

Sufficiency and Wolfe Type Duality for Nonsmooth Multiobjective Programming Problems

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Abstract

In this paper, a class of nonsmooth multiobjective programming problems is considered. We introduce the new concept of invex of order $\sigma(B, \varphi)$ - V -type II for nondifferentiable locally Lipschitz functions using the tools of Clarke subdifferential. The new functions are used to derive the sufficient optimality condition for a class of nonsmooth multiobjective programming problems. Utilizing the sufficient optimality conditions, weak and strong duality theorems are established for Wolfe type duality model.

Keywords

Multiobjective Programming, Optimality Condition, Locally Lipschitz Function, Wolfe Type Dual Problem

1. Introduction

The field of multiobjective programming, also called vector programming, has grown remarkably in different directions in the settings of optimality conditions and duality theory since the 1980s. It has been enriched by the applications of various types of generalizations of convexity theory, with and without differentiability assumptions. The Clarke subdifferential [1] (also called the Clarke generalized gradient) is an important tool to derive optimality conditions for nonsmooth optimization problems. Together with the Clarke's subdifferential, many generalized convexity or invexity functions were generalized to locally Lipschitz functions. Based upon the generalized functions, several sufficient optimality conditions and duality results were established for the optimization problems. We can see for examples [2]-[8]. In [9] Upadhyay introduced some

new generalizations of the concept of (ϕ, ρ) -invexity and established the necessary and sufficient optimality conditions for a class of nonsmooth semi-infinite minmax programming problems. In [10] the new concepts of $(\phi, \rho) - V$ -type I were introduced. Sufficient optimality conditions and Mond-Weir duality results were obtained for nonsmooth multiobjective programming problems. Recently, many researchers have been interested in other types of solution concepts. One of them is higher order strict minimizer. In [11] and [12] some sufficient conditions and duality results were obtained for the new concept of strict minimizer of higher order for a multiobjective optimization problem.

In this paper, we consider the nonsmooth multiobjective programming including the locally Lipschitz functions. The new concepts of invex of order $\sigma(B, \varphi) - V$ -type II functions are introduced. Then, a sufficient optimality condition is obtained for the nondifferentiable multiobjective programming problem under the new functions and the Wolfe type duality results are obtained.

2. Preliminaries and Definitions

Let R^n be the n -dimensional Euclidean space and let X be a nonempty open subset of R^n . For $x = (x_1, x_2, \dots, x_n)^T, y = (y_1, y_2, \dots, y_n)^T \in R^n$, we denote:

$$\begin{aligned} x = y &\Leftrightarrow x_i = y_i, \quad i = 1, 2, \dots, n; \\ x \leq y &\Leftrightarrow x_i \leq y_i, \quad i = 1, 2, \dots, n; \\ x \leq y &\Leftrightarrow x_i \leq y_i, \quad i = 1, 2, \dots, n \text{ and } x \neq y; \\ x < y &\Leftrightarrow x_i < y_i, \quad i = 1, 2, \dots, n; \\ x \in R_+^n &\Leftrightarrow x \geq 0. \end{aligned}$$

Definition 2.1. [1] The function $f : X \rightarrow R$ is said to be locally Lipschitz at $x \in X$, if there exist scalars $k > 0$ and $\varepsilon > 0$, such that

$$|f(y) - f(z)| \leq k \|y - z\|, \quad \text{for all } y, z \in B(x, \varepsilon). \tag{1}$$

where $B(x, \varepsilon)$ is the open ball of radius ε about x .

Definition 2.2. [1] The generalized directional derivative of a locally Lipschitz function f at x in the direction d , denoted by $f^0(x; d)$, is as follows:

$$f^0(x; d) = \limsup_{\substack{y \rightarrow x \\ \lambda \rightarrow 0^+}} \frac{f(y + \lambda d) - f(y)}{\lambda}. \tag{2}$$

Definition 2.3. [1] The generalized gradient of $f : X \rightarrow R$ at $x \in X$, denoted by $\partial f(x)$, is defined as follows:

$$\partial f(x) = \{ \xi \in R^n : f^0(x; d) \geq \langle \xi, d \rangle, \forall d \in R^n \}. \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in R^n .

