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Applications for Certain Classes of Spirallike Functions Defined by the Srivastava-Attiya Operator

Jae Ho Choi

Department of Mathematics Education, Daegu National University of Education, Daegu, Korea
Email: choijh@dnue.ac.kr

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Abstract

Let the function f be analytic in $\mathbb{U}=\{z: z\in\mathbb{C} \text{ and } |z|<1\}$ and be given by $f(z)=z+\sum_{k=2}^{\infty}a_k z^k$. In this paper, making use of the Srivastava-Attiya operator $\mathcal{L}_{s,b}$, we introduce two classes of analytic functions and investigate some convolution properties and coefficient estimates for these classes. Furthermore, several inclusion properties involving these and other families of integral operators are also considered.

Keywords

Analytic Function, Hadamard Product, Subordination, Srivastava-Attiya Operator

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z: z\in\mathbb{C} \text{ and } |z|<1\}$. Also let f and g be analytic in $\mathbb{U}$ with $f(0)=g(0)$. Then we say that f is *subordinate* to g in $\mathbb{U}$, written $f\prec g$ or $f(z)\prec g(z)$, if there exists the Schwarz function w, analytic in $\mathbb{U}$ such that $w(0)=0$, $|w(z)|<1$ and $f(z)=g(w(z))$ $(z\in\mathbb{U})$. We also observe that

$$f(z)\prec g(z) \quad \text{in } \mathbb{U}$$

if and only if

$$f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U})$$

whenever g is univalent in $\mathbb{U}$.

For functions $f_j(z) \in \mathcal{A}$, given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (j = 1, 2),$$

we define the *Hadamard product* (*or convolution*) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \quad (z \in \mathbb{U}).$$

Making use of the principle of subordination between analytic functions, Bhoosnurmath and Devadas [1] considered the subclasses $\mathcal{S}^{\alpha}[A,B]$ and $\mathcal{K}^{\alpha}[A,B]$ of the class $\mathcal{A}$ for $|\alpha| < \frac{\pi}{2}$ and $-1 \le B < A \le 1$ as following (see also [2] and [3]):

$$\mathcal{S}^{\alpha}[A,B] = \left\{ f \in \mathcal{A} : \mathrm{e}^{i\alpha} \frac{zf'(z)}{f(z)} \prec \cos\alpha \left(\frac{1+Az}{1+Bz} \right) + i \sin\alpha \ (z \in \mathbb{U}) \right\}, \tag{1.2}$$

and

$$\mathcal{K}^{\alpha}[A,B] = \left\{ f \in \mathcal{A} : \mathrm{e}^{i\alpha} \frac{(zf'(z))'}{f(z)} \prec \cos\alpha \left(\frac{1+Az}{1+Bz} \right) + i \sin\alpha \ (z \in \mathbb{U}) \right\}. \tag{1.3}$$

We note that

$$\mathcal{S}^{0}[A,B] = \mathcal{S}[A,B], \quad \mathcal{K}^{0}[A,B] = \mathcal{K}[A,B] \quad (-1 \le B < A \le 1),$$

where the classes $\mathcal{S}[A,B]$ and $\mathcal{K}[A,B]$ are introduced and studied by many authors (see [4] [5] and [6]). Furthermore, $\mathcal{S}^{\alpha}[1,-1] \equiv \mathcal{S}(\alpha)$ denote the α-spirallike functions studied by Spacek [7], which are univalent in $\mathbb{U}$.

With a view to define the Srivastava-Attiya transform, we recall here a general Hurwitz-Lerch zeta function, which is defined in [8] by the following series:

$$\Phi(z,s,a) := \frac{1}{a^s} + \sum_{k=1}^{\infty} \frac{z^k}{(k+a)^s}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^- = \{0,-1,-2,\cdots\}; s \in \mathbb{C} \text{ when } z \in \mathcal{U}; \mathrm{Re}(s) > 1 \text{ when } |z| = 1 \right)$$

For further interesting properties and characteristics of the Hurwitz-Lerch Zeta and other related functions $\Phi(z,s,a)$ see [9] [10] and [11].

