

Martingales and Super-Martingales Relative to a Convex Set of Equivalent Measures

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Abstract

In the paper, the martingales and super-martingales relative to a convex set of equivalent measures are systematically studied. The notion of local regular super-martingale relative to a convex set of equivalent measures is introduced and the necessary and sufficient conditions of the local regularity of it in the discrete case are founded. The description of all local regular super-martingales relative to a convex set of equivalent measures is presented. The notion of the complete set of equivalent measures is introduced. We prove that every bounded in some sense super-martingale relative to the complete set of equivalent measures is local regular. A new definition of the fair price of contingent claim in an incomplete market is given and the formula for the fair price of Standard Option of European type is found. The proved Theorems are the generalization of the famous Doob decomposition for super-martingale onto the case of super-martingales relative to a convex set of equivalent measures.

Keywords

Random Process, Convex Set of Equivalent Measures, Optional Doob Decomposition, Local Regular Super-Martingale, Martingale, Fair Price of Contingent Claim

1. Introduction

In the paper, a new method of investigation of martingales and super-martingales relative to a convex set of equivalent measures is developed. A new proof that the essential supremum over the set of regular martingales, generated by a certain nonnegative random value and a convex set of equivalent measures, is a super-martingale with respect to this set of measures, is given.

A notion of local regular super-martingale is introduced and the necessary and sufficient conditions are found under that the above defined super-martingales

are local regular ones. The last fact allowed us to describe the local regular super-martingales. It is proved that the existence of a nontrivial martingale relative to a convex set of equivalent measures, generally speaking, does not guarantee for a nonnegative super-martingale to be a local regular one.

An important notion of the complete convex set of equivalent measures is introduced. It is proved that any super-martingale relative to the complete convex set of equivalent measures on a measurable space with the finite set of elementary events is a local regular one. The notion of the complete convex set of equivalent measures is generalized onto an arbitrary space of elementary events. It is proved that the nonnegative and the majorized from below super-martingales are local regular ones.

The definition of the fair price of contingent claim is introduced. The sufficient conditions of the existence of the fair price of contingent claim are presented. The conditions that the introduced notion coincides with classical one are given.

All these notions are used in the case as the convex set of equivalent measures is a set of equivalent martingale measures for the evolution of both risk and non-risk assets. The formula for the fair price of Standard Contract with Option of European type in an incomplete market is found.

The notion of the complete convex set of equivalent measures permits us to give a new proof of the optional decomposition for a nonnegative super-martingale. This proof does not use the no-arbitrage arguments and the measurable choice [1] [2] [3] [4].

First, the optional decomposition for diffusion processes super-martingale was opened by El Karoui N. and Quenez M. C. [5]. After that, Kramkov D. O. and Follmer H. [1] [2] proved the optional decomposition for the nonnegative bounded super-martingales. Folmer H. and Kabanov Yu. M. [3] [4] proved analogous result for an arbitrary super-martingale. Recently, Bouchard B. and Nutz M. [6] considered a class of discrete models and proved the necessary and sufficient conditions for the validity of the optional decomposition.

The optional decomposition for super-martingales plays the fundamental role for the risk assessment in incomplete markets [1] [2] [5] [7] [8] [9] [10] [11]. Considered in the paper problem is a generalization of the corresponding one that appeared in mathematical finance about the optional decomposition for a super-martingale and which is related with the construction of the superhedge strategy in incomplete financial markets.

Our statement of the problem unlike the above-mentioned one and it is more general: a super-martingale relative to a convex set of equivalent measures is given and it is necessary to find the conditions for the super-martingale and the set of measures under that the optional decomposition exists.

The generality of our statement of the problem is that we do not require that the considered set of measures was generated by the random process that is a local martingale as it is done in the papers [1] [4] [5] [6] and that is important

for the proof of the optional decomposition in these papers.

2. Local Regular Super-Martingales Relative to a Convex Set of Equivalent Measures

We assume that on a measurable space $\{\Omega, \mathcal{F}\}$ a filtration $\mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \mathcal{F}, m = \overline{0, \infty}$, and a family of convex set of equivalent measures M on \mathcal{F} are given. Further, we assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and the σ -algebra $\mathcal{F} = \sigma\left(\bigvee_{n=1}^{\infty} \mathcal{F}_n\right)$ is a minimal σ -algebra generated by the algebra $\bigvee_{n=1}^{\infty} \mathcal{F}_n$. A random process $\psi = \{\psi_m\}_{m=0}^{\infty}$ is said to be adapted one relative to the filtration $\{\mathcal{F}_m\}_{m=0}^{\infty}$ if ψ_m is a \mathcal{F}_m measurable random value, $m = \overline{0, \infty}$.

Definition 1. An adapted random process $f = \{f_m\}_{m=0}^{\infty}$ is said to be a super-martingale relative to the filtration $\mathcal{F}_m, m = \overline{0, \infty}$, and the convex family of equivalent measures M if $E^P |f_m| < \infty, m = \overline{1, \infty}, P \in M$, and the inequalities

$$E^P \{f_m | \mathcal{F}_k\} \leq f_k, \quad 0 \leq k \leq m, \quad m = \overline{1, \infty}, \quad P \in M, \tag{1}$$

are valid.

Further, for an adapted process f we use both the denotation $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ and the denotation $\{f_m\}_{m=0}^{\infty}$.

Definition 2. A super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ relative to a convex set of equivalent measures M is a local regular one if $\sup_{P \in M} E^P |f_m| < \infty, m = \overline{1, \infty}$, and there exists an adapted nonnegative increasing random process

$\{g_m, \mathcal{F}_m\}_{m=0}^{\infty}, g_0 = 0, \sup_{P \in M} E^P |g_m| < \infty, m = \overline{1, \infty}$, such that $\{f_m + g_m, \mathcal{F}_m\}_{m=0}^{\infty}$ is a martingale relative to every measure from M .

The next elementary Theorem 1 will be very useful later.

Theorem 1. Let a super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$, relative to a convex set of equivalent measures M be such that $\sup_{P \in M} E^P |f_m| < \infty, m = \overline{1, \infty}$. The necessary and sufficient condition for it to be a local regular one is the existence of an adapted nonnegative random process $\{g_m^0, \mathcal{F}_m\}_{m=0}^{\infty}, \sup_{P \in M} E^P |g_m^0| < \infty, m = \overline{1, \infty}$, such that

$$f_{m-1} - E^P \{f_m | \mathcal{F}_{m-1}\} = E^P \{g_m^0 | \mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}, \quad P \in M. \tag{2}$$

Proof. Necessity. If $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ is a local regular super-martingale, then there exist a martingale $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^{\infty}$ and a non-decreasing nonnegative random process $\{g_m, \mathcal{F}_m\}_{m=0}^{\infty}, g_0 = 0$, such that

$$f_m = \bar{M}_m - g_m, \quad m = \overline{1, \infty}. \tag{3}$$

From here we obtain the equalities

$$\begin{aligned} & E^P \{f_{m-1} - f_m | \mathcal{F}_{m-1}\} \\ &= E^P \{g_m - g_{m-1} | \mathcal{F}_{m-1}\} = E^P \{g_m^0 | \mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}, \quad P \in M, \end{aligned} \tag{4}$$

where we introduced the denotation $\bar{g}_m^0 = g_m - g_{m-1} \geq 0$. It is evident that $E^P \bar{g}_m^0 \leq \sup_{P \in M} E^P g_m + \sup_{P \in M} E^P g_{m-1} < \infty$.