Consider the following nonsmooth multiobjective programming problem:

$$\begin{aligned} \text{Minimize } & f(x) = (f_1(x), f_2(x), \dots, f_k(x)), \\ \text{(MP)} \quad & \text{s.t. } g_j(x) \leq 0, \quad j = 1, 2, \dots, m, \\ & x \in X. \end{aligned}$$

where $f_i : X \rightarrow R (i \in K = \{1, 2, \dots, k\})$ and $g_j : X \rightarrow R (j \in M = \{1, 2, \dots, m\})$ are locally Lipschitz functions and X is a convex set in R^n .

Let $X_0 = \{x | g_j(x) \leq 0, j \in M\}$ be the set of feasible solutions of (MP), and

$$J(x) = \{j | g_j(x) = 0, x \in X_0, j \in M\}.$$

Definition 2.4. A point $\bar{x} \in X_0$ is a strict minimizer of order σ for (MP) with respect to a nonlinear function $\psi : X \times X \rightarrow R^n$, if for a constant $\rho \in \text{int } R_+^k$, such that

$$f(\bar{x}) \leq f(x) + \rho \|\psi(x, \bar{x})\|^\sigma, \text{ for all } x \in X_0. \quad (4)$$

Throughout the paper, we suppose that $\eta : X \times X \rightarrow R^n; b_0, b_1 : X \times X \rightarrow R_+;$
 $\varphi_0, \varphi_1 : R \rightarrow R; \alpha, \beta : X \times X \rightarrow R_+ \setminus \{0\}; \rho_i, \tau \in R_+, i \in K$.

Definition 2.5. (f, g) is said to be invex of order $\sigma (B, \varphi) - V$ -type II at $\bar{x} \in X$, if there exist $\eta, b_0, b_1, \varphi_0, \varphi_1, \alpha, \beta, \rho_i (i \in K), \tau$ and some vectors $\lambda \in R_+^k$ and $\mu \in R_+^m$ such that for all $x \in X$ the following inequalities hold:

$$b_0(x, \bar{x}) \varphi_0 \left[\sum_{i=1}^k \lambda_i \left(f_i(x) - f_i(\bar{x}) + \rho_i \|\psi(x, \bar{x})\|^\sigma \right) \right] \geq \alpha(x, \bar{x}) \left\langle \sum_{i=1}^k \lambda_i \xi_i, \eta(x, \bar{x}) \right\rangle, \forall \xi_i \in \partial f_i(\bar{x}), i \in K \quad (5)$$

$$-b_1(x, \bar{x}) \varphi_1 \left[\sum_{j=1}^m \mu_j g_j(\bar{x}) \right] \geq \beta(x, \bar{x}) \left\langle \sum_{j=1}^m \mu_j \zeta_j, \eta(x, \bar{x}) \right\rangle + \tau \|\psi(x, \bar{x})\|^\sigma, \quad (6)$$

$$\forall \zeta_j \in \partial g_j(\bar{x}), j \in M.$$

Definition 2.6. (f, g) is said to be (pseudo, quasi) invex of order $\sigma (B, \varphi) - V$ -type II at $\bar{x} \in X$, if there exist $\eta, b_0, b_1, \varphi_0, \varphi_1, \alpha, \beta, \rho_i (i \in K), \tau$ and some vectors $\lambda \in R_+^k$ and $\mu \in R_+^m$ such that for all $x \in X$ the following inequalities hold:

$$\alpha(x, \bar{x}) \left\langle \sum_{i=1}^k \bar{\lambda}_i \xi_i, \eta(x, \bar{x}) \right\rangle \geq 0, \forall \xi_i \in \partial f_i(\bar{x}), i \in K \quad (7)$$

$$\Rightarrow b_0(x, \bar{x}) \varphi_0 \left[\sum_{i=1}^k \lambda_i \left(f_i(x) - f_i(\bar{x}) + \rho_i \|\psi(x, \bar{x})\|^\sigma \right) \right] \geq 0,$$

$$-b_1(x, \bar{x}) \varphi_1 \left[\sum_{j=1}^m \mu_j g_j(\bar{x}) \right] \leq 0 \quad (8)$$

$$\Rightarrow \beta(x, \bar{x}) \left\langle \sum_{j=1}^m \mu_j \zeta_j, \eta(x, \bar{x}) \right\rangle + \tau \|\psi(x, \bar{x})\|^\sigma \leq 0, \forall \zeta_j \in \partial g_j(\bar{x}), j \in M.$$

3. Optimality Condition

In this section, we establish sufficient optimality conditions for a strict minimizer of (MP).