Recently, Srivastava and Attiya [12] have introduced the linear operator $\mathcal{L}_{s,b} : \mathcal{A} \to \mathcal{A}$, defined in terms of the Hadamard product by

$$\mathcal{L}_{s,b}(f)(z) = \mathcal{G}_{s,b}(z) * f(z) \quad \left(z \in \mathbb{U}; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \right), \tag{1.4}$$

where

$$\mathcal{G}_{s,b} = (1+b)^s \left[\Phi(z,s,b) - b^{-s} \right] \quad (z \in \mathbb{U}). \tag{1.5}$$

The operator $\mathcal{L}_{s,b}$ is now popularly known in the literature as the Srivastava-Attiya operator. Various class-mapping properties of the operator $\mathcal{L}_{s,b}$ (and its variants) are discussed in the recent works of Srivastava and Attiya

[12], Liu [13], Murugusundaramoorthy [14], Yuan and Liu [15] and others.

It is easy to observe from (1.1) and (1.4) that

$$\mathcal{L}_{s,b}(f)(z) = z + \sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^{s} a_k z^k. \tag{1.6}$$

We note that:

1) $\mathcal{L}_{0,b}(f)(z) = f(z)$;

2) $\mathcal{L}_{1,0}(f)(z) = \mathcal{L}(f)(z) = \int_0^z \frac{f(t)}{t}\mathrm{d}t \quad (f \in \mathcal{A})$ (see Alexander [16]);

3) $\mathcal{L}_{m,1}(f)(z) = \mathcal{I}^m f(z) \quad (m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\} = \{0,1,2,3,\cdots\})$ (see Flett [17]);

4) $\mathcal{L}_{\gamma,1}(f)(z) = \mathcal{Q}^{\gamma} f(z) \quad (\gamma > 0)$ (see Jung *et al.* [18]);

5) $\mathcal{L}_{m,0}(f)(z) = \mathcal{L}^m f(z) \quad (m \in \mathbb{N}_0)$ (see Sălăgean [19]).

It is easily verified from (1.6) that

$$z\left(\mathcal{L}_{s,b}(f)(z)\right)' = (b+1)\mathcal{L}_{s-1,b}(f)(z) - b\mathcal{L}_{s,b}(f)(z) \tag{1.7}$$

$$\left(f \in \mathcal{A}; b \in \mathbb{C}\setminus\mathbb{Z}_0^-; s \in \mathbb{C}\right)$$

Next, by using the linear operator $\mathcal{L}_{s,b}$, we introduce the following new classes of analytic functions for $b \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $s \in \mathbb{C}$, $|\alpha| < \frac{\pi}{2}$ and $-1 \le B < A \le 1$:

$$\mathcal{S}_{s,b}^{\alpha}[A,B] := \left\{f \in \mathcal{A} : \mathcal{L}_{s,b}(f)(z) \in \mathcal{S}^{\alpha}[A,B]\,(z \in \mathbb{U})\right\} \tag{1.8}$$

and

$$\mathcal{K}_{s,b}^{\alpha}[A,B] := \left\{f \in \mathcal{A} : \mathcal{L}_{s,b}(f)(z) \in \mathcal{K}^{\alpha}[A,B]\,(z \in \mathbb{U})\right\}. \tag{1.9}$$

It follows from the definitions (1.8) and (1.9) that

$$f(z) \in \mathcal{K}_{s,b}^{\alpha}[A,B] \Leftrightarrow zf'(z) \in \mathcal{S}_{s,b}^{\alpha}[A,B] \quad (z \in \mathbb{U}). \tag{1.10}$$

In this article, we investigate some convolution properties and coefficient estimates for the classes $\mathcal{S}_{s,b}^{\alpha}[A,B]$ and $\mathcal{K}_{s,b}^{\alpha}[A,B]$. Furthermore, several inclusion properties and relevant connections of the results presented here with those obtained in earlier works are also discussed.

