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Modified Generalized Degree Distance of Some Graph Operations

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Abstract

The modified generality degree distance, is defined as: $H_\lambda^*(G)=\sum_{\{u,v\}\subseteq V(G)} d^\lambda(u,v)(d_G(u)d_G(v))$, which is a modification of the generality degree distance. In this paper, we give some computing formulas of the modified generality degree distance of some graph operations, such as, composition, join, etc.

Keywords

Degree Distance, Modified Generalized Degree Distance, Graph Operations

1. Introduction

Throughout this paper all graphs considered are finite and simple graphs. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$, and denoted by n and m the values of $|V(G)|$ and $|E(G)|$, respectively. For vertices $u,v\in V(G)$, the **distance** between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G, and let $d_G(v)$ be the degree of a vertex $v\in V(G)$. The **complement** of G, denoted by $\overline{G}$, is the graph with vertex set $V(G)$, in which two distinct vertices are adjacent if and only if they are not adjacent in G, and denoted by $\overline{m}$ the value of $|E(\overline{G})|$. We use C_n, P_n and K_n to denote the cycle, path and complete graph on n vertices, respectively. Other terminology and notation will be introduced where it is needed or can be found in [1].

A **Topological index** of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In chemistry, topological index is used for modeling physicochemical, pharmacologic, biological and other properties of chemical compounds [2]. One of the oldest and well-studied distance-based graph invariants is the Wiener number $W(G)$, also termed as **Wiener index** in chemical or mathematical chemistry literature, which is defined [3] as the sum of distances over all unordered vertex pairs in G, namely,

$$W(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v).$$

Dobrynin and Kochetova [4] and Gutman [5] introduced a new graph invariant with the name **degree distance** or **Schultz molecular topological index**, which is defined as follows:

$$DD(G)=\sum_{\{u,v\}\subseteq V(G)} \left(d_G(u)+d_G(v)\right)d_G(u,v).$$

In [6], Gutman and Klavzar defined product-degree distance as follow:

$$DD^*(G)=\sum_{\{u,v\}\subseteq V(G)} \left(d_G(u)\cdot d_G(v)\right)d_G(u,v).$$

Note that the degree distance and product-degree distance are degree-weight versions of the Wiener index. We encourage the interested readers to consult [7] [8] [9] for Wiener index. In [4] [10] [11] [12] [13] [14], there are more conclusions for degree distance, which shows that the research of degree distance is a hot topic. In [15], Sagan *et al.* computed some exact formulae for the Wiener polynomial of various graph operations containing Cartesian product, composition, join, disjunction and symmetric difference of graphs, whose concepts will be presented in later part. In [16], Hamzeh *et al.* consider the generalized degree distances of some graph operations. The generalized degree distance of a graph is defined as follow [17]:

$$H_\lambda(G)=\sum_{\{u,v\}\subseteq V(G)} d^\lambda(u,v)\left(d_G(u)+d_G(v)\right). \tag{1}$$

For a real number λ, the modified generalized degree distance, denoted by $H_\lambda^*(G)$, is also defined in [17]:

$$H_\lambda^*(G)=\sum_{\{u,v\}\subseteq V(G)} d^\lambda(u,v)\left(d_G(u)d_G(v)\right). \tag{2}$$

If $\lambda=0$, $H_\lambda(G)=4m$ and $H_\lambda^*(G)=4m^2$. When $\lambda=1$, $H_\lambda(G)=DD(G)$ and $H_\lambda^*(G)=DD^*(G)$, which implies that the generalized degree distance is equal to the degree distance (or Schultz index), and the modified generalized degree distance is equal to the product-degree distance. Therefore the study of this new topological index is important and we try to obtain some new results related to this topological index.

In this paper, we show that the explicit formulas for $H_\lambda^*(G)$ of some graph operations containing the composition, join, disjunction and symmetric difference of graphs, and we apply the results to compute the modified generality degree distance of some special graphs.

Next, we introduce four types of graph operations:

The **join** $G=G_1+G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1,

V_2 and edge sets E_1, E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

The **composition** $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2), see [18].

The **disjunction** $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent with (u_2, v_2) whenever $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$.

The **symmetric difference** $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G) \text{ or } u_2v_2 \in E(H) \text{ but not both}\}$.

In the final of this section, we present several well-known indices: the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ [19], the first Zagreb conindex $\bar{M}_1(G)$ and the second Zagreb coindex $\bar{M}_2(G)$ [20], which will be used in our results.