Theorem 3.1. Let $\bar{x} \in X_0$. Suppose that

- 1) There exist $\lambda_i \geq 0, i \in K, \sum_{i=1}^k \lambda_i = 1, \mu_j \geq 0, j \in J(\bar{x})$, such that

$$0 \in \sum_{i=1}^k \lambda_i \partial f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \mu_j \partial g_j(\bar{x}),$$

- 2) (f, g_J) is invex of order σ (B, φ) - V -type II at \bar{x} ,
 - 3) $b_0(x, \bar{x}) > 0, b_1(x, \bar{x}) \geq 0; a < 0 \Rightarrow \varphi_0(a) < 0, a = 0 \Rightarrow \varphi_1(a) = 0$.
- Then \bar{x} is a strict minimizer of order σ for (MP).

Proof: Since $0 \in \sum_{i=1}^k \lambda_i \partial f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \mu_j \partial g_j(\bar{x})$, there exists $\xi_i \in \partial f_i(\bar{x}), i \in K, \zeta_j \in \partial g_j(\bar{x}), j \in J(\bar{x})$, such that

$$\sum_{i=1}^k \lambda_i \xi_i + \sum_{j \in J(\bar{x})} \mu_j \zeta_j = 0. \tag{9}$$

whence

$$\left\langle \sum_{i=1}^k \lambda_i \xi_i + \sum_{j \in J(\bar{x})} \mu_j \zeta_j, \eta(x, \bar{x}) \right\rangle = 0. \tag{10}$$

Suppose that \bar{x} is not a strict minimizer of order σ for (MP). Then there exists $x \in X_0$ and $\rho \in R_+^k$, such that

$$f(\bar{x}) > f(x) + \rho \|\psi(x, \bar{x})\|^\sigma. \tag{11}$$

By $\lambda_i \geq 0, i \in K, \sum_{i=1}^k \lambda_i = 1$ and hypothesis 3), we have

$$b_0(x, \bar{x}) \varphi_0 \left[\sum_{i=1}^k \lambda_i \left(f_i(x) - f_i(\bar{x}) + \rho_i \|\psi(x, \bar{x})\|^\sigma \right) \right] < 0. \tag{12}$$

Since $g_j(\bar{x}) = 0, j \in J(\bar{x})$ and $\mu_j \geq 0, j \in J(\bar{x})$, and hypothesis 3), we get

$$-b_1(x, \bar{x}) \varphi_1 \left[\sum_{j \in J(\bar{x})} \mu_j g_j(\bar{x}) \right] = 0. \tag{13}$$

In view of the hypothesis 1), one finds from (12) and (13) that

$$\alpha(x, \bar{x}) \left\langle \sum_{i=1}^k \lambda_i \xi_i, \eta(x, \bar{x}) \right\rangle < 0, \forall \xi_i \in \partial f_i(\bar{x}), i \in K. \tag{14}$$

$$\beta(x, \bar{x}) \left\langle \sum_{j \in J(\bar{x})} \mu_j \zeta_j, \eta(x, \bar{x}) \right\rangle + \tau \|\psi(x, \bar{x})\|^\sigma \leq 0, \forall \zeta_j \in \partial g_j(\bar{x}), j \in J(\bar{x}). \tag{15}$$

From $\alpha(x, \bar{x}) > 0, \beta(x, \bar{x}) > 0$ and $\tau \geq 0$, we obtain

$$\left\langle \sum_{i=1}^k \lambda_i \xi_i, \eta(x, \bar{x}) \right\rangle < 0, \forall \xi_i \in \partial f_i(\bar{x}), i \in K. \tag{16}$$

$$\left\langle \sum_{j \in J(\bar{x})} \mu_j \zeta_j, \eta(x, \bar{x}) \right\rangle \leq 0, \forall \zeta_j \in \partial g_j(\bar{x}), j \in J(\bar{x}). \tag{17}$$

Also

$$\left\langle \sum_{i=1}^k \lambda_i \xi_i + \sum_{j \in J(\bar{x})} \mu_j \zeta_j, \eta(x, \bar{x}) \right\rangle < 0, \forall \xi_i \in \partial f_i(\bar{x}), i \in K, \forall \zeta_j \in \partial g_j(\bar{x}), j \in J(\bar{x}) \tag{18}$$

which contradicts (10). Hence the result is true.