Sufficiency. Suppose that there exists an adapted nonnegative random process $\bar{g}^0 = \{\bar{g}_m^0\}_{m=0}^\infty$, $\bar{g}_0^0 = 0$, $E^P \bar{g}_m^0 < \infty$, $m = \overline{1, \infty}$, such that the equalities (2) hold. Let us consider the random process $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$, where

$$\bar{M}_0 = f_0, \bar{M}_m = f_m + \sum_{i=1}^m \bar{g}_i^0, m = \overline{1, \infty}. \tag{5}$$

It is evident that $E^P |\bar{M}_m| < \infty$ and

$$E^P \{\bar{M}_{m-1} - \bar{M}_m | \mathcal{F}_{m-1}\} = E^P \{f_{m-1} - f_m - \bar{g}_m^0 | \mathcal{F}_{m-1}\} = 0. \tag{6}$$

Theorem 1 is proved. □

Lemma 1. Any super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a family of measures M for which there hold equalities $E^P f_m = f_0, m = \overline{1, \infty}, P \in M$ is a martingale with respect to this family of measures and the filtration $\mathcal{F}_m, m = \overline{1, \infty}$.

Proof. The proof of Lemma 1 see [12]. □

3. Description of Local Regular Super-Martingales Relative to a Convex Set of Equivalent Measures Generated by the Finite Set of Equivalent Measures

Below, we describe the local regular super-martingales relative to a convex set of equivalent measures M generated by the finite set of equivalent measures. For this we need some auxiliary statements.

Lemma 2. On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_m on it, let G be a sub σ -algebra of the σ -algebra \mathcal{F} and let $f_s, s \in S$ be a finite family of nonnegative bounded random values. Then for every measure P from M .

$$E^P \left\{ \max_{s \in S} f_s | G \right\} \geq \max_{s \in S} E^P \{ f_s | G \}, P \in M. \tag{7}$$

Proof. We have the inequalities

$$\max_{s \in S} f_s \geq f_t, t \in S. \tag{8}$$

Therefore,

$$E^P \left\{ \max_{s \in S} f_s | G \right\} \geq E^P \{ f_t | G \}, t \in S, P \in M. \tag{9}$$

The last implies

$$E^P \left\{ \max_{s \in S} f_s | G \right\} \geq \max_{s \in S} E^P \{ f_s | G \}, P \in M. \tag{10}$$

□

In the next Lemma, we present the formula for calculation of the conditional expectation relative to another measure from M .

Lemma 3. On a measurable space $\{\Omega, \mathcal{F}\}$ with a filtration \mathcal{F}_n on it, let M be a convex set of equivalent measures and let ξ be a bounded random value. Then the following formulas

$$E^n \{ \xi | \mathcal{F}_n \} = E^{P_n} \{ \xi \varphi_n^R | \mathcal{F}_n \}, n = \overline{1, \infty}, \tag{11}$$

are valid, where

$$\varphi_n^{P_1} = \frac{dP_1}{dP_2} \left[E^{P_2} \left\{ \frac{dP_1}{dP_2} \mid \mathcal{F}_n \right\} \right]^{-1}, \quad P_1, P_2 \in M. \tag{12}$$

Proof. The proof of Lemma 3 is evident. □

Let P_1, \dots, P_k be a family of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$ and let us introduce the denotation M for a convex set of equivalent measures

$$M = \left\{ Q, Q = \sum_{i=1}^k \alpha_i P_i, \alpha_i \geq 0, i = \overline{1, k}, \sum_{i=1}^k \alpha_i = 1 \right\}. \tag{13}$$

Lemma 4. If ξ is an integrable random value relative to the set of equivalent measures P_1, \dots, P_k , then the formula

$$\operatorname{ess\,sup}_{Q \in M} E^Q \{ \xi \mid \mathcal{F}_n \} = \max_{1 \leq i \leq k} E^{P_i} \{ \xi \mid \mathcal{F}_n \} \tag{14}$$

is valid almost everywhere relative to the measure P_1 .

Proof. The definition of $\operatorname{ess\,sup}$ for non countable family of random variables see [13]. Using the formula

$$E^Q \{ \xi \mid \mathcal{F}_n \} = \frac{\sum_{i=1}^k \alpha_i E^{P_i} \{ \varphi_i \mid \mathcal{F}_n \} E^{P_i} \{ \xi \mid \mathcal{F}_n \}}{\sum_{i=1}^k \alpha_i E^{P_i} \{ \varphi_i \mid \mathcal{F}_n \}}, \quad Q \in M, \tag{15}$$

where $\varphi_i = \frac{dP_i}{dP_1}$, $Q = \sum_{i=1}^k \alpha_i P_i$, we obtain the inequality

$$E^Q \{ \xi \mid \mathcal{F}_n \} \leq \max_{1 \leq i \leq k} E^{P_i} \{ \xi \mid \mathcal{F}_n \}, \quad Q \in M, \tag{16}$$

or,

$$\operatorname{ess\,sup}_{Q \in M} E^Q \{ \xi \mid \mathcal{F}_n \} \leq \max_{1 \leq i \leq k} E^{P_i} \{ \xi \mid \mathcal{F}_n \}. \tag{17}$$

On the other side [13],

$$E^{P_i} \{ \xi \mid \mathcal{F}_n \} \leq \operatorname{ess\,sup}_{Q \in M} E^Q \{ \xi \mid \mathcal{F}_n \}, \quad i = \overline{1, k}. \tag{18}$$

Therefore,

$$\max_{1 \leq i \leq k} E^{P_i} \{ \xi \mid \mathcal{F}_n \} \leq \operatorname{ess\,sup}_{Q \in M} E^Q \{ \xi \mid \mathcal{F}_n \}. \tag{19}$$

Lemma 4 is proved. □

Lemma 5. On a measurable space $\{\Omega, \mathcal{F}\}$ with a filtration \mathcal{F}_n on it, let ξ be a nonnegative bounded random value. If $\frac{dP_l}{dP_i}, i, l = \overline{1, k}$ are \mathcal{F}_1 measurable

and $P_1 \left(\frac{dP_l}{dP_i} > 0 \right) = 1, i, l = \overline{1, k}$, then the inequalities

$$E^{P_i} \left\{ \max_{1 \leq l \leq k} E^{P_l} \{ \xi \mid \mathcal{F}_n \} \mid \mathcal{F}_m \right\} \leq \max_{1 \leq l \leq k} E^{P_l} \{ \xi \mid \mathcal{F}_m \}, \quad l = \overline{1, k}, \quad n > m, \tag{20}$$

are valid.