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2, \quad \bar{M}_1(G) = \sum_{uv \notin E(G)} \left(d_G(u) + d_G(v)\right).$$

$$M_2(G) = \sum_{uv \in E(G)} \left(d_G(u) d_G(v)\right), \quad \bar{M}_2(G) = \sum_{uv \notin E(G)} \left(d_G(u) d_G(v)\right).$$

In fact, $M_1(G)$ can be also expressed as a sum over edges of G,

$$M_1(G) = \sum_{uv \in E(G)} \left(d_G(u) + d_G(v)\right).$$

2. Main Results

The purpose of this section is to compute the modified generalized degree distance for four graph operations. We begin with the following crucial lemma related to distance properties of some graph operations.

Lemma 2.1. [18] [21] *Let G and H be two graphs, then we have*:

1) $|V(G \vee H)| = |V(G[H])| = |V(G \oplus H)| = |V(G)| \cdot |V(H)|$,

$$|E(G \vee H)| = |E(G)| \cdot |V(H)|^2 + |E(H)| \cdot |V(G)|^2 - 2|E(G)| \cdot |E(H)|,$$

$$|E(G + H)| = |E(G)| + |E(H)| + |V(G)| \cdot |V(H)|,$$

$$|E(G \oplus H)| = |E(G)| \cdot |V(H)|^2 + |E(H)| \cdot |V(G)|^2 - 4|E(G)| \cdot |E(H)|,$$

$$|E(G[H])| = |E(G)| \cdot |V(H)|^2 + |E(H)| \cdot |V(G)|.$$

2) The join, composition, disjunction and symmetric difference of graphs are associative and all of them are commutative except from composition.

3) $$d_{G+H}(u, v) = \begin{cases} 0 & u = v \\ 1 & uv \in E(G) \text{ or } uv \in E(H) \text{ or } \left(u \in V(G) \,\&\, V \in V(H)\right) \\ 2 & \text{otherwise,} \end{cases}$$

4) $d_{G[H]}\left((a,b),(c,d)\right)=\begin{cases} d_G(a,c) & a\neq c \\ 0 & a=c\,\&\,b=d \\ 1 & a=c\,\&\,bd\in E(H) \\ 2 & a=c\,\&\,bd\notin E(H), \end{cases}$

5) $d_{G\vee H}\left((a,b),(c,d)\right)=\begin{cases} 0 & a=c\,\&\,b=d \\ 1 & ac\in E(G) \text{ or } bd\in E(H) \\ 2 & \text{otherwise}, \end{cases}$

6) $d_{G\oplus H}\left((a,b),(c,d)\right)=\begin{cases} 0 & a=c\,\&\,b=d \\ 1 & ac\in E(G) \text{ or } bd\in E(H) \text{ but not both} \\ 2 & \text{otherwise}, \end{cases}$

7) $d_{G+H}(a)=\begin{cases} d_G(a)+|V(H)| & a\in V(G) \\ d_H(a)+|V(G)| & a\in V(H) \end{cases},$

8) $d_{G[H]}\left((a,b)\right)=|V(H)|d_G(a)+d_H(b),$

9) $d_{G\oplus H}\left((a,b)\right)=|V(H)|d_G(a)+|V(G)|d_H(b)-2d_G(a)d_H(b),$

10) $d_{G\vee H}\left((a,b)\right)=|V(H)|d_G(a)+|V(G)|d_H(b)-d_G(a)d_H(b).$

Proof. The parts 1) - 5) are consequence of definitions and some well-known results of the book of Imrich and Klavzar [18]. For the proof of 6) - 10) we refer to [21]. ■

For a given graph G_i, we denote n_i and m_i by the number of vertices and edges, respectively. Then we can obtain the modified generalized degree distance of the join graph G_1+G_2 as following:

Theorem 2.1. *Let* G_1 *and* G_2 *be two graphs. Then*

$$\begin{aligned} H_\lambda^*(G_1+G_2) &= 2^\lambda\left(n_2\bar{M}_1(G_1)+n_1\bar{M}_1(G_2)+\bar{M}_2(G_1)+\bar{M}_2(G_2)\right) \\ &\quad +2^\lambda\left(n_2^2\bar{m}_1+n_1^2\bar{m}_2\right)+n_2M_1(G_1)+n_1M_1(G_2)+M_2(G_1) \\ &\quad +M_2(G_2)+4m_1m_2+n_1n_2(2m_1+n_1n_2+2m_2)+n_2^2m_1+n_1^2m_2. \end{aligned}$$