4. Wolfe Type Duality

In this section, we consider the Wolfe type dual for the primal problem (MP) and establish various duality theorems. Let e be the vector of R^k whose components are all ones.

$$\begin{aligned}
 \text{(MD)} \quad & \text{Maximize } F(u) = f(u) + \bar{\mu}^T g(u)e \\
 & = \left(f_1(u) + \sum_{j=1}^m \bar{\mu}_j g_j(u), \dots, f_k(u) + \sum_{j=1}^m \bar{\mu}_j g_j(u) \right) \\
 & \text{subject to } 0 \in \sum_{i=1}^k \bar{\lambda}_i \partial f_i(u) + \sum_{j=1}^m \bar{\mu}_j \partial g_j(u), \\
 & \bar{\lambda}_i \geq 0, \sum_{i=1}^k \bar{\lambda}_i = 1, \quad i \in K, \\
 & \bar{\mu}_j \geq 0, \quad j \in M.
 \end{aligned}$$

Let

$$U = \left\{ (u, \bar{\lambda}, \bar{\mu}) \in X \times R_+^k \times R_+^m \mid 0 \in \sum_{i=1}^k \bar{\lambda}_i \partial f_i(u) + \sum_{j=1}^m \bar{\mu}_j \partial g_j(u), \bar{\lambda} \geq 0, \sum_{i=1}^k \bar{\lambda}_i = 1, \bar{\mu} \geq 0 \right\}$$

be the set of all feasible solutions in problem (MD).

Theorem 4.1. (weak duality) Let $\forall x \in X_0$ and $\forall (u, \bar{\lambda}, \bar{\mu}) \in W$ be feasible solutions for (MP) and (MD), respectively. Moreover, assume that

- 1) (f, g) is invex of order σ (B, φ) - V -type II at u ,
- 2) $b_1(x, u) > b_0(x, u) > 0$; $a < b \Rightarrow \varphi_0(a) < \varphi_1(b)$; $\alpha(x, u) = \beta(x, u)$.

Then the following can hold:

$$f(x) \geq F(u) - \rho \|\psi(x, u)\|^\sigma. \quad (19)$$

Proof: Suppose contrary to the result that $f(x) < F(u) - \rho \|\psi(x, u)\|^\sigma$ holds, then we have

$$f_i(x) < f_i(u) + \sum_{j=1}^m \bar{\mu}_j g_j(u) - \rho_i \|\psi(x, u)\|^\sigma, \quad i \in K. \quad (20)$$

which implies

$$f_i(x) - f_i(u) + \rho_i \|\psi(x, u)\|^\sigma < \sum_{j=1}^m \bar{\mu}_j g_j(u), \quad i \in K. \quad (21)$$

Using $\bar{\lambda}_i \geq 0, i \in K, \sum_{i=1}^k \bar{\lambda}_i = 1$, we have

$$\sum_{i=1}^k \bar{\lambda}_i \left(f_i(x) - f_i(u) + \rho_i \|\psi(x, u)\|^\sigma \right) < \sum_{j=1}^m \bar{\mu}_j g_j(u). \quad (22)$$

By hypothesis 2), we have

$$b_0(x, u) \varphi_0 \left[\sum_{i=1}^k \bar{\lambda}_i \left(f_i(x) - f_i(u) + \rho_i \|\psi(x, u)\|^\sigma \right) \right] < b_1(x, u) \varphi_1 \left[\sum_{j=1}^m \bar{\mu}_j g_j(u) \right]. \quad (23)$$

with hypothesis 1) and 2), the above inequality yields

$$\alpha(x, u) \left\langle \sum_{i=1}^k \bar{\lambda}_i \xi_i, \eta(x, u) \right\rangle < -\alpha(x, u) \left\langle \sum_{j=1}^m \bar{\mu}_j \zeta_j, \eta(x, u) \right\rangle - \tau \|\psi(x, u)\|^\sigma, \quad (24)$$

$$\forall \xi_i \in \partial f_i(u), i \in K, \forall \zeta_j \in \partial g_j(u), j \in M.$$