Proof. From Lemma 3 and Lemma 5 conditions relative to the density of one measure with respect to another, we have

$$\max_{1 \leq l \leq k} E^{P_l} \{ \xi | \mathcal{F}_n \} = E^{P_l} \{ \xi | \mathcal{F}_n \}, \quad l = \overline{1, k}. \tag{21}$$

From the equality (21) we obtain the inequality

$$E^{P_l} \left\{ \max_{1 \leq l \leq k} E^{P_l} \{ \xi | \mathcal{F}_n \} | \mathcal{F}_m \right\} \leq \max_{1 \leq l \leq k} E^{P_l} \{ \xi | \mathcal{F}_m \}, \quad l = \overline{1, k}. \tag{22}$$

Lemma 5 is proved. □

In this section, we assume that the conditions of Lemma 5 relative to the density of one measure with respect to another are true.

Lemma 6. On a measurable space $\{\Omega, \mathcal{F}\}$ with a filtration \mathcal{F}_n on it, let ξ be a nonnegative random value which is integrable relative to the set of equivalent measures P_1, \dots, P_k . Then the inequalities

$$E^Q \left\{ \max_{1 \leq l \leq k} E^{P_l} \{ \xi | \mathcal{F}_n \} | \mathcal{F}_m \right\} \leq \max_{1 \leq l \leq k} E^{P_l} \{ \xi | \mathcal{F}_m \}, \quad n > m, \quad Q \in M, \tag{23}$$

are valid.

Proof. Using Lemma 5 inequalities for the nonnegative bounded ξ and the formula

$$E^Q \{ \Phi | \mathcal{F}_m \} = \frac{\sum_{i=1}^k \alpha_i E^{P_i} \{ \varphi_i | \mathcal{F}_m \} E^{P_i} \{ \Phi | \mathcal{F}_m \}}{\sum_{i=1}^k \alpha_i E^{P_i} \{ \varphi_i | \mathcal{F}_m \}}, \quad Q \in M, \tag{24}$$

where $\Phi = \max_{1 \leq l \leq k} E^{P_l} \{ \xi | \mathcal{F}_n \}$, $\varphi_i = \frac{dP_i}{dP_1}$, $i = \overline{1, k}$, we prove Lemma 6 inequalities.

Let us consider the case, as $\max_{1 \leq l \leq k} E^{P_l} \xi < \infty$. Let $\xi_s, s = \overline{1, \infty}$ be a sequence of bounded random values converging to ξ monotonously. Then

$$E^Q \left\{ \max_{1 \leq l \leq k} E^{P_l} \{ \xi_s | \mathcal{F}_n \} | \mathcal{F}_m \right\} \leq \max_{1 \leq l \leq k} E^Q \{ \xi_s | \mathcal{F}_m \}, \quad l = \overline{1, k}. \tag{25}$$

Due to the monotony convergence of ξ_s to ξ , as $s \rightarrow \infty$, we can pass to the limit under the conditional expectations on the left and right sides in the inequalities (25) that proves Lemma 6. □

Lemma 7. On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_n on it, for every nonnegative integrable random value ξ relative to a set of equivalent measures $\{P_1, \dots, P_k\}$ the inequalities

$$E^Q \left\{ \operatorname{ess\,sup}_{P \in M} E^P \{ \xi | \mathcal{F}_n \} | \mathcal{F}_m \right\} \leq \operatorname{ess\,sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \quad Q \in M, \quad n > m, \tag{26}$$

are valid.

Lemma 7 is a consequence of Lemma 6.

Lemma 8. On a measurable space $\{\Omega, \mathcal{F}\}$ with a filtration \mathcal{F}_m on it, let ξ be a nonnegative integrable random value with respect to a set of equivalent measures $\{P_1, \dots, P_k\}$ and such that

$$E^{P_i} \xi = M_0, \quad i = \overline{1, k}, \tag{27}$$

then the random process $\left\{ M_m = \operatorname{ess\,sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \right\}_{m=0}^\infty$ is a martingale relative to a convex set of equivalent measures M .

Proof. Due to Lemma 7, a random process

$$\left\{ M_m = \operatorname{ess\,sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \right\}_{m=0}^\infty \text{ is a super-martingale, that is,}$$

$$E^P \{ M_m | \mathcal{F}_{m-1} \} \leq M_{m-1}, \quad m = \overline{1, \infty}, \quad P \in M. \tag{28}$$

Or, $E^P M_m \leq M_0$. From the other side,

$$E^{P_s} \left[\max_{1 \leq i \leq k} E^{P_i} \{ \xi | \mathcal{F}_m \} \right] \geq \max_{1 \leq i \leq k} E^{P_s} E^{P_i} \{ \xi | \mathcal{F}_m \} \geq M_0, \quad s = \overline{1, k}. \tag{29}$$

The above inequalities imply $E^{P_s} M_m = M_0, m = \overline{1, \infty}, s = \overline{1, k}$. The last equalities lead to the equalities $E^P M_m = M_0, m = \overline{1, \infty}, P \in M$. The fact that M_m is a super-martingale relative to the set of measures M and the above equalities prove Lemma 8, since the Lemma 1 conditions are valid. \square

In the next Theorem we denote $\mathcal{F} = \sigma \left(\bigvee_{i=1}^\infty \mathcal{F}_i \right)$ the minimal σ -algebra generated by the algebra $\bigvee_{i=1}^\infty \mathcal{F}_i$.

Theorem 2. Let $\{\Omega, \mathcal{F}\}$ be a measurable space with a filtration \mathcal{F}_m on it and let ξ be a nonnegative integrable random value with respect to a set of equivalent measures P_1, \dots, P_k . The necessary and sufficient conditions of the local regularity of the super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$, where

$$f_m = \operatorname{ess\,sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \quad m = \overline{1, \infty}, \quad \max_{1 \leq i \leq k} E^{P_i} \xi < \infty, \tag{30}$$

is its uniform integrability relative to the set of measure P_1, \dots, P_k and the fulfillment of the equalities

$$E^{P_i} \xi = f_0, \quad i = \overline{1, k}. \tag{31}$$

Proof. The necessity. Let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a local regular super-martingale. Then

$$f_n = M_n - g_n, \quad n = \overline{0, \infty}, \quad g_0 = 0, \quad f_0 = E^{P_i} M_n, \quad i = \overline{1, k}. \tag{32}$$

From here we obtain $E^{P_i} g_n \leq f_0, i = \overline{1, k}$. Due to the uniform integrability of f_n and g_n we obtain

$$E^{P_i} (f_\infty + g_\infty) = f_0, \quad i = \overline{1, k}, \tag{33}$$

where $f_\infty = \xi, g_\infty = \lim_{n \rightarrow \infty} g_n$, since $\mathcal{F} = \sigma \left(\bigvee_{i=1}^\infty \mathcal{F}_i \right)$. But $f_0 = \max_{1 \leq i \leq k} E^{P_i} \xi = E^{P_{i_0}} \xi$.

From (33) we have $E^{P_{i_0}} g_\infty = 0$. The last equality gives $g_\infty = 0$, or

$$E^{P_i} \xi = E^{P_{i_0}} \xi, \quad i = \overline{1, k}. \tag{34}$$

The sufficiency. If the conditions of Theorem 2 are satisfied, then $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale, where $\bar{M}_m = \sup_{P \in M} E^P \{ \xi | \mathcal{F}_m \}$. The last implies the

local regularity of $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$. Theorem 2 is proved. □

4. Description of Local Regular Super-Martingales Relative to an Arbitrary Convex Set of Equivalent Measures

Below, in the paper we assume that an arbitrary convex set of equivalent measures M on a measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it satisfies the conditions: the density $\frac{dP}{dQ}$ is \mathcal{F}_1 measurable one and $P_0\left(\frac{dP}{dQ} > 0\right) = 1$ for all $P, Q \in M$, where the fixed measure $P_0 \in M$. Such a class of equivalent measures is sufficiently wide. It contains the class of equivalent martingale measures generated by a local martingale.