Proof. In the graph G_1+G_2, we can partition the set of pairs of vertices of G_1+G_2 into three subsets A_1, A_2 and A_3. In A_1, we collect all pairs of vertices u and v such that u is in G_1 and v is in G_2. Hence, they are adjacent in G_1+G_2. The sets A_2 and A_3 are the set of pairs of vertices u and v which are in G_1 and G_2, respectively. Therefore, we can partition the sum in the formula of $H_\lambda^*(G_1+G_2)$ into three sums S_i such that S_i is over A_i for $i=1,2,3$. By 3) and 7) of Lemma 2.1, we have

$$\begin{aligned} S_1^* &= \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_1+G_2}^\lambda(u,v)\left(\left(d_{G_1}(u)+n_2\right)\left(d_{G_2}(v)+n_1\right)\right) \\ &= 4m_1m_2+n_2n_2(2m_1+n_1n_2+2m_2). \end{aligned}$$

$$\begin{aligned} S_2^* &= \sum_{\{u,v\}\subseteq V(G_1)} d_{G_1+G_2}^\lambda(u,v)\left(\left(d_{G_1}(u)+n_2\right)\left(d_{G_1}(v)+n_2\right)\right) \\ &= \sum_{uv\in E(G_1)}\left(\left(d_{G_1}(u)+n_2\right)\left(d_{G_1}(v)+n_2\right)\right)+\sum_{uv\notin E(G_1)}2^\lambda\left(\left(d_{G_1}(u)+n_2\right)\left(d_{G_1}(v)+n_2\right)\right) \\ &= 2^\lambda\left(\bar{M}_2(G_1)+n_2\bar{M}_1(G_1)+n_2^2\bar{m}_1\right)+M_2(G_1)+n_2M_1(G_1)+n_2^2m_1. \end{aligned}$$

$$S_3^* = \sum_{\{u,v\}\subseteq V(G_2)} d_{G_1+G_2}^{\lambda}(u,v)\left(\left(d_{G_2}(u)+n_1\right)\left(d_{G_2}(v)+n_1\right)\right)$$
$$= \sum_{uv\in E(G_2)} \left(\left(d_{G_2}(u)+n_1\right)\left(d_{G_2}(v)+n_1\right)\right) + \sum_{uv\notin E(G_2)} 2^{\lambda}\left(\left(d_{G_2}(u)+n_1\right)\left(d_{G_2}(v)+n_1\right)\right)$$
$$= 2^{\lambda}\left(\bar{M}_2(G_2)+n_1\bar{M}_1(G_2)+n_1^2\bar{m}_2\right)+M_2(G_2)+n_1M_1(G_2)+n_1^2m_2.$$

Therefore,

$$H_\lambda^*(G_1+G_2) = S_1^*+S_2^*+S_3^*$$
$$= 2^{\lambda}\left(n_2\bar{M}_1(G_1)+n_1\bar{M}_1(G_2)+\bar{M}_2(G_1)+\bar{M}_2(G_2)\right)+2^{\lambda}\left(n_2^2\bar{m}_1+n_1^2\bar{m}_2\right)$$
$$+n_2M_1(G_1)+n_1M_1(G_2)+M_2(G_1)+M_2(G_2)+4m_1m_2$$
$$+n_1n_2(2m_1+n_1n_2+2m_2)+n_2^2m_1+n_1^2m_2.$$

■

In the above theorem, if $\lambda=1$, then we can obtain $DD^*(G_1+G_2)$. Replace separately G_1 and G_2 by K_1 and G in Theorem 2.1, we can obtain the following result.

Corollary 2.2. *Let G be a connected graph with n vertices and m edges. Then*

$$H_\lambda^*(K_1+G) = 2^{\lambda}\left(\bar{M}_1(G)+\bar{M}_2(G)+\bar{m}\right)+\left(M_1(G)+M_2(G)\right)+n^2+2nm+m.$$

We can observe that $M_1(C_n)=4n$ for $n\ge 3$, $M_1(P_n)=4n-6$ for $n>1$, and $\bar{M}_1(C_n)=2n(n-3)$, $\bar{M}_1(P_n)=2(n-2)^2$. Hence, we can compute the formulae for modified generalized degree distance of fan graph K_1+P_n and wheel graph K_1+C_n (see Figure 1) in the following.