2. Convolution Properties and Coefficient Estimates

Unless otherwise mentioned, we will assume in the reminder of this paper that $-1 \le B < A \le 1$, $|\alpha| < \frac{\pi}{2}$ and $|\zeta| = 1$. In order to establish our convolution properties, we shall need the following lemmas due to Bhoosnurnath and Devadas [1] [2].

Lemma 2.1 ([1]). *The function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{S}^{\alpha}[A,B]$ *if and only if*

$$\frac{1}{z}\left\{f(z) * (1-Mz)\frac{z}{(1-z)^2}\right\} \ne 0 \quad (z \in \mathbb{U}), \tag{2.1}$$

where

$$M=\frac{e^{i\alpha}+\left(A\cos\alpha+iB\sin\alpha\right)\zeta}{\left(A-B\right)\zeta\cos\alpha}. \tag{2.2}$$

Lemma 2.2 ([2] Lemma 3 with $n=1$). *The function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{K}^{\alpha}[A,B]$ *if and only if*

$$\frac{1}{z}\left\{f(z)*(1-Nz)\frac{z}{(1-z)^3}\right\}\neq 0 \quad (z\in\mathbb{U}), \tag{2.3}$$

where

$$N=\frac{2e^{i\alpha}+\left[\left(A+B\right)\cos\alpha+i2B\sin\alpha\right]\zeta}{\left(A-B\right)\zeta\cos\alpha}. \tag{2.4}$$

We begin by proving the following theorem.

Theorem 2.3 *The function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{K}^{\alpha}_{s,b}[A,B]$ *if and only if*

$$1-\sum_{k=2}^{\infty}\frac{\left(k-1+kB\zeta\right)e^{i\alpha}-\left(A\cos\alpha+iB\sin\alpha\right)\zeta}{\left(A-B\right)\zeta\cos\alpha}\left(\frac{1+b}{k+b}\right)^{s}a_k z^{k-1}\neq 0 \ \ (z\in\mathbb{U}).$$

Proof. From Lemma 2.1, we find that $f(z)\in\mathcal{S}^{\alpha}_{s,b}[A,B]$ if and only if

$$\frac{1}{z}\left[\mathcal{L}_{s,b}(f)(z)*(1-Mz)\frac{z}{(1-z)^2}\right]\neq 0 \quad (z\in\mathbb{U}), \tag{2.5}$$

where M is given by (2.2). Then, by applying (1.6), the left hand side of (2.5) becomes

$$\begin{aligned}&\frac{1}{z}\left[\mathcal{L}_{s,b}(f)(z)*\left(\frac{z}{(1-z)^2}-\frac{Mz}{(1-z)^2}\right)\right]\\&=\frac{1}{z}\left[z\left(\mathcal{L}_{s,b}(f)(z)\right)'-M\left\{z\left(\mathcal{L}_{s,b}(f)(z)\right)'-\mathcal{L}_{s,b}(f)(z)\right\}\right]\\&=1-\sum_{k=2}^{\infty}\frac{\left(k-1+kB\zeta\right)e^{i\alpha}-\left(A\cos\alpha+iB\sin\alpha\right)\zeta}{\left(A-B\right)\zeta\cos\alpha}\left(\frac{1+b}{k+b}\right)^{s}a_k z^{k-1},\end{aligned}$$

which completes the proof of Theorem 2.3.

Theorem 2.4 *The function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{K}^{\alpha}_{s,b}[A,B]$ *if and only if*

$$1-\sum_{k=2}^{\infty}k\frac{(k-1)e^{i\alpha}-\left[\left(A-kB\right)\cos\alpha-i\left(k-1\right)B\sin\alpha\right]\zeta}{\left(A-B\right)\zeta\cos\alpha}\left(\frac{1+b}{k+b}\right)^{s}a_k z^{k-1}\neq 0 \ \ (z\in\mathbb{U}).$$

