In spite of its simplicity, the Theorem 1 appeared very useful for the description of the local regular super-martingales.

Section 3 contains the important Definition 3 of the set of equivalent measures consistent with the filtration. In Lemma 3, we give an example of the set of equivalent measures consistent with the filtration. Theorem 2 contains the sufficient conditions under that there exists a nonnegative super-martingale on a measurable space with the set of measures consistent with the filtration. In Theorem 3, the sufficient conditions are founded which guarantee the existence on a measurable space a regular martingale.

Lemma 4 gives the sufficient conditions of the existence of a set of measures consistent with the filtration and such that the mean value of a nonnegative random value relative to all measures equals one.

Further the problem of constructing the set of equivalent measures satisfying the conditions of Lemma 4 is investigated. In Lemma 5 on a probability space we describe the set of equivalent measures satisfying the condition: the mean value of a random value relative to this set of measures equals zero. At last in this section, Theorem 4 gives the necessary and sufficient conditions of the local regularity of a nonnegative super-martingale relative to the equivalent set of measures.

In Section 4, in Lemma 6, we investigate the closure of the set of considered set of measure in the case of the countable space of elementary events. It is proved that in metrics (69) the closure of the set of considered set of measures contains the set of measures (70).

Further, we introduce the notion of the exhaustive decomposition of a measurable space. Lemma 7 states: if the measurable space is a complete separable metric space with the Borel σ-algebra on it then has an exhaustive decomposition.

Using this notion, in Lemma 8, we describe the closure of the considered set of measures relative to the pointwise convergence of measures and the closure of expectation values relative to this set of measures. Theorem 5 is a consequence of Lemma 5 and contains the description of the set of measures, being equivalent to the given measure, expectations relative to which equals one. Theorem 6 states the necessary and sufficient conditions when the set of measures (95) is consistent with filtration.

Theorem 7 states the necessary and sufficient conditions of the consistency with the filtration of the set of measure (95).

Using Lemma 5, in Lemma 9, we construct an example of the set of equivalent measures consistent with the filtration.

Further we consider an evolution of risk asset with memory. In Theorem 8, we describe completely the set of martingale measures for the considered evolution and prove that every martingale measure of this family is an integral over some measure on the set of extreme points of the set of martingale measures. Theorem 9 describes the closure of the mean value of integrable random values relative to

the contractions of martingale measures onto the filtration. In Theorem 10 the bound for every nonnegative $\mathcal{F}_n$ measurable random value the mathematical expectation for which relative to every martingale measure is bounded by 1 is found. In Theorem 11, it is proved that every nonnegative super-martingale relative to the regular set of measures is a local regular one. The same statement, as in Theorem 11, it is proved in Theorem 12 in the case, as a super-martingale is bounded from below.

Section 5 contains the description of the local regular super-martingales. Using Theorem 1, we prove Theorem 14, giving the possibility to describe the local regular super-martingales. Further, we introduce a class K of the local regular super-martingales relative to a regular set of measures. Theorem 15 states that every nonnegative uniformly integrable super-martingale relative to a regular set of measures belongs to the class K. The next Theorem 16 states that all super-martingales that are majorized by elements from the set A_0 also belong to the class K. At last, in corollary 3, we give an example of the local regular super-martingale playing an important role in the definition of the fair price of the contingent claim [5].

Section 6 contains the application of the results obtained above to calculation of the fair price of super-hedge, when the risk asset evolves by the discrete geometric Brownian motion. In this case, we describe the set of regular measures. We find the set of extreme points of the regular set of measures. It is proved that the fair price of the super-hedge is given by the formula (208).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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