Example 2.1. $H_\lambda^*(K_1+P_n)=2^{\lambda-1}\left(9n^2-39n+44\right)+\left(3n^2+7n-15\right).$

$$H_\lambda^*(K_1+C_n)=2^{\lambda-1}9n(n-3)+3n(n+3).$$

Next, we compute the exact formula for the modified generalized degree distance of the composition of two graphs. Before starting the discussion, we first denote by $A(G)$ the sum $\sum_{i,j=1, i\neq j}^{n} d_G^{\lambda}(u_i,u_j)d_G(u_i)$. It is easy to deduce that $\sum_{i,j=1, i\neq j}^{n} d_G^{\lambda}(u_i,u_j)d_G(u_i)=\sum_{i,j=1, i\neq j}^{n} d_G^{\lambda}(u_i,u_j)d_G(u_j)$.

By calculations we obtain the expressions for $A(P_n)$ and $A(C_n)$ as following:

$$A(P_n)=\sum_{i=1}^{n-1}2(2n-2i-1)i^{\lambda},$$

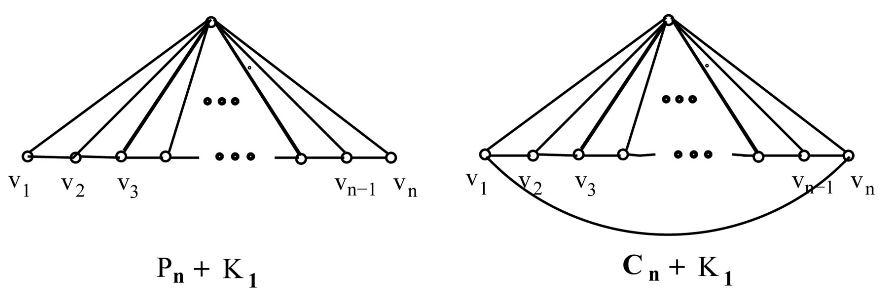

Figure 1. The graphs P_n+K_1 and C_n+K_1.

$$A(C_n)=\begin{cases}\sum_{i=1}^{\frac{n-1}{2}}4ni^{\lambda}, & n \text{ is odd},\\ 2^{1-\lambda}n^{1+\lambda}+\sum_{i=1}^{\frac{n-2}{2}}4ni^{\lambda}, & n \text{ is even}.\end{cases}$$

we can also obtain the expressions for $H_{\lambda}^{*}(P_n)$ and $H_{\lambda}^{*}(C_n)$:

$$H_{\lambda}^{*}(P_n)=2(n-1)^{\lambda}+\sum_{i=2}^{n-1}8(i-1)(n-i)^{\lambda}.$$

$$H_{\lambda}^{*}(C_n)=\begin{cases}\sum_{i=1}^{\frac{n-1}{2}}8ni^{\lambda}, & n \text{ is odd},\\ 4n\left(\frac{n}{2}\right)^{\lambda}+\sum_{i=1}^{\frac{n-2}{2}}8ni^{\lambda}, & n \text{ is even}.\end{cases}$$

These formulae are similar to the known results in [22]:

$$W_{\lambda}(P_n)=n\sum_{i=1}^{n-1}i^{\lambda}-\sum_{i=1}^{n-1}i^{\lambda+1},$$

$$W_{\lambda}(C_n)=\begin{cases}n\sum_{i=1}^{\frac{n-1}{2}}i^{\lambda}, & n \text{ is odd},\\ \left(\frac{n}{2}\right)^{\lambda+1}+n\sum_{i=1}^{\frac{n-2}{2}}i^{\lambda}, & n \text{ is even}.\end{cases}$$

Theorem 2.3. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned}H_{\lambda}^{*}(G_1[G_2])&=2^{\lambda}\left(n_2^2\bar{m}_2M_1(G_1)+n_1\bar{M}_2(G_2)+2n_2m_1\bar{M}_1(G_2)\right)\\&+2n_2\left(m_2M_1(G_1)+m_1M_1(G_2)\right)+n_1M_2(G_2)\\&+4n_2^2m_2A(G_1)+4m_2^2W_{\lambda}(G_1)+n_2^4H_{\lambda}^{*}(G_1).\end{aligned}$$

