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Brief Investigation on Square Root of a Node of T_3 Tree

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Abstract

The article investigates some properties of square root of T_3 tree's nodes. It first proves several inequalities that are helpful to estimate the square root of a node, and then proves several theorems to describe the distribution of the square root of the nodes on T_3 tree.

Keywords

Square Root, Node, Valuated Binary Tree, Inequality

1. Introduction

Article [1] introduced the T_3 tree and showed a number of properties of tree, including divisibility, multiples and divisors and multiplications of the nodes. Looking through the other papers that are related with the article [1], such as articles [2]-[7], one can see that the T_3 tree is really a new attempt to study integers. However, one can also see that, there has not been an article that concerns the square root of a node in the T_3 tree. As is known, a divisor of integer N must be no bigger than $\sqrt{N}$. Hence the location where $\sqrt{N}$ lies in the T_3 tree is important for finding N's divisor. Accordingly, this article makes an investigation on the issue and presents the results.

2. Preliminaries

2.1. Symbols and Notations

Symbol T_3 is the T_3 tree that was introduced in [1] and [2] and symbol $N_{(k,j)}$ is by default the node at position j on level k of T_3, where $k \ge 0$ and $0 \le j \le 2^k - 1$. An integer X is said to be *clamped* on level k of T_3 if $2^{k+1} + 1 \le X \le 2^{k+2} - 1$ and symbol $X \triangleq k$ indicates X is clamped on level k. An odd interval $[a,b]$ is a set of consecutive odd numbers that take a as lower bound and b as upper bound, for example, $[3,11]=\{3,5,7,9,11\}$. Intervals in this whole article are by default the odd ones unless particularly mentioned. Symbol $\lfloor x \rfloor$ is the floor function, an integer function of real number x that satisfies inequality $x-1<\lfloor x \rfloor \le x$, or equivalently $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$. Symbol $A \Rightarrow B$ means conclusion B can be derived from condition A.

2.2. Lemmas

Lemma 1 (See in [1]). Let $N_{(k,j)}$ be the node at the j^{th} position on the k^{th} level of T_3 with $k \ge 0$ and $0 \le j \le 2^k - 1$; then $2^{k+1}+1 \le N_{(k,j)} \le 2^{k+2}-1$.

Lemma 2 (See in [8]). For real numbers x and y, it holds

(P2) $\lfloor x \rfloor - \lfloor y \rfloor - 1 \le \lfloor x-y \rfloor \le \lfloor x \rfloor - \lfloor y \rfloor < \lfloor x \rfloor - \lfloor y \rfloor + 1$

(P8) $n\lfloor x \rfloor \le \lfloor nx \rfloor$ with n being a positive integer

(P13) $x \le y \Rightarrow \lfloor x \rfloor \le \lfloor y \rfloor$

(P17) $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor$, $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor$

3. Main Results and Proofs

Theorem 1. Let $a>1$ be an integer and $x>0$ be a real number; then it holds

$$a^{\lfloor x \rfloor - 1} < a^{\lfloor x \rfloor} \le a^x < a^{\lfloor x \rfloor + 1} \quad (1)$$

and

$$a^{\lfloor x \rfloor - 1} < a^{\lfloor x \rfloor} \le \lfloor a^x \rfloor \le a^{\lfloor x \rfloor + 1} \quad (2)$$

Proof. Since $a>1$, the definition $x-1<\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ immediately yields

$$a^{\lfloor x \rfloor - 1} < a^{\lfloor x \rfloor} \le a^x < a^{\lfloor x \rfloor + 1}$$

Since a and $\lfloor x \rfloor$ are integers, it yields by Lemma 2 (**P13**)

$$a^{\lfloor x \rfloor} \le \lfloor a^x \rfloor \le a^{\lfloor x \rfloor + 1}$$

Considering $\frac{a^{\lfloor x \rfloor}}{a^{\lfloor x \rfloor - 1}} = a > 1$, it knows $a^{\lfloor x \rfloor - 1} < a^{\lfloor x \rfloor}$; consequently

$$a^{\lfloor x \rfloor - 1} < a^{\lfloor x \rfloor} \le \lfloor a^x \rfloor \le a^{\lfloor x \rfloor + 1}$$

□

Theorem 1. Let n be a positive integer and

$$b_0 = n^2, b_1 = n^2+1, \cdots, b_i = n^2+i, \cdots, b_{2n} = n^2+2n, b_{2n+1} = n^2+2n+1$$

then

$$\left\lfloor \sqrt{b_i} \right\rfloor_{(i=1,\cdots,2n)} = \left\lfloor \sqrt{b_0} \right\rfloor = n, \ \left\lfloor \sqrt{b_{2n+1}} \right\rfloor = n+1 \quad (3)$$