Introduce into consideration a set A_0 of all integrable nonnegative random values ξ relative to a convex set of equivalent measures M satisfying conditions

$$E^P \xi = 1, \quad P \in M. \tag{35}$$

It is evident that the set A_0 is not empty, since contains the random value $\xi = 1$. More interesting case is as A_0 contains more then one element.

Lemma 9. On a measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. If the nonnegative random value ξ is such that $\sup_{P \in M} E^P \xi < \infty$, then $\left\{f_m = \operatorname{ess\,sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}, \mathcal{F}_m\right\}_{m=0}^\infty$ is a super-martingale relative to the convex set of equivalent measures M .

Proof. From the definition of $\operatorname{ess\,sup}$ [13], for every $\operatorname{ess\,sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}$ there exists a countable set D_m such that

$$\operatorname{ess\,sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\} = \sup_{P \in D_m} E^P \{\xi | \mathcal{F}_m\}, \quad m = \overline{0, \infty}. \tag{36}$$

The set $D = \bigcup_{m=0}^\infty D_m$ is also countable one and the equality

$$\operatorname{ess\,sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\} = \sup_{P \in D} E^P \{\xi | \mathcal{F}_m\} \tag{37}$$

is true. Really, since

$$\sup_{P \in D} E^P \{\xi | \mathcal{F}_m\} \geq \sup_{P \in D_M} E^P \{\xi | \mathcal{F}_m\} = \operatorname{ess\,sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}. \tag{38}$$

From the other side,

$$\operatorname{ess\,sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\} \geq E^Q \{\xi | \mathcal{F}_m\}, \quad Q \in M. \tag{39}$$

The last gives

$$\operatorname{ess\,sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\} \geq \sup_{P \in D} E^P \{\xi | \mathcal{F}_m\}. \tag{40}$$

The inequalities (38), (40) prove the needed statement. So, for all m we can choose the common set D . Let $D = \{\bar{P}_1, \dots, \bar{P}_n, \dots\}$. Due to Lemma 7, for every $Q \in \bar{M}_k$, we have

$$E^Q \left\{ \max_{1 \leq i \leq k} E^{\bar{P}_i} \{ \xi | \mathcal{F}_n \} | \mathcal{F}_m \right\} \leq \max_{1 \leq i \leq k} E^{\bar{P}_i} \{ \xi | \mathcal{F}_m \}, \quad n > m, \quad Q \in \bar{M}_k, \quad (41)$$

where

$$\bar{M}_k = \left\{ P \in M, P = \sum_{i=1}^k \alpha_i \bar{P}_i, \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1 \right\}. \quad (42)$$

It is evident that $\max_{1 \leq i \leq k} E^{\bar{P}_i} \{ \xi | \mathcal{F}_n \}$ tends to $\sup_{P \in D} E^P \{ \xi | \mathcal{F}_n \}$ monotonously increasing, as $k \rightarrow \infty$. Fixing $Q \in \bar{M}_k \subset \bar{M}_{k+1}$ and tending k to the infinity in the inequalities (41), we obtain

$$E^Q \left\{ \sup_{P \in D} E^P \{ \xi | \mathcal{F}_n \} | \mathcal{F}_m \right\} \leq \sup_{P \in D} E \{ \xi | \mathcal{F}_m \}, \quad n > m, \quad Q \in \bar{M}_k. \quad (43)$$

The last inequality implies that for every measure Q , belonging to the convex span, constructed on the set D , $\left\{ f_m = \text{ess sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \right\}_{m=0}^\infty$ is a super-martingale relative to the convex set of equivalent measures, generated by the set D . Now, if a measure Q_0 does not belong to the convex span, constructed on the set D , then we can add it to the set D and repeat the proof made above. As a result, we proved that $\left\{ f_m = \text{ess sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \right\}_{m=0}^\infty$ is also a super-martingale relative to the measure Q_0 . Zorn Lemma [14] complete the proof of Lemma 9. \square

Theorem 3. On a measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. For a random value $\xi \in A_0$, the random process $\left\{ E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \right\}_{m=0}^\infty, P \in M$ is a local regular martingale relative to the convex set of equivalent measures M .

Proof. Let P_1, \dots, P_n be a certain subset of measures from M . Denote M_n a convex set of equivalent measures

$$M_n = \left\{ P \in M, P = \sum_{i=1}^n \alpha_i P_i, \alpha_i \geq 0, i = \overline{1, n}, \sum_{i=1}^n \alpha_i = 1 \right\}. \quad (44)$$

Due to Lemma 8, $\left\{ \bar{M}_m, \mathcal{F}_m \right\}_{m=0}^\infty$ is a martingale relative to the set of measures M_n , where $\bar{M}_m = \text{ess sup}_{P \in M_n} E^P \{ \xi | \mathcal{F}_m \}, \xi \in A_0$. Let us consider an arbitrary measure $P_0 \in M$ and let

$$M_n^{P_0} = \left\{ P \in M, P = \sum_{i=0}^n \alpha_i P_i, \alpha_i \geq 0, i = \overline{0, n}, \sum_{i=0}^n \alpha_i = 1 \right\}. \quad (45)$$

Then $\left\{ \bar{M}_m^{P_0}, \mathcal{F}_m \right\}_{m=0}^\infty$, where $\bar{M}_m^{P_0} = \text{ess sup}_{P \in M_n^{P_0}} E^P \{ \xi | \mathcal{F}_m \}$, is a martingale relative to the set of measures $M_n^{P_0}$. It is evident that

$$\bar{M}_m \leq \bar{M}_m^{P_0}, \quad m = \overline{0, \infty}. \quad (46)$$

Since $E^P \bar{M}_m = E^P \bar{M}_m^{P_0} = 1, m = \overline{0, \infty}, P \in M_n$, the inequalities (46) give $\bar{M}_m = \bar{M}_m^{P_0}$. Analogously, $E^{P_0} \{ \xi | \mathcal{F}_m \} \leq \bar{M}_m^{P_0}$. From the equalities $E^{P_0} E^{P_0} \{ \xi | \mathcal{F}_m \} = E^{P_0} \bar{M}_m^{P_0} = 1$ we obtain $E^{P_0} \{ \xi | \mathcal{F}_m \} = \bar{M}_m^{P_0} = \bar{M}_m$. Since the

measure P_0 is an arbitrary one it implies that $\{E^P\{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale relative to all measures from M . Due to Theorem 1, it is a local regular super-martingale with the random process $\bar{g}_m^0 = 0, m = \overline{0, \infty}$. Theorem 3 is proved. \square

Theorem 4. On a measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. If $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is an adapted random process satisfying conditions

$$f_m \leq f_{m-1}, E^P \xi | f_m | < \infty, P \in M, m = \overline{1, \infty}, \xi \in A_0, \tag{47}$$

then the random process

$$\{f_m E^P\{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty, P \in M, \tag{48}$$

is a local regular super-martingale relative to the convex set of equivalent measures M .