Proof. Set $V(G_1)=\{u_1,u_2,\cdots,u_{n_1}\}$ and $V(G_2)=\{v_1,v_2,\cdots,v_{n_2}\}$. By 4), 8) of Lemma 2.1 and definition of $H_{\lambda}^{*}(G)$, we have

$$\begin{aligned}H_{\lambda}^{*}(G_1[G_2])&=\sum_{\{u,v\}\subseteq V(G_1[G_2])}d_{G_1[G_2]}^{\lambda}(u,v)\left(d_{G_1[G_2]}(u)d_{G_1[G_2]}(v)\right)\\&=\frac{1}{2}\sum_{(u_i,v_k)}\sum_{(u_j,v_l)}d_{G_1[G_2]}^{\lambda}\left((u_i,v_k),(u_j,v_l)\right)\left(n_2d_{G_1}(u_i)+d_{G_2}(v_k)\right)\\&\cdot\left(n_2d_{G_1}(u_j)+d_{G_2}(v_l)\right)\\&=\sum_{p=1}^{n_1}\sum_{\substack{k,l=1\\ v_kv_l\in E(G_2)}}^{n_2}d_{G_1[G_2]}^{\lambda}\left((u_p,v_k),(u_p,v_l)\right)\left(n_2^2d_{G_1}^2(u_p)+n_2d_{G_1}(u_p)\right.\\&\left.\cdot\left(d_{G_2}(v_k)+d_{G_2}(v_l)\right)+d_{G_2}(v_k)d_{G_2}(v_l)\right)\\&+\sum_{k,l=1}^{n_2}\sum_{\substack{i,j=1\\ i\neq j}}^{n_1}d_{G_1[G_2]}^{\lambda}\left((u_i,v_k),(u_j,v_l)\right)\left(n_2^2d_{G_1}(u_i)d_{G_1}(u_j)\right.\\&\left.+n_2d_{G_1}(u_i)d_{G_2}(v_l)+n_2d_{G_1}(u_j)d_{G_2}(v_k)+d_{G_2}(v_k)d_{G_2}(v_l)\right)\\&+\sum_{p=1}^{n_1}\sum_{\substack{k,l=1\\ v_kv_l\notin E(G_2)}}^{n_2}d_{G_1[G_2]}^{\lambda}\left((u_i,v_k),(u_j,v_l)\right)\left(n_2^2d_{G_1}^2(u_p)\right.\\&\left.+n_2d_{G_1}(u_p)\left(d_{G_2}(v_k)+d_{G_2}(v_l)\right)+d_{G_2}(v_k)d_{G_2}(v_l)\right)\\&=2^{\lambda}\left(n_2^2\bar{m}_2M_1(G_1)+n_1\bar{M}_2(G_2)+2n_2m_1\bar{M}_1(G_2)\right)+2n_2\left(m_2M_1(G_1)\right.\\&\left.+m_1M_1(G_2)\right)+n_1M_2(G_2)+4n_2^2m_2A(G_1)+4m_2^2W_{\lambda}(G_1)+n_2^4H_{\lambda}^{*}(G_1)\end{aligned}$$

■

In Theorem 2.3, $H_\lambda^*(G_1[G_2]) = DD^*(G_1[G_2])$ if $\lambda = 1$, and in the above proof, when $u_i = u_j$ & $v_k v_l \in E(G_2)$, $d_{G_1[G_2]}((u_i,v_k),(u_j,v_l)) = 1$; when $u_i \neq u_j$, $d_{G_1[G_2]}((u_i,v_k),(u_j,v_l)) = d_{G_1}(u_i,u_j)$; when $u_i = u_j$ & $v_k v_l \notin E(G_2)$, $d_{G_1[G_2]}((u_i,v_k),(u_j,v_l)) = 2$ by Lemma 2.1 4).

By composing paths or cycles with various small graphs, we can obtain classes of polymer-like graphs. As an application, we give the formulae of $H_\lambda^*(P_n[K_2])$ and $H_\lambda^*(C_n[K_2])$, where $P_n[K_2]$ and $C_n[K_2]$ are open fence graph and closed fence graph (see Figure 2), respectively.

Example 2.2. $H_\lambda^*(P_n[K_2]) = 25n - 32 + 2^4(H_\lambda^*(P_n) + A(P_n)) + 4W_\lambda(P_n)$,

$$H_\lambda^*(C_n[K_2]) = 25n - 8 + 2^4(H_\lambda^*(C_n) + A(C_n)) + 4W_\lambda(C_n).$$

The following theorem characterizes the modified generalized degree distance of the disjunction of two graphs.