Proof. $\left\lfloor \sqrt{b_0} \right\rfloor = n$ and $\left\lfloor \sqrt{b_{2n+1}} \right\rfloor = n+1$ obviously hold. Now consider that, for $i = 1, 2, \cdots, 2n$, it holds

$$n = \sqrt{b_0} < \sqrt{b_i} = n\sqrt{1 + \frac{i}{n^2}} < \sqrt{b_{2n+1}} = n + 1$$

This is to say that, $\sqrt{b_i}\Big|_{(i=1,2,\cdots,2n)} \in (n, n+1)$; since n is an integer, it is sure by definition of the floor function

$$\left\lfloor \sqrt{b_i} \right\rfloor_{(i=1,\cdots,2n)} = n$$

□

Corollary 1. Let n and α be a positive integers and

$b_0 = n^{2\alpha}, b_1 = n^{2\alpha} + 1, \cdots, b_i = n^{2\alpha} + i, \cdots, b_{2n^\alpha} = n^{2\alpha} + 2n^\alpha, b_{2n^\alpha+1} = n^{2\alpha} + 2n^\alpha + 1$

then

$$\sqrt{b_i}\Big|_{(i=0,1,2,\cdots,2n^\alpha)} = n^\alpha, \sqrt{b_{2n^\alpha+1}} = n^\alpha + 1 \tag{4}$$

Proof. (Omitted)

□

Example 1. Take $b_0 = 2^8, b_1 = 2^8 + 1 = 257, \cdots, b_{2\times 2^4} = 2^8 + 2\times 2^4 = 288$, $b_{2\times 2^4+1} = 2^8 + 2\times 2^4 + 1 = 289$; then $\left\lfloor \sqrt{b_0} \right\rfloor = 2^4, \left\lfloor \sqrt{b_1} \right\rfloor = 16, \cdots, \left\lfloor \sqrt{b_{2\times 2^4}} \right\rfloor = 16, \left\lfloor \sqrt{b_{2\times 2^4+1}} \right\rfloor = 17$.

Theorem 2. Let n and α be a positive integers and

$$b_0 = n^{2\alpha+1}, b_1 = n^{2\alpha+1} + 1, \cdots, b_i = n^{2\alpha+1} + i, \cdots, b_{2n^\alpha} = n^{2\alpha+1} + 2n^\alpha\sqrt{n},$$
$$b_{2n^\alpha+1} = n^{2\alpha+1} + 2n^\alpha\sqrt{n} + 1$$

then

$$\left\lfloor n^\alpha\sqrt{n} \right\rfloor \le \left\lfloor \sqrt{b_i} \right\rfloor\Big|_{(i=0,1,2,\cdots,2n^\alpha+1)} \le \left\lfloor n^\alpha\sqrt{n} \right\rfloor + 1$$

Proof. See the following deductions.

1) $b_{2n^\alpha+1} = n^{2\alpha+1} + 2n^\alpha\sqrt{n} + 1 = \left(n^\alpha\sqrt{n} + 1\right)^2 \Rightarrow \sqrt{b_{2n^\alpha+1}} = n^\alpha\sqrt{n} + 1$
$\Rightarrow \left\lfloor \sqrt{b_{2n^\alpha+1}} \right\rfloor = \left\lfloor n^\alpha\sqrt{n} \right\rfloor + 1$

2) By Lemma 2(**P13**)

$$n^{2\alpha+1} = b_0 < b_1 = n^{2\alpha+1} + 1 < \cdots < b_{2n^\alpha} = n^{2\alpha+1} + 2n^\alpha\sqrt{n} < b_{2n^\alpha+1} = \left(n^\alpha\sqrt{n} + 1\right)^2$$
$$\Rightarrow n^\alpha\sqrt{n} = \sqrt{b_0} < \sqrt{b_i}\Big|_{(i=1,2,\cdots,2n^\alpha)} < n^\alpha\sqrt{n} + 1$$
$$\Rightarrow \left\lfloor n^\alpha\sqrt{n} \right\rfloor = \sqrt{b_0} \le \sqrt{b_i}\Big|_{(i=1,2,\cdots,2n^\alpha)} \le \left\lfloor n^\alpha\sqrt{n} \right\rfloor + 1$$

□

Example 2. Take $b_0, b_1, \cdots, b_{2n^\alpha}$ and $b_{2n^\alpha+1}$ by

$b_0 = 2^9, b_1 = 2^9 + 1 = 513, \cdots, b_{2^5-1} = 2^9 + 2^5 - 1 = 543$, $b_{2^5} = 2^9 + 2^5 = 544$,