Proof. Due to Theorem 3, the random process $\{E^P\{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale relative to the convex set of equivalent measures M . Therefore,

$$\begin{aligned} & f_{m-1} E^P\{\xi | \mathcal{F}_{m-1}\} - E^P\{f_m E^P\{\xi | \mathcal{F}_m\} | \mathcal{F}_{m-1}\} \\ &= E^P\{(f_{m-1} - f_m) E^P\{\xi | \mathcal{F}_m\} | \mathcal{F}_{m-1}\}, m = \overline{1, \infty}. \end{aligned} \tag{49}$$

So, if to put $\bar{g}_m^0 = (f_{m-1} - f_m) E^P\{\xi | \mathcal{F}_m\}, m = \overline{1, \infty}$, then $\bar{g}_m^0 \geq 0$, it is \mathcal{F}_m -measurable and $E^P \bar{g}_m^0 \leq E^P \xi (|f_{m-1}| + |f_m|) < \infty$. It proves the needed statement. \square

Corollary 1. If $f_m = \alpha, m = \overline{1, \infty}, \alpha \in R^1, \xi \in A_0$, then $\{\alpha E^P\{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular martingale. Assume that $\xi = 1$, then $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular super-martingale relative to a convex set of equivalent measures M .

Denote F_0 the set of adapted processes

$$F_0 = \left\{ f = \{f_m\}_{m=0}^\infty, P(|f_m| < \infty) = 1, P \in M, f_m \leq f_{m-1} \right\}. \tag{50}$$

For every $\xi \in A_0$ let us introduce the set of adapted processes

$$L_\xi = \left\{ \bar{f} = \{f_m E^P\{\xi | \mathcal{F}_m\}\}_{m=0}^\infty, \{f_m\}_{m=0}^\infty \in F_0, E^P \xi | f_m | < \infty, P \in M \right\}, \tag{51}$$

and

$$V = \bigcup_{\xi \in A_0} L_\xi. \tag{52}$$

Corollary 2. Every random process from the set K , where

$$K = \left\{ \sum_{i=1}^m C_i \bar{f}_i, \bar{f}_i \in V, C_i \geq 0, i = \overline{1, m}, m = \overline{1, \infty} \right\}, \tag{53}$$

is a local regular super-martingale relative to the convex set of equivalent measures M on a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_m on it.

Proof. The proof is evident. \square

Theorem 5. On a measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. Suppose that $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a nonnegative uniformly integrable super-martingale relative to a convex set of

equivalent measures M , then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set K .

Proof. Necessity. It is evident that if $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ belongs to K , then it is a local regular super-martingale.

Sufficiency. Suppose that $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular super-martingale. Then there exists nonnegative adapted process $\{\bar{g}_m\}_{m=1}^\infty$, $E^P \bar{g}_m < \infty$, $m = \overline{1, \infty}$, and a martingale $\{M_m\}_{m=0}^\infty$, such that

$$f_m = M_m - \sum_{i=1}^m \bar{g}_i^0, \quad m = \overline{0, \infty}. \tag{54}$$

Then $M_m \geq 0$, $m = \overline{0, \infty}$, $E^P M_m < \infty$, $P \in M$. Since $0 < E^P M_m = f_0 < \infty$ we have $E^P \sum_{i=1}^m \bar{g}_i^0 < f_0$. Let us put $g_\infty = \lim_{m \rightarrow \infty} \sum_{i=1}^m \bar{g}_i^0$. Using the uniform integrability of f_m , we can pass to the limit in the equality

$$E^P \left(f_m + \sum_{i=1}^m \bar{g}_i^0 \right) = f_0, \quad P \in M, \tag{55}$$

as $m \rightarrow \infty$. Passing to the limit in the last equality, as $m \rightarrow \infty$, we obtain

$$E^P (f_\infty + g_\infty) = f_0, \quad P \in M. \tag{56}$$

Introduce into consideration a random value $\xi = \frac{f_\infty + g_\infty}{f_0}$. Then $E^P \xi = 1$, $P \in M$. From here we obtain that $\xi \in A_0$ and

$$M_m = f_0 E^P \{ \xi | \mathcal{F}_m \}, \quad m = \overline{0, \infty}. \tag{57}$$

Let us put $\bar{f}_m^2 = -\sum_{i=1}^m \bar{g}_i^0$. It is easy to see that the adapted random process $\bar{f}_2 = \{ \bar{f}_m^2, \mathcal{F}_m \}_{m=0}^\infty$ belongs to F_0 . Therefore, for the super-martingale $f = \{ f_m, \mathcal{F}_m \}_{m=0}^\infty$ the representation

$$f = \bar{f}_1 + \bar{f}_2,$$

is valid, where $\bar{f}_1 = \{ f_0 E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \}_{m=0}^\infty$ belongs to L_ξ with $\xi = \frac{f_\infty + g_\infty}{f_0}$ and $f_m^1 = f_0$, $m = \overline{0, \infty}$. The same is valid for \bar{f}_2 with $\xi = 1$. This implies that f belongs to the set K . Theorem 5 is proved. \square

Theorem 6. On a measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. Suppose that the super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to the convex set of equivalent measures M satisfy conditions

$$|f_m| \leq C \xi_0, \quad m = \overline{1, \infty}, \quad \xi_0 \in A_0, \quad 0 < C < \infty, \tag{58}$$

then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set K .

Proof. The necessity is evident.

Sufficiency. Suppose that $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular super-martingale.

Then there exists a nonnegative adapted random process

$\{\bar{g}_m\}_{m=1}^\infty$, $E^P \bar{g}_m^0 < \infty$, $m = \overline{1, \infty}$, and a martingale

$\{M_m\}_{m=0}^\infty$, $E^P |M_m| < \infty$, $m = \overline{1, \infty}$, $P \in M$, such that

$$f_m = M_m - \sum_{i=1}^m \bar{g}_i^0, \quad m = \overline{0, \infty}. \tag{59}$$

The inequalities $f_m + C\xi_0 \geq 0$, $m = \overline{1, \infty}$, give the inequalities

$$f_m + CE^P \{\xi_0 | \mathcal{F}_m\} \geq 0, \quad m = \overline{0, \infty}. \tag{60}$$

From the inequalities (58) it follows that the super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a uniformly integrable one relative to the convex set of equivalent measures M . The martingale $\{E^P \{\xi_0 | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ relative to the convex set of equivalent measures M is also uniformly integrable one.