Theorem 2.4. *Let* G_1 *and* G_2 *be two graphs. Then*

$$\begin{aligned}
&H_\lambda^*(G_1 \vee G_2) \\
&= 2^\lambda \Big(n_2\big(n_2^2 + 2n_2\bar{m}_2 - 4m_2\big)\bar{M}_2(G_1) + n_1\big(n_1^2 + 2n_1\bar{m}_1 - 4m_1\big)\bar{M}_2(G_2) \\
&\quad + n_2^2\bar{m}_2 M_1(G_1) + n_1^2\bar{m}_1 M_1(G_2) + 2n_1n_2\big(m_2\bar{M}_1(G_1) + m_1\bar{M}_1(G_2)\big)\Big) \\
&\quad + 2^\lambda\Big(\big(M_1(G_2) - n_2\bar{M}_1(G_2)\big)\bar{M}_2(G_1) + \big(M_1(G_1) - n_1\bar{M}_1(G_1)\big)\bar{M}_2(G_2) \\
&\quad + \big(n_1n_2\bar{M}_1(G_1)\bar{M}_1(G_2) - n_1\bar{M}_1(G_1)M_1(G_2) - n_2\bar{M}_1(G_2)M_1(G_1)\big) \\
&\quad + \bar{M}_2(G_1)\bar{M}_2(G_2)\Big) + 4n_2^2m_1^2m_2 + 4n_1^2m_2^2m_1 + 2n_1m_2\big(n_2^2 - 2m_2\big)M_1(G_1) \\
&\quad + 2n_2m_1\big(n_1^2 - 2m_1\big)M_1(G_2) + \big(n_2^2 - m_2\big)\big(n_2^2 - 4m_2\big)M_2(G_1) \\
&\quad + \big(n_1^2 - m_1\big)\big(n_1^2 - 4m_1\big)M_2(G_2) - n_1n_2M_1(G_1)M_1(G_2) \\
&\quad + n_1M_1(G_1)M_2(G_2) + n_2M_1(G_2)M_2(G_1) - M_2(G_1)M_2(G_2).
\end{aligned}$$

Proof. By the definition of $G_1 \vee G_2$, we first present the following four sums:

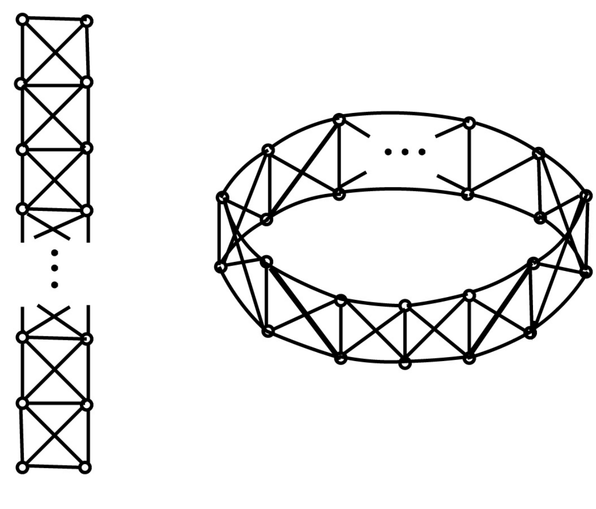

Figure 2. The graphs $P_n[K_2]$ and $C_n[K_2]$.

$$
\begin{aligned}
S_1^* &= \sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)\\
&\quad\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)\\
&= \sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)} n_2^2 d_{G_1}(x)d_{G_1}(y)\\
&\quad+\sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)} n_1^2 d_{G_2}(u)d_{G_2}(v)\\
&\quad+\sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)} n_1 n_2\left(d_{G_1}(x)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)\right)\\
&\quad-\sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)} n_2 d_{G_1}(x)d_{G_1}(y)\left(d_{G_2}(u)+d_{G_2}(v)\right)\\
&\quad-\sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)} n_1 d_{G_2}(u)d_{G_2}(v)\cdot\left(d_{G_1}(x)+d_{G_1}(y)\right)\\
&\quad+\sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= 4n_2^2 m_1^2 m_2 + 2n_2 m_1\left(n_1^2-2m_1\right)M_1(G_2)+\left(n_1^2-2m_1\right)^2 M_2(G_2)
\end{aligned}
$$

$$
\begin{aligned}
S_2^* &= \sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)\\
&\quad\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)\\
&= \sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)} n_2^2 d_{G_1}(x)d_{G_1}(y)\\
&\quad+\sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)} n_1^2\cdot d_{G_2}(u)d_{G_2}(v)\\
&\quad+\sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)} n_1 n_2\left(d_{G_1}(x)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)\right)\\
&\quad-\sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)} n_2 d_{G_1}(x)d_{G_1}(y)\left(d_{G_2}(u)+d_{G_2}(v)\right)\\
&\quad-\sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)} n_1 d_{G_2}(u)d_{G_2}(v)\cdot\left(d_{G_1}(x)+d_{G_1}(y)\right)\\
&\quad+\sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= 4n_1^2 m_2^2 m_1 + 2n_1 m_2\left(n_2^2-2m_2\right)M_1(G_1)+\left(n_2^2-2m_2\right)^2 M_2(G_1)
\end{aligned}
$$