$b_{2^5+1} = 2^9 + 2^5 + 1 = 545$ then

$$\lfloor\sqrt{b_0}\rfloor = \lfloor\sqrt{512}\rfloor = 22,$$
$$\lfloor\sqrt{b_1}\rfloor = \lfloor\sqrt{513}\rfloor = 22,$$
$$\lfloor\sqrt{b_2}\rfloor = \lfloor\sqrt{514}\rfloor = 22,$$
$$\cdots,$$
$$\lfloor\sqrt{b_{2^4}}\rfloor = \lfloor\sqrt{2^9+2^4}\rfloor = \lfloor\sqrt{528}\rfloor = 22,$$
$$\lfloor\sqrt{b_{2^4+1}}\rfloor = \lfloor\sqrt{2^9+17}\rfloor = \lfloor\sqrt{529}\rfloor = 23,$$
$$\cdots,$$
$$\lfloor\sqrt{b_{2\times 2^4-1}}\rfloor = \lfloor\sqrt{543}\rfloor = 23,$$
$$\lfloor\sqrt{b_{2\times 2^4}}\rfloor = \lfloor\sqrt{544}\rfloor = 23,$$
$$\lfloor\sqrt{b_{2\times 2^4+1}}\rfloor = \lfloor\sqrt{545}\rfloor = 23$$

Theorem 4 Suppose integer k satisfies $k > 2$ and $N_{(k,0)}$ be the leftmost node on level k of $\boldsymbol{T}_3$; then $\lfloor\sqrt{N_{(k,0)}}\rfloor$ is even if k is odd, whereas, it can be either odd or even if k is even.

Proof. Since $N_{(k,0)} = 2^{k+1} + 1$, it knows by Corollary 1 $\lfloor\sqrt{N_{(k,0)}}\rfloor = 2^{\frac{k+1}{2}}$ for an odd k. If k is even, let it be $k = 2\alpha + 1$; then by Theorem 2 $\lfloor\sqrt{N_{(k,0)}}\rfloor = \lfloor 2^\alpha\sqrt{2}\rfloor$ or $\lfloor\sqrt{N_{(k,0)}}\rfloor = \lfloor 2^\alpha\sqrt{2}\rfloor + 1$, which indicates $\lfloor\sqrt{N_{(k,0)}}\rfloor$ can be either odd or even.

□

Example 3. Taking $N_{(7,0)}, N_{(11,0)}, N_{(19,0)}, N_{(8,0)}, N_{(10,0)}$ and $N_{(16,0)}$ as examples results in the following results.

$$N_{(7,0)} = 2^{7+1} + 1 = 257 \Rightarrow \lfloor\sqrt{N_{(7,0)}}\rfloor = 16$$
$$N_{(11,0)} = 2^{11+1} + 1 = 4097 \Rightarrow \lfloor\sqrt{N_{(11,0)}}\rfloor = 64$$
$$N_{(19,0)} = 2^{19+1} + 1 = 1048577 \Rightarrow \lfloor\sqrt{N_{(19,0)}}\rfloor = 1024$$
$$N_{(8,0)} = 2^{8+1} + 1 = 513 \Rightarrow \lfloor\sqrt{N_{(8,0)}}\rfloor = 22$$
$$N_{(10,0)} = 2^{10+1} + 1 = 1025 \Rightarrow \lfloor\sqrt{N_{(10,0)}}\rfloor = 45$$
$$N_{(16,0)} = 2^{16+1} + 1 = 131073 \Rightarrow \lfloor\sqrt{N_{(16,0)}}\rfloor = 362$$

Theorem 5. Suppose integers k and j satisfy $k > 2$ and $0 \le j \le 2^k - 1$; let $N_{(k,j)}$ be the node at position j on level k of $\boldsymbol{T}_3$; then it holds

$$2^{\left\lfloor\frac{k+1}{2}\right\rfloor} - 1 < 2^{\left\lfloor\frac{k+1}{2}\right\rfloor} \le \lfloor\sqrt{N_{(k,j)}}\rfloor \le 2^{\left\lfloor\frac{k}{2}\right\rfloor+2} < 2^{\left\lfloor\frac{k}{2}\right\rfloor+2} + 1 \tag{5}$$

Proof. Since $2^{k+1} + 1 \le N_{(k,j)} \le 2^{k+2} - 1$, it yields $2^{k+1} < N_{(k,j)} < 2^{k+2}$; hence it holds

$$2^{\frac{k+1}{2}} < \sqrt{N_{(k,j)}} < 2^{\frac{k}{2}+1}$$

By Lemma 2 (**P13**), it yields

$$\left\lfloor 2^{\frac{k+1}{2}} \right\rfloor \le \left\lfloor \sqrt{N_{(k,j)}} \right\rfloor \le \left\lfloor 2^{\frac{k}{2}+1} \right\rfloor \tag{6}$$