Proof. From Lemma 2.2, we observe that $f(z)\in\mathcal{K}^{\alpha}_{s,b}[A,B]$ if and only if

$$\frac{1}{z}\left[\mathcal{L}_{s,b}(f)(z)*(1-Nz)\frac{z}{(1-z)^3}\right]\neq 0 \quad (z\in\mathbb{U}), \tag{2.6}$$

where N is given by (2.4). Then, by using (1.6), the left hand side of (2.6) may be written as

$$\frac{1}{z}\left[\mathcal{L}_{s,b}(f)(z)*\left(\frac{z}{(1-z)^3}-\frac{Nz^2}{(1-z)^3}\right)\right]$$
$$=\frac{1}{z}\left[\frac{1}{2}z\left(z\mathcal{L}_{s,b}(f)(z)\right)''-N\left\{\frac{1}{2}z\left(z\mathcal{L}_{s,b}(f)(z)\right)''-z\left(\mathcal{L}_{s,b}(f)(z)\right)'\right\}\right]$$
$$=1-\sum_{k=2}^{\infty}k\frac{(k-1)e^{i\alpha}-\left[(A-kB)\cos\alpha-i(k-1)B\sin\alpha\right]\zeta}{(A-B)\zeta\cos\alpha}\left(\frac{1+b}{k+b}\right)^s a_k z^{k-1},$$

which evidently proves Theorem 2.4.

Next, we determine coefficients estimates for a function of the form (1.1) to be in the classes $\mathcal{S}_{s,b}^{\alpha}[A,B]$ and $\mathcal{K}_{s,b}^{\alpha}[A,B]$.

Theorem 2.5 *Let* $b>-1$ *and* $s\ge 0$. *The function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{S}_{s,b}^{\alpha}[A,B]$ *if its coefficients satisfy the condition*

$$\sum_{k=2}^{\infty}\left(k-1+\left|A\cos\alpha+iB\sin\alpha-kBe^{i\alpha}\right|\right)\left(\frac{1+b}{k+b}\right)^s|a_k|\le(A-B)\cos\alpha.$$

Proof. Since

$$\left|1-\sum_{k=2}^{\infty}\frac{(k-1+kB\zeta)e^{i\alpha}-(A\cos\alpha+iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}\left(\frac{1+b}{k+b}\right)^s a_k z^{k-1}\right|$$
$$\ge 1-\sum_{k=2}^{\infty}\left|\frac{(k-1+kB\zeta)e^{i\alpha}-(A\cos\alpha+iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}\right|\left(\frac{1+b}{k+b}\right)^s|a_k|,$$

and

$$\left|\frac{(k-1+kB\zeta)e^{i\alpha}-(A\cos\alpha+iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}\right|$$
$$=\frac{\left|(k-1)e^{i\alpha}-\left(A\cos\alpha+iB\sin\alpha-kBe^{i\alpha}\right)\zeta\right|}{(A-B)\cos\alpha}$$
$$\le\frac{(k-1)+\left|A\cos\alpha+iB\sin\alpha-kBe^{i\alpha}\right|}{(A-B)\cos\alpha},$$

by virtue of Theorem 2.3, we conclude that $f(z)\in\mathcal{S}_{s,b}^{\alpha}[A,B]$. Thus, the proof of Theorem 2.5 is completed.

By using arguments similar to those above with Theorem 2.4, we can prove the following theorem.

Theorem 2.6 *Let* $b>-1$ *and* $s\ge 0$. *The function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{K}_{s,b}^{\alpha}[A,B]$ *if its coefficients satisfy the condition*

$$\sum_{k=2}^{\infty}k\left\{k-1+\left|(A-kB)\cos\alpha-i(k-1)B\sin\alpha\right|\right\}\left(\frac{1+b}{k+b}\right)^s|a_k|\le(A-B)\cos\alpha.$$

3. Inclusion Properties and Applications

To prove the inclusion properties for the classes $\mathcal{S}_{s,b}^{\alpha}[A,B]$ and $\mathcal{K}_{s,b}^{\alpha}[A,B]$, we shall require the following lemma due to Eenigenburg *et al.* [20].