That is

$$\left\langle \sum_{i=1}^k \bar{\lambda}_i \xi_i + \sum_{j=1}^m \bar{\mu}_j \zeta_j, \eta(x, u) \right\rangle < -\frac{\tau}{\alpha(x, u)} \|\psi(x, u)\|^\sigma, \quad (25)$$

$$\forall \xi_i \in \partial f_i(u), i \in K, \forall \zeta_j \in \partial g_j(u), j \in M.$$

From $\tau \geq 0$, which implies

$$\left\langle \sum_{i=1}^k \bar{\lambda}_i \xi_i + \sum_{j=1}^m \bar{\mu}_j \zeta_j, \eta(x, u) \right\rangle < 0, \forall \xi_i \in \partial f_i(u), i \in K, \forall \zeta_j \in \partial g_j(u), j \in M. \quad (26)$$

On the other hand, by using the constraint conditions of (MD), there exist

$$\xi_i \in \partial f_i(u), i \in K \text{ and } \zeta_j \in \partial g_j(u), j \in M,$$

such that

$$\sum_{i=1}^k \bar{\lambda}_i \xi_i + \sum_{j=1}^m \bar{\mu}_j \zeta_j = 0. \quad (27)$$

Also,

$$\left\langle \sum_{i=1}^k \bar{\lambda}_i \xi_i + \sum_{j=1}^m \bar{\mu}_j \zeta_j, \eta(x, u) \right\rangle = 0. \quad (28)$$

which contradicts (26). Then the result is true.

Theorem 4.2. (weak duality) Let $\forall x \in X_0$ and $\forall (u, \bar{\lambda}, \bar{\mu}) \in W$ be feasible solutions for (MP) and (MD), respectively. Moreover, assume that

- 1) (f, g) is (pseudo, quasi) invex of order σ $(B, \varphi) - V -$ type II at u ,
- 2) $b_1(x, u) > b_0(x, u) > 0; a < b \Rightarrow \varphi_0(a) < \varphi_1(b) = 0$.

Then the following can hold:

$$f(x) \geq F(u) - \rho \|\psi(x, u)\|^\sigma. \quad (29)$$

Proof: Suppose contrary to the result that $f(x) < F(u) - \rho \|\psi(x, u)\|^\sigma$ holds, then we have

$$f_i(x) < f_i(u) + \sum_{j=1}^m \bar{\mu}_j g_j(u) - \rho_i \|\psi(x, u)\|^\sigma, i \in K. \quad (30)$$

Also

$$f_i(x) - f_i(u) + \rho_i \|\psi(x, u)\|^\sigma < \sum_{j=1}^m \bar{\mu}_j g_j(u), i \in K. \quad (31)$$

Since $\bar{\lambda}_i \geq 0, i \in K, \sum_{i=1}^k \bar{\lambda}_i = 1$, which yields

$$\sum_{i=1}^k \bar{\lambda}_i (f_i(x) - f_i(u) + \rho_i \|\psi(x, \bar{x})\|^\sigma) < \sum_{j=1}^m \bar{\mu}_j g_j(u). \quad (32)$$

It follows from hypothesis 2) that

$$b_0(x, u) \varphi_0 \left[\sum_{i=1}^k \bar{\lambda}_i \left(f_i(x) - f_i(u) + \rho_i \|\psi(x, u)\|^\sigma \right) \right] < b_1(x, u) \varphi_1 \left[\sum_{j=1}^m \bar{\mu}_j g_j(u) \right] = 0. \quad (33)$$

In the view of hypothesis 1), one finds from (33) that

$$\alpha(x, u) \left\langle \sum_{i=1}^k \bar{\lambda}_i \bar{\xi}_i, \eta(x, u) \right\rangle < 0, \forall \bar{\xi}_i \in \partial f_i(u), i \in K. \quad (34)$$

For $\alpha(x, u) > 0$, we have

$$\left\langle \sum_{i=1}^k \bar{\lambda}_i \bar{\xi}_i, \eta(x, u) \right\rangle < 0, \forall \bar{\xi}_i \in \partial f_i(u), i \in K. \quad (35)$$