Then $M_m + CE^P \{\xi_0 | \mathcal{F}_m\} \geq 0$, $m = \overline{0, \infty}$. Since

$0 < E^P [M_m + CE^P \{\xi_0 | \mathcal{F}_m\}] = f_0 + C < \infty$ we have $E^P \sum_{i=1}^m \bar{g}_i^0 < f_0 + C$. Let us

put $g_\infty = \lim_{m \rightarrow \infty} \sum_{i=1}^m \bar{g}_i^0$. Using the uniform integrability of f_m and $\sum_{i=1}^m \bar{g}_i^0$ we can pass to the limit in the equality

$$E^P \left(f_m + \sum_{i=1}^m \bar{g}_i^0 \right) = f_0, \quad P \in M, \tag{61}$$

as $m \rightarrow \infty$. Passing to the limit in the last equality, as $m \rightarrow \infty$, we obtain

$$E^P (f_\infty + g_\infty) = f_0, \quad P \in M. \tag{62}$$

Introduce into consideration a random value $\xi_1 = \frac{f_\infty + C\xi_0 + g_\infty}{f_0 + C} \geq 0$. Then

$E^P \xi_1 = 1$, $P \in M$. From here we obtain that $\xi_1 \in A_0$ and for the super-martingale $f = \{f_m, \mathcal{F}_m\}_{m=0}^\infty$ the representation

$$f_m = f_m^0 E^P \{\xi_0 | \mathcal{F}_m\} + f_m^1 E^P \{\xi_1 | \mathcal{F}_m\} + f_m^2 E^P \{\xi_2 | \mathcal{F}_m\}, \quad m = \overline{0, \infty}, \tag{63}$$

is valid, where $f_m^0 = -C$, $f_m^1 = f_0 + C$, $f_m^2 = -\sum_{i=1}^m \bar{g}_i^0$, $m = \overline{0, \infty}$, $\xi_2 = 1$. From the last

representation it follows that the super-martingale $f = \{f_m, \mathcal{F}_m\}_{m=0}^\infty$ belongs to the set K . Theorem 6 is proved. \square

Corollary 3. Let f_N , $N < \infty$ be a \mathcal{F}_N -measurable integrable random value, $\sup_{P \in M} E^P |f_N| < \infty$, and let there exist $\alpha_0 \in R^1$ such that

$$-\alpha_0 M_N + f_N \leq 0, \quad \omega \in \Omega,$$

where $\{M_m, \mathcal{F}_m\}_{m=0}^\infty = \{E^P \{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$, $\xi \in A_0$. Then a super-martingale

$\{f_m^0 + \bar{f}_m\}_{m=0}^\infty$ is a local regular one relative to the convex set of equivalent measures M , where

$$f_m^0 = \alpha_0 M_m, \tag{64}$$

$$\bar{f}_m = \begin{cases} 0, & m < N, \\ f_N - \alpha_0 M_N, & m \geq N. \end{cases} \tag{65}$$

Proof. It is evident that $\bar{f}_{m-1} - \bar{f}_m \geq 0, m = \overline{0, \infty}$. Therefore, the super-martingale

$$f_m^0 + \bar{f}_m = \begin{cases} \alpha_0 M_m, & m < N, \\ f_N, & m = N, \\ f_N - \alpha_0 M_N + \alpha_0 M_m, & m > N \end{cases} \tag{66}$$

is a local regular one relative to the convex set of equivalent measures M . Corollary 3 is proved. □

5. Optional Decomposition for Super-Martingales Relative to the Complete Convex Set of Equivalent Measures

In this section we introduce the notion of complete set of equivalent measures and prove that non negative super-martingales are local regular ones with respect to this set of measures. For this purpose we are needed the next auxiliary statement.

Theorem 7. The necessary and sufficient conditions of the local regularity of the nonnegative super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures M are the existence of \mathcal{F}_m -measurable random values $\xi_m^0 \in A_0, m = \overline{1, \infty}$, such that

$$\frac{f_m}{f_{m-1}} \leq \xi_m^0, E^P \{ \xi_m^0 | \mathcal{F}_{m-1} \} = 1, P \in M, m = \overline{1, \infty}. \tag{67}$$

Proof. The necessity. Without loss of generality, we assume that $f_m \geq a$ for a certain real number $a > 0$. Really, if it is not so, then we can come to the consideration of the super-martingale $\{f_m + a, \mathcal{F}_m\}_{m=0}^\infty$. Thus, let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a nonnegative local regular super-martingale. Then there exists a nonnegative adapted random process $\{g_m\}_{m=0}^\infty, g_0 = 0$, such that $\sup_{P \in M} E^P g_m < \infty$,

$$f_{m-1} - E^P \{ f_m | \mathcal{F}_{m-1} \} = E^P \{ g_m | \mathcal{F}_{m-1} \}, P \in M, m = \overline{1, \infty}. \tag{68}$$

Let us put $\xi_m^0 = \frac{f_m + g_m}{f_{m-1}}, m = \overline{1, \infty}$. Then $\xi_m^0 \in A_0$ and from the equalities (68) we obtain $E^P \{ \xi_m^0 | \mathcal{F}_{m-1} \} = 1, P \in M, m = \overline{1, \infty}$. It is evident that the inequalities (67) are valid.

The sufficiency. Suppose that the conditions of Theorem 7 are valid. Then $f_m \leq f_{m-1} + f_{m-1}(\xi_m^0 - 1)$. Introduce the denotation $g_m = -f_m + f_{m-1}\xi_m^0$. Then $g_m \geq 0, \sup_{P \in M} E^P g_m \leq \sup_{P \in M} E^P f_m + \sup_{P \in M} E^P f_{m-1} < \infty, m = \overline{1, \infty}$. The last equality and inequalities give

$$f_m = f_0 + \sum_{i=1}^m f_{i-1} (\xi_i^0 - 1) - \sum_{i=1}^m g_i, m = \overline{1, \infty}. \tag{69}$$

Let us consider the random process $\{M_m, \mathcal{F}_m\}_{m=0}^\infty$, where

$M_m = f_0 + \sum_{i=1}^m f_{i-1} (\xi_i^0 - 1)$. Then $E^P \{M_m | \mathcal{F}_{m-1}\} = M_{m-1}, P \in M, m = \overline{1, \infty}$. Theorem 7 is proved. □

5.1. Space of Finite Set of Elementary Events

In this subsection we assume that a space of elementary events Ω is finite one, that is, $N_0 = |\Omega| < \infty$, and we give a new proof of the optional decomposition for super-martingales relative to the complete convex set of equivalent measures. This proof does not use topological arguments as in [15].

Let \mathcal{F} be a certain algebra of subsets of the set Ω and let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be an increasing set of algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F}$. Denote M a convex set of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$. Further, we assume that the set A_0 contains an element $\xi_0 \neq 1$. It is evident that every algebra \mathcal{F}_n is generated by sets $A_i^n, i = \overline{1, N_n}$, $A_i^n \cap A_j^n = \emptyset, i \neq j$, $N_n < \infty$,