Lemma 3.1 ([20]). *Let* $h(z)$ *be convex univalent in* $\mathbb{U}$ *with*

$\mathrm{Re}\{\beta h(z)+\nu\}>0$ *for all* $z\in\mathbb{U}$. *If* $p(z)$ *is analytic in* $\mathbb{U}$ *with* $p(0)=h(0)$, *then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\nu}\prec h(z)\quad(z\in\mathbb{U})$$

implies that $p(z)\prec h(z)\quad(z\in\mathbb{U})$.

By applying Lemma 3.1, we prove

Theorem 3.2 *Let* $b>-1$ *and* $s\geq 0$. *If*

$$\mathrm{Re}\left\{\mathrm{e}^{-i\alpha}\frac{z}{1+Bz}\right\}>-\frac{b+1}{(A-B)\cos\alpha}\quad(z\in\mathbb{U}),\tag{3.1}$$

then

$$\mathcal{S}^{\alpha}_{s-1,b}[A,B]\subset\mathcal{S}^{\alpha}_{s,b}[A,B].$$

Proof. Let $f(z)\in\mathcal{S}^{\alpha}_{s-1,b}[A,B]$ for $b>-1$ and $s\geq 0$, and set

$$p(z)=\mathrm{e}^{i\alpha}\frac{z\left(\mathcal{L}_{s,b}(f)(z)\right)'}{\mathcal{L}_{s,b}(f)(z)}\quad(z\in\mathbb{U}),\tag{3.2}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=\mathrm{e}^{i\alpha}$. By applying the identity (1.7), we obtain

$$\mathrm{e}^{-i\alpha}p(z)+b=(b+1)\frac{\mathcal{L}_{s-1,b}(f)(z)}{\mathcal{L}_{s,b}(f)(z)}.\tag{3.3}$$

Making use of the logarithmic differentiation on both side in (3.3), we have

$$p(z)+\frac{zp'(z)}{\mathrm{e}^{-i\alpha}p(z)+b}\prec\cos\alpha\left(\frac{1+Az}{1+Bz}\right)+i\sin\alpha=h(z).\tag{3.4}$$

Since the function $h(z)$ is convex univalent in $\mathbb{U}$ with $h(0)=\mathrm{e}^{i\alpha}$, from (3.1) we see that

$$\mathrm{Re}\left\{\mathrm{e}^{-i\alpha}h(z)+b\right\}>0\quad(z\in\mathbb{U}).$$

Thus, by using Lemma 3.1 and (3.4), we observe that $p(z)\prec h(z)$ in $\mathbb{U}$, so that $f(z)\in\mathcal{S}^{\alpha}_{s,b}[A,B]$. This completes the proof of theorem 3.2.

Theorem 3.3 *Let* $b>-1$ *and* $s\geq 0$. *Suppose that* (3.1) *holds for all* $z\in\mathbb{U}$. *Then*

$$\mathcal{K}^{\alpha}_{s-1,b}[A,B]\subset\mathcal{K}^{\alpha}_{s,b}[A,B].$$

Proof. Applying (1.10) and Theorem 3.2, we observe that

$$\begin{aligned}f(z)\in\mathcal{K}^{\alpha}_{s-1,b}[A,B]&\Leftrightarrow zf'(z)\in\mathcal{S}_{s-1,b}[A,B]\\&\Rightarrow zf'(z)\in\mathcal{S}^{\alpha}_{s,b}[A,B]\\&\Leftrightarrow f(z)\in\mathcal{K}^{\alpha}_{s,b}[A,B],\end{aligned}$$

which evidently proves Theorem 3.3.

Putting $b=1$ and $A=-B=1$ in Theorem 3.2 and 3.3, we have the following corollary.