Since $(u, \bar{\lambda}, \bar{\mu}) \in M$ is a feasible solution for (MD), there exist $\bar{\xi}_i \in \partial f_i(u), i \in K$ and $\bar{\zeta}_j \in \partial g_j(u), j \in M$ such that

$$\sum_{i=1}^k \bar{\lambda}_i \bar{\xi}_i + \sum_{j=1}^m \bar{\mu}_j \bar{\zeta}_j = 0. \quad (36)$$

whence

$$\left\langle \sum_{i=1}^k \bar{\lambda}_i \bar{\xi}_i, \eta(x, u) \right\rangle + \left\langle \sum_{j=1}^m \bar{\mu}_j \bar{\zeta}_j, \eta(x, u) \right\rangle = 0. \quad (37)$$

It follows from (35) that

$$\left\langle \sum_{j=1}^m \bar{\mu}_j \bar{\zeta}_j, \eta(x, u) \right\rangle > 0. \quad (38)$$

For $\beta(x, u) > 0$ and $\tau \geq 0$, which yields

$$\beta(x, u) \left\langle \sum_{j=1}^m \bar{\mu}_j \bar{\zeta}_j, \eta(x, u) \right\rangle + \tau \|\psi(x, u)\|^\sigma > 0. \quad (39)$$

From hypothesis 1), it follows that

$$-b_1(x, u) \varphi_1 \left[\sum_{j=1}^m \bar{\mu}_j g_j(u) \right] > 0. \quad (40)$$

whence

$$b_1(x, u) \varphi_1 \left[\sum_{j=1}^m \bar{\mu}_j g_j(u) \right] < 0. \quad (41)$$

which contradicts (33). Then the result is true.

The following definition is needed in the proof of the strong duality theorem.

Definition 4.1. A point $u \in X$ is called a strict maximizer of order σ for (MD) with respect to a nonlinear function $\psi: X \times X \rightarrow R^n$, if there exists a constant $\rho \in \text{int } R_+^k$ such that

$$F(u) + \rho \|\psi(x, u)\|^\sigma \geq F(x), \forall x \in X. \quad (42)$$

Theorem 4.3. (strong duality) Assume that $\bar{x} \in X_0$ is a strict minimizer of order σ with respect to ψ for (MP), also there exist $\bar{\lambda} \geq 0, \sum_{i=1}^k \bar{\lambda}_i = 1$ and

$\bar{\mu} \geq 0$, such that $0 \in \sum_{i=1}^k \bar{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j \partial g_j(\bar{x})$ and $\sum_{j=1}^m \bar{\mu}_j g_j(\bar{x}) = 0$. Fur-

thermore, if all the hypothesis of Theorem 4.1 are satisfied for all feasible solutions of (MP) and (MD), then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a strict maximizer of order σ for (MD) with respect to ψ .

Proof: The hypothesis implies that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible solution of (MD). By Theorem 4.1, for any feasible (y, λ, μ) of (MD), we have

$$f(\bar{x}) \geq F(y) - \rho \|\psi(\bar{x}, y)\|^\sigma. \quad (43)$$

That is

$$f_i(\bar{x}) \geq f_i(y) + \sum_{j=1}^m \mu_j g_j(y) - \rho_i \|\psi(\bar{x}, y)\|^\sigma, i \in K. \quad (44)$$

Using $\sum_{j=1}^m \bar{\mu}_j g_j(\bar{x}) = 0$, which yields

$$f_i(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j g_j(\bar{x}) + \rho_i \|\psi(\bar{x}, y)\|^\sigma \geq f_i(y) + \sum_{j=1}^m \mu_j g_j(y), i \in K. \quad (45)$$

whence

$$F(\bar{x}) + \rho \|\psi(\bar{x}, y)\|^\sigma \geq F(y). \quad (46)$$

Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a strict maximizer of order σ for (MD) with respect to ψ .

5. Conclusion

In this paper, we have defined a class of new generalized functions. By using the new functions, we have presented a sufficient optimality condition and Wolfe type duality results for a nondifferentiable multiobjective problem. The present results can be further generalized for other programming problems.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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