$$
\begin{aligned}
S_3^* &= \sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)\\
&\quad\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)\\
&= \sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)} n_2^2 d_{G_1}(x)d_{G_1}(y)\\
&\quad+\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)} n_1^2 d_{G_2}(u)d_{G_2}(v)\\
&\quad+\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)} n_1 n_2\left(d_{G_1}(x)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)\right)\\
&\quad-\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)} n_2 d_{G_1}(x)d_{G_1}(y)\left(d_{G_2}(u)+d_{G_2}(v)\right)\\
&\quad-\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)} n_1 d_{G_2}(u)d_{G_2}(v)\left(d_{G_1}(x)+d_{G_1}(y)\right)\\
&\quad+\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2 m_2 M_2(G_1)+n_1 n_2 M_1(G_1)M_1(G_2)-n_2 M_1(G_2)M_2(G_1)\\
&\quad+n_1^2 m_1 M_2(G_2)-n_1 M_1(G_1)M_2(G_2)+M_2(G_1)M_2(G_2)
\end{aligned}
$$

$$
\begin{aligned}
S_4^* &= \sum_{\substack{xy\notin E(G_1)\\ x\neq y}}\sum_{\substack{uv\notin E(G_2)\\ u\neq v}} 2^{\lambda}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)\\
&\quad\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)\\
&\quad+\sum_{xy\notin E(G_1)}\sum_{u\in V(G_2)} 2^{\lambda}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)
\end{aligned}
$$

$$\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)$$
$$+\sum_{x\in V(G_1)}\sum_{uv\notin E(G_2)}2^{\lambda}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)$$
$$\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)$$
$$=2^{\lambda}\left(\left(n_2^3+2n_2^2\bar{m}_2-4n_2m_2\right)\bar{M}_2(G_1)+\left(n_1^3+2n_1^2\bar{m}_1-4n_1m_1\right)\bar{M}_2(G_2)\right.$$
$$\left.+2n_1n_2\left(m_2\bar{M}_1(G_1)+m_1\bar{M}_1(G_2)\right)+n_1^2\bar{m}_1M_1(G_2)+n_2^2\bar{m}_2M_1(G_1)\right)$$
$$+2^{\lambda}\left(\left(n_1n_2\bar{M}_1(G_1)\bar{M}_1(G_2)-n_1M_1(G_2)\bar{M}_1(G_1)-n_2M_1(G_1)\bar{M}_1(G_2)\right)\right.$$
$$\left.+\left(M_1(G_2)-n_2\bar{M}_1(G_2)\right)\bar{M}_2(G_1)+\left(M_1(G_1)-n_1\bar{M}_1(G_1)\right)\bar{M}_2(G_2)\right)$$
$$+2^{\lambda}\bar{M}_2(G_1)\bar{M}_2(G_2).$$

Thus, we can obtain the result of Theorem 2.4 by the formula $H_{\lambda}^{*}(G_1\vee G_2)=S_1^{*}+S_2^{*}+S_4^{*}-S_3^{*}$. ■

Finally, we consider the modified generalized degree distance of the symmetric difference of two graphs.

Theorem 2.5. *Let* G_1 *and* G_2 *be two graphs. Then*

$$H_{\lambda}^{*}(G_1\oplus G_2)$$
$$=2^{\lambda}\left(n_2\left(n_2^2+2n_2\bar{m}_2-8m_2\right)\bar{M}_2(G_1)+n_1\left(n_1^2+2n_1\bar{m}_1-8m_1\right)\bar{M}_2(G_2)\right.$$
$$\left.+n_2^2\bar{m}_2M_1(G_1)+n_1^2\bar{m}_1M_1(G_2)+2n_1n_2\left(m_2\bar{M}_1(G_1)+m_1\bar{M}_1(G_2)\right)\right)$$
$$+2^{\lambda}\left(2\left(2M_1(G_2)-n_2\bar{M}_1(G_2)\right)\bar{M}_2(G_1)+2\left(2M_1(G_1)-n_1\bar{M}_1(G_1)\right)\bar{M}_2(G_2)\right.$$
$$+n_1n_2\bar{M}_1(G_1)\bar{M}_2(G_2)-2n_1\bar{M}_1(G_1)M_1(G_2)-2n_2\bar{M}_1(G_2)M_1(G_1)$$
$$\left.+2^2\bar{M}_2(G_1)\bar{M}_2(G_2)\right)+4n_2^2m_1^2m_2+4n_1^2m_2^2m_1+2n_1m_2\left(n_2^2-4m_2\right)M_1(G_1)$$
$$+2n_2m_1\left(n_1^2-4m_1\right)M_1(G_2)+\left(n_2^2-2m_2\right)\left(n_2^2-8m_2\right)M_2(G_1)$$
$$+\left(n_1^2-2m_1\right)\left(n_1^2-8m_1\right)M_2(G_2)-2n_1n_2M_1(G_1)M_1(G_2)$$
$$+4n_1M_1(G_1)M_2(G_2)+4n_2M_1(G_2)M_2(G_1)-8M_2(G_1)M_2(G_2).$$