By Theorem 1, it holds

$$2^{\left\lfloor \frac{k+1}{2} \right\rfloor} \le 2^{\frac{k+1}{2}}$$

and

$$2^{\frac{k}{2}+1} < 2^{\left\lfloor \frac{k}{2} \right\rfloor + 2}$$

Hence it results in

$$2^{\left\lfloor \frac{k+1}{2} \right\rfloor} = \left\lfloor 2^{\left\lfloor \frac{k+1}{2} \right\rfloor} \right\rfloor \le \left\lfloor 2^{\frac{k+1}{2}} \right\rfloor \le \left\lfloor \sqrt{N_{(k,j)}} \right\rfloor \le \left\lfloor 2^{\frac{k}{2}+1} \right\rfloor \le \left\lfloor 2^{\left\lfloor \frac{k}{2} \right\rfloor + 2} \right\rfloor = 2^{\left\lfloor \frac{k}{2} \right\rfloor + 2}$$

That is

$$2^{\left\lfloor \frac{k+1}{2} \right\rfloor} \le \left\lfloor \sqrt{N_{(k,j)}} \right\rfloor \le 2^{\left\lfloor \frac{k}{2} \right\rfloor + 2} \tag{7}$$

or **equivalently**

$$2^{\left\lfloor \frac{k+1}{2} \right\rfloor} - 1 < \left\lfloor \sqrt{N_{(k,j)}} \right\rfloor < 2^{\left\lfloor \frac{k}{2} \right\rfloor + 2} + 1 \tag{8}$$

□

Corollary 2. $\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor$ is clamped in T_3 on level $\left\lfloor \frac{k+1}{2} \right\rfloor - 1$ and or level $\left\lfloor \frac{k}{2} \right\rfloor$.

Proof. Since $2^{\left\lfloor \frac{k+1}{2} \right\rfloor} - 1$ the biggest node on level $\left\lfloor \frac{k+1}{2} \right\rfloor - 2$ and $2^{\left\lfloor \frac{k}{2} \right\rfloor + 2} + 1$ is the smallest node on level $\left\lfloor \frac{k}{2} \right\rfloor + 1$, it knows by (8) $\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor$ may be clamped on levels from $\left\lfloor \frac{k+1}{2} \right\rfloor - 1$ to $\left\lfloor \frac{k}{2} \right\rfloor$, totally $\left\lfloor \frac{k}{2} \right\rfloor - \left(\left\lfloor \frac{k+1}{2} \right\rfloor - 1 \right) + 1$ levels.

By Lemma 2 (**P2**) $\left\lfloor \frac{k}{2} \right\rfloor - \left(\left\lfloor \frac{k+1}{2} \right\rfloor - 1 \right) + 1 \ge 2 + \left\lfloor \frac{k}{2} - \frac{k+1}{2} \right\rfloor = 2 - 1 = 1$

By Lemma 2 (**P17** & **P8**)

$$\left\lfloor \frac{k}{2} \right\rfloor - \left(\left\lfloor \frac{k+1}{2} \right\rfloor - 1 \right) + 1 = 2 + \left\lfloor \frac{k}{2} \right\rfloor - \left(\lfloor k \rfloor - \left\lfloor \frac{k}{2} \right\rfloor \right) = 2 + 2\left\lfloor \frac{k}{2} \right\rfloor - \lfloor k \rfloor \le 2$$

Hence the corollary holds

□

Example 4. Taking the smallest nodes and the biggest nodes on level 7 and level 10 respectively, it can see that $\left\lfloor\sqrt{N_{(7,*)}}\right\rfloor$ is clamped on 2 levels, whereas $\left\lfloor\sqrt{N_{(10,*)}}\right\rfloor$ is clamped on 1 level.

$$N_{(7,0)} = 2^{7+1} + 1 = 257 \Rightarrow \left\lfloor\sqrt{N_{(7,0)}}\right\rfloor = 16 \triangleq 2$$

$$N_{\left(7,2^7-1\right)} = 2^{7+2} - 1 = 511 \Rightarrow \left\lfloor\sqrt{N_{\left(7,2^7-1\right)}}\right\rfloor = 22 \triangleq 3$$

$$N_{(10,0)} = 2^{11} + 1 = 2047 \Rightarrow \left\lfloor\sqrt{N_{(10,0)}}\right\rfloor = 45 \triangleq 4$$

$$N_{\left(10,2^{10}-1\right)} = 2^{12} - 1 = 4095 \Rightarrow \left\lfloor\sqrt{N_{\left(10,2^{10}-1\right)}}\right\rfloor = 63 \triangleq 4$$

4. Conclusion

Elementary number theory shows that an integer must have a divisor smaller than the square root of the integer itself. Hence the square root is undoubtedly an important issue of an integer. Since T_3 tree is considered to be a new tool to study integers, the square root of a node is certainly helpful to know the distribution of the node's divisors. The properties proved in this article are sure to provide a know-about the square root of the nodes. We hope it will be useful in the future.

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