$\bigcup_{i=1}^{N_n} A_i^n = \Omega, n = \overline{1, N}$. Let $m_n = E^P \{\xi_0 | \mathcal{F}_n\}, P \in M, n = \overline{1, N}$. Then for m_n the representation

$$m_n = \sum_{i=1}^{N_n} m_i^n \chi_{A_i^n}(\omega), \quad n = \overline{1, N}, \tag{70}$$

is valid. Consider the difference $d^n(\omega) = m_n - m_{n-1}$. Then

$$d^n(\omega) = \sum_{j=1}^{N_n} d_j^n \chi_{A_j^n}(\omega) = \sum_{j \in I_n^-} d_j^n \chi_{A_j^n}(\omega) + \sum_{j \in I_n^+} d_j^n \chi_{A_j^n}(\omega), \tag{71}$$

$$\sum_{j \in I_n^-} \chi_{A_j^n}(\omega) + \sum_{j \in I_n^+} \chi_{A_j^n}(\omega) = 1, \tag{72}$$

where $d_j^n \leq 0$, as $j \in I_n^-$, and $d_j^n > 0$ for $j \in I_n^+$. From the equalities (71), (72) we obtain

$$E^P d^n(\omega) = \sum_{j \in I_n^-} d_j^n P(A_j^n) + \sum_{j \in I_n^+} d_j^n P(A_j^n) = 0, \quad P \in M, \tag{73}$$

$$\sum_{j \in I_n^-} P(A_j^n) + \sum_{j \in I_n^+} P(A_j^n) = 1, \quad P \in M. \tag{74}$$

Denote M_n the contraction of the set of measures M on the algebra \mathcal{F}_n . Introduce into the set M_n the metrics

$$\rho_n(P_1, P_2) = \max_B \sum_{s=1}^k |P_1(B_s^n) - P_2(B_s^n)|, \quad P_1, P_2 \in M_n, n = \overline{1, N}, \tag{75}$$

where $B = \{B_1^n, \dots, B_k^n\}$ is a partition of Ω on k subsets, that is, $B_i^n \in \mathcal{F}_n, i = \overline{1, k}$, $B_i^n \cap B_j^n = \emptyset, i \neq j$, $\bigcup_{i=1}^k B_i^n = \Omega$. The maximum in the formula (75) is all over the partitions of the set Ω , belonging to the σ -algebra \mathcal{F}_n .

Definition 3. On a measurable space $\{\Omega, \mathcal{F}\}$, a convex set of equivalent measure M we call complete if for every $1 \leq n \leq N$ the closure of the set of measures M_n in the metrics (75) contains the measures

$$P_{ij}^n(A) = \begin{cases} 0, & A \neq A_i^n, A_j^n, \\ \frac{d_j^n}{-d_i^n + d_j^n}, & A = A_i^n, \\ \frac{-d_i^n}{-d_i^n + d_j^n}, & A = A_j^n \end{cases} \tag{76}$$

for every $i \in I_n^-$ and $j \in I_n^+$.

Lemma 10. Let a convex family of equivalent measures M be a complete one and the set A_0 contains an element $\xi_0 \neq 1$. Then for every non negative \mathcal{F}_n -measurable random value $\xi_n = \sum_{i=1}^{N_n} C_i^n \chi_{A_i^n}$ there exists a real number α_n such that

$$\frac{\sum_{i=1}^{N_n} C_i^n \chi_{A_i^n}}{\sup_{P \in M_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n)} \leq 1 + \alpha_n (m_n - m_{n-1}), \quad n = \overline{1, N}. \tag{77}$$

Proof. On the set \bar{M}_n , the functional $\varphi(P) = \sum_{i=1}^{N_n} C_i^n P(A_i^n)$ is a continuous one, where \bar{M}_n is the closure of the set M_n in the metrics $\rho_n(P_1, P_2)$. From this it follows that the equality

$$\sup_{P \in M_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n) = \sup_{P \in \bar{M}_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n) \tag{78}$$

is valid. Denote $f_i^n = \frac{C_i^n}{\sup_{P \in M_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n)}$, $i = \overline{1, N_n}$. Then

$$\sum_{i=1}^{N_n} f_i^n P(A_i^n) \leq 1, \quad P \in \bar{M}_n. \tag{79}$$

For those $i \in I_n^-$ for which $d_i^n < 0$ and those $j \in I_n^+$ for which $d_j^n > 0$ the inequality (79) is as follows

$$f_i^n \frac{d_j^n}{-d_i^n + d_j^n} + \frac{-d_i^n}{-d_i^n + d_j^n} f_j^n \leq 1, \quad d_i^n < 0, i \in I_n^-, d_j^n > 0, j \in I_n^+. \tag{80}$$

From (80) we obtain the inequalities

$$f_j^n \leq 1 + \frac{1 - f_i^n}{-d_i^n} d_j^n, \quad d_i^n < 0, i \in I_n^-, d_j^n > 0, j \in I_n^+. \tag{81}$$

Since the inequalities (81) are valid for every $\frac{1 - f_i^n}{-d_i^n}$, as $d_i^n < 0$, and since the set of such elements is finite, then if to denote

$$\alpha_n = \min_{\{i, d_i^n < 0\}} \frac{1 - f_i^n}{-d_i^n}, \tag{82}$$

then we have

$$f_j^n \leq 1 + \alpha_n d_j^n, \quad d_j^n > 0, \quad j \in I_n^+. \tag{83}$$

From the definition of α_n we obtain the inequalities

$$f_i^n \leq 1 + \alpha_n d_i^n, \quad d_i^n < 0, \quad i \in I_n^-. \tag{84}$$

Now if $d_i^n = 0$ for some $i \in I_n^-$, then in this case $f_i^n \leq 1$. All these inequalities give

$$f_i^n \leq 1 + \alpha_n d_i^n, \quad i \in I_n^- \cup I_n^+. \tag{85}$$

Multiplying on $\chi_{A_i^n}$ the inequalities (85) and summing over all $i \in I_n^- \cup I_n^+$ we obtain the needed inequality. Lemma 10 is proved. \square

Theorem 8. Suppose that the conditions of Lemma 10 are valid. Then every non negative super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^N$ relative to a convex set of equivalent measures M , satisfying conditions

$$\frac{f_n}{f_{n-1}} \leq C_n < \infty, \quad n = \overline{1, N}, \tag{86}$$

is a local regular one, where $C_n, n = \overline{1, N}$ are constants.

Proof. Consider the random value $\xi_n = \frac{f_n}{f_{n-1}}$. Due to Lemma 10

$$\frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq 1 + \alpha_n (m_n - m_{n-1}) = \xi_n^0, \quad n = \overline{1, N}. \tag{87}$$

It is evident that $E^P \{\xi_n^0 | \mathcal{F}_{n-1}\} = 1, P \in M, n = \overline{1, N}$. Since $\sup_{P \in M} E^P \xi_n \leq 1$, then

$$\frac{f_n}{f_{n-1}} \leq \xi_n^0, \quad n = \overline{1, N}. \tag{88}$$

Theorem 7 and the inequalities (88) prove Theorem 8. \square

Theorem 9. On a finite space of elementary events $\{\Omega, \mathcal{F}\}$ with a filtration \mathcal{F}_n on it, every super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^N$ relative to the complete convex set of equivalent measures M is a local regular one if the set A_0 contains $\xi_0 \neq 1$.