Proof. Similar to the proof of Theorem 2.4, we consider four sums:

$$S_1^{*}=\sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)$$
$$\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)$$
$$=4n_2^2m_1^2m_2+2n_2m_1\left(n_1^2-4m_1\right)M_1(G_2)+\left(n_1^2-4m_1\right)^2M_2(G_2).$$

$$S_2^{*}=\sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)$$
$$\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)$$
$$=4n_1^2m_2^2m_1+2n_1m_2\left(n_2^2-4m_2\right)M_1(G_1)+\left(n_2^2-4m_2\right)^2M_2(G_1).$$

$$S_3^{*}=\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right)$$
$$\cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)$$
$$=n_2^2m_2M_2(G_1)+n_1n_2M_1(G_1)M_1(G_2)-2n_2M_1(G_2)M_2(G_1)$$
$$+n_1^2m_1M_2(G_2)-2n_1M_1(G_1)M_2(G_2)+4M_2(G_1)M_2(G_2).$$

$$
\begin{aligned}
S_4^* = & \sum_{\substack{xy\notin E(G_1)\\ x\neq y}}\sum_{\substack{uv\notin E(G_2)\\ u\neq v}} 2^{\lambda}\left(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\right) \\
& \cdot\left(n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right) \\
= & 2^{\lambda}\left(\left(n_2^3+2n_2^2\bar{m}_2-8n_2m_2\right)\bar{M}_2(G_1)+\left(n_1^3+2n_1^2\bar{m}_1-8n_1m_1\right)\bar{M}_2(G_2)\right. \\
& \left.+2n_1n_2\left(m_2\bar{M}_1(G_1)+m_1\bar{M}_1(G_2)\right)+n_1^2\bar{m}_1M_1(G_2)+n_2^2\bar{m}_2M_1(G_1)\right) \\
& +2^{\lambda}\left(\left(n_1n_2\bar{M}_1(G_1)\bar{M}_1(G_2)-2n_1M_1(G_2)\bar{M}_1(G_1)-2n_2M_1(G_1)\bar{M}_1(G_2)\right)\right. \\
& \left.+2\left(2M_1(G_2)-n_2\bar{M}_1(G_2)\right)\bar{M}_2(G_1)+2\left(2M_1(G_1)-n_1\bar{M}_1(G_1)\right)\bar{M}_2(G_2)\right) \\
& +2^{\lambda+2}\bar{M}_2(G_1)\bar{M}_2(G_2).
\end{aligned}
$$

Therefore, we can obtain the result of Theorem 2.5 by
$H_\lambda^*(G_1\oplus G_2)=S_1^*+S_2^*+S_4^*-2S_3^*$. ■

Remark. In Section 2, we present the explicit formulae of the modified generalized degree distance for four types of graph operations containing G_1+G_2, $G_1[G_2]$, $G_1\vee G_2$ and $G_1\oplus G_2$, and we give some examples. It implies that our results are convenient to compute the modified generalized degree distance of these graph operations. Moreover, if $\lambda=1$, then $H_\lambda^*(G)=DD^*(G)$. This implies that our results are related to the product-degree distance. In [16] Hamzeh *et al.* construct graph polynomial as following:

$$
H_\lambda^*(G,x)=\sum_{\{u,v\}\subseteq V(G)}\left(d_G(u)d_G(v)\right)x^{d^{\lambda}(u,v)}.
$$

It is easy to see that the results of our Theorems 2.1, 2.3, 2.4 and 2.5 are exactly the first derivatives at point $x=1$ of the graph polynomial $H_\lambda^*(G,x)$. Thus we obtain the relation between the modified generalized degree distance polynomial and Wiener-type invariant polynomial for graphs.

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