Corollary 3.4 *Suppose that* $s \geq 0$ *and*

$$\mathrm{Re}\left\{ \mathrm{e}^{-i\alpha} \frac{z}{1-z} \right\} > -\frac{1}{\cos\alpha} \quad (z \in \mathbb{U}). \tag{3.5}$$

Then

$$\mathcal{S}_{s-1,1}^{\alpha}[1,-1] \subset \mathcal{S}_{s,1}^{\alpha}[1,-1]$$

and

$$\mathcal{K}_{s-1,1}^{\alpha}[1,-1] \subset \mathcal{K}_{s,1}^{\alpha}[1,-1].$$

Finally, we consider the generalized Bernardi-Libera-Livingston integral operator $\mathcal{J}_\sigma(f)$ defined by (*cf.* [21] [22] and [23])

$$\mathcal{J}_\sigma(f) \equiv \mathcal{J}_\sigma(f)(z) := \frac{\sigma+1}{z^\sigma} \int_0^z t^{\sigma-1} f(t)\,\mathrm{d}t \quad (f \in \mathcal{A}; \sigma > -1). \tag{3.6}$$

Theorem 3.5 *Let* $b > -1$, $s \geq 0$ *and* $\sigma > -1$. *Suppose that*

$$\mathrm{Re}\left\{ \mathrm{e}^{-i\alpha} \frac{z}{1+Bz} \right\} > -\frac{\sigma+1}{(A-B)\cos\alpha} \quad (z \in \mathbb{U}). \tag{3.7}$$

If $f(z) \in \mathcal{S}_{s,b}^{\alpha}[A,B]$, then $\mathcal{J}_\sigma(f)(z) \in \mathcal{S}_{s,b}^{\alpha}[A,B]$.

Proof. If we set

$$p(z) = \mathrm{e}^{i\alpha} \frac{z\left(\mathcal{L}_{s,b}\mathcal{J}_\sigma(f)(z)\right)'}{\mathcal{L}_{s,b}\mathcal{J}_\sigma(f)(z)} \quad (z \in \mathbb{U}), \tag{3.8}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = \mathrm{e}^{i\alpha}$. By virtue of (3.5), we observe that

$$z\left(\mathcal{L}_{s,b}\mathcal{J}_\sigma(f)(z)\right)' = (\sigma+1)\mathcal{L}_{s,b}(f)(z) - \sigma\mathcal{L}_{s,b}\mathcal{J}_\sigma(f)(z) \quad (z \in \mathbb{U}). \tag{3.9}$$

In view of (3.7) and (3.8), we have

$$\mathrm{e}^{-i\alpha} p(z) + \sigma = (\sigma+1) \frac{\mathcal{L}_{s,b}(f)(z)}{\mathcal{L}_{s,b}\mathcal{J}_\sigma(f)(z)}.$$

By using same argument as in the proof of Theorem 3.2 with (3.6), we conclude that $\mathcal{J}_\sigma(f)(z) \in \mathcal{S}_{s,b}^{\alpha}[A,B]$. This evidently completes the proof of Theorem 3.5.

Theorem 3.6 *Let* $b > -1$, $s \geq 0$ *and* $\sigma > -1$. *Suppose that* (3.6) *holds for all* $z \in \mathbb{U}$. *If* $f(z) \in \mathcal{K}_{s,b}^{\alpha}[A,B]$, *then* $\mathcal{J}_\sigma(f)(z) \in \mathcal{K}_{s,b}^{\alpha}[A,B]$.

Proof. By using Theorem 3.4, it follows that

$$\begin{aligned} f(z) \in \mathcal{K}_{s,b}[A,B] &\Leftrightarrow zf'(z) \in \mathcal{S}_{s,b}[A,B] \\ &\Rightarrow \mathcal{J}_\sigma(zf'(z)) \in \mathcal{S}_{s,b}[A,B] \\ &\Leftrightarrow z\left(\mathcal{J}_\sigma(f)(z)\right)' \in \mathcal{S}_{s,b}[A,B] \\ &\Rightarrow \mathcal{J}_\sigma(f)(z) \in \mathcal{K}_{s,b}[A,B], \end{aligned}$$

which completes the proof of Theorem 3.6.

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