Proof. It is evident that every super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^N$ is bounded. Therefore, there exists a constant $C_0 > 0$ such that

$\frac{3C_0}{2} > f_m + C_0 > \frac{C_0}{2}, \omega \in \Omega, m = \overline{0, N}$. From this it follows that the super-martingale $\{f_m + C_0, \mathcal{F}_m\}_{m=0}^N$ is a nonnegative one and satisfies the conditions

$$\frac{f_n + C_0}{f_{n-1} + C_0} \leq 3, \quad n = \overline{1, N}. \tag{89}$$

It implies that the conditions of Theorem 8 are satisfied. Theorem 9 is proved. \square

Theorem 10. Let M be a complete convex set of equivalent measure on a measurable space $\{\Omega, \mathcal{F}\}$ with a filtration \mathcal{F}_m on it. Suppose that $\xi_0 \in A_0, \xi_0 \neq 1$, and $m_n = E^P \{\xi_0 | \mathcal{F}_n\}$ is a martingale relative to the set of measures M . Let M_0^a be a set of all martingale measures absolutely continuous

relative to any measure $P \in M$. Then the inclusion $\bar{M} \subseteq M_0^a$ is valid, where \bar{M} is a closure of the set of measures M in metrics $\rho_N(P_1, P_2)$, defined in (75).

Proof. Let the sequence $P_s \in M$ be a convergent one to the measure $P_0 \in \bar{M}$, then for $D \in \mathcal{F}_{n-1}$

$$\int_D m_n dP_s = \int_D m_{n-1} dP_s, \quad s = \overline{1, \infty}. \tag{90}$$

The functionals $\int m_n dP, \int m_{n-1} dP$ on the set \bar{M} for all $D \in \mathcal{F}_{n-1}$ are continuous ones relative^D to the^D metrics $\rho_N(P_1, P_2)$, defined by the formula (75). Going to the limit in the equality (90), as $s \rightarrow \infty$, we obtain

$$\int_D m_n dP_0 = \int_D m_{n-1} dP_0, \quad n = \overline{1, N}, \quad D \in \mathcal{F}_{n-1}. \tag{91}$$

The last implies that $P_0 \in M_0^a$. Theorem 10 is proved. □

5.2. Countable Set of Elementary Events

In this subsection, we generalize the results of the previous subsection onto the countable space of elementary events. Let \mathcal{F} be a certain σ -algebra of subsets of the countable set of elementary events Ω and let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be a certain increasing set of σ -algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Denote M a set of equivalent measures on the measurable space $\{\Omega, \mathcal{F}\}$. Further, we assume that the set A_0 contains an element $\xi_0 \neq 1$. Suppose that the σ -algebra \mathcal{F}_n is generated by the sets $A_i^n, i = \overline{1, \infty}, A_i^n \cap A_j^n = \emptyset, i \neq j, \bigcup_{i=1}^{\infty} A_i^n = \Omega, n = \overline{1, \infty}$.

Introduce into consideration the martingale $m_n = E^P \{\xi_0 | \mathcal{F}_n\}, P \in M, n = \overline{1, \infty}$. Then for m_n the representation

$$m_n = \sum_{i=1}^{\infty} m_i^n \chi_{A_i^n}(\omega), \quad n = \overline{1, \infty}, \tag{92}$$

is valid. Consider the difference $d^n(\omega) = m_n - m_{n-1}$. Then

$$d^n(\omega) = \sum_{j=1}^{\infty} d_j^n \chi_{A_j^n}(\omega) = \sum_{j \in I_n^-} d_j^n \chi_{A_j^n}(\omega) + \sum_{j \in I_n^+} d_j^n \chi_{A_j^n}(\omega), \tag{93}$$

$$\sum_{j \in I_n^-} \chi_{A_j^n}(\omega) + \sum_{j \in I_n^+} \chi_{A_j^n}(\omega) = 1, \tag{94}$$

where $d_j^n \leq 0$, as $j \in I_n^-$, and $d_j^n > 0$, $j \in I_n^+$. From the equalities (93), (94) we obtain

$$E^P d^n(\omega) = \sum_{j \in I_n^-} d_j^n P(A_j^n) + \sum_{j \in I_n^+} d_j^n P(A_j^n) = 0, \quad P \in M, \tag{95}$$

$$\sum_{j \in I_n^-} P(A_j^n) + \sum_{j \in I_n^+} P(A_j^n) = 1, \quad P \in M. \tag{96}$$

Denote M_n the contraction of the set of measures M on the σ -algebra \mathcal{F}_n . Introduce into the set M_n the metrics

$$\rho_n(P_1, P_2) = \sup_B \sum_{s=1}^k |P_1(B_s^n) - P_2(B_s^n)|, \quad P_1, P_2 \in M_n, \quad n = \overline{1, \infty}, \tag{97}$$

where $B = \{B_1^n, \dots, B_k^n\}$ is a partition of Ω on k subsets, that is, $B_i^n \in \mathcal{F}_n, i = \overline{1, k}$, $B_i^n \cap B_j^n = \emptyset, i \neq j, \bigcup_{i=1}^k B_i^n = \Omega$. The supremum in the formula (97) is all over the partitions of the set Ω , belonging to the σ -algebra \mathcal{F}_n .

Definition 4. On a measurable space $\{\Omega, \mathcal{F}\}$ with a filtration \mathcal{F}_n on it, a convex set of equivalent measure M we call complete one if for every $1 \leq n < \infty$ the closure of the set of measures M_n in the metrics (97) contains the measures

$$P_{ij}^n(A) = \begin{cases} 0, & A \neq A_i^n, A_j^n, \\ \frac{d_j^n}{-d_i^n + d_j^n}, & A = A_i^n, \\ \frac{-d_i^n}{-d_i^n + d_j^n}, & A = A_j^n \end{cases} \tag{98}$$

for every $i \in I_n^-$ and $j \in I_n^+$.

Lemma 11. Let a family of measures M be complete and the set A_0 contains an element $\xi_0 \neq 1$. Then for every non-negative bounded \mathcal{F}_n -measurable random value $\xi_n = \sum_{i=1}^{\infty} C_i^n \chi_{A_i^n}$ there exists a real number α_n such that

$$\frac{\sum_{i=1}^{\infty} C_i^n \chi_{A_i^n}}{\sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n)} \leq 1 + \alpha_n (m_n - m_{n-1}), \quad n = \overline{1, \infty}. \tag{99}$$

Proof. On the set \overline{M}_n , the functional $\varphi(P) = \sum_{i=1}^{\infty} C_i^n P(A_i^n)$ is a continuous one relative to the metrics $\rho_n(P_1, P_2)$, where \overline{M}_n is the closure of the set M_n in this metrics. From this it follows that the equality

$$\sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n) = \sup_{P \in \overline{M}_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n) \tag{100}$$

is valid. Denote $f_i^n = \frac{C_i^n}{\sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n)}, i = \overline{1, \infty}$. Then

$$\sum_{i=1}^{\infty} f_i^n P(A_i^n) \leq 1, \quad P \in \overline{M}_n. \tag{101}$$

The last inequalities can be written in the form

$$\sum_{i \in I^-} f_i^n P(A_i^n) + \sum_{i \in I^+} f_i^n P(A_i^n) \leq 1, \quad P \in \overline{M}_n. \tag{102}$$

For those $i \in I_n^-$ for which $d_i^n < 0$ and those $j \in I_n^+$ for which $d_j^n > 0$ the inequality (102) is as follows

$$f_i^n \frac{d_j^n}{-d_i^n + d_j^n} + \frac{-d_i^n}{-d_i^n + d_j^n} f_j^n \leq 1, \quad d_i^n < 0, d_j^n > 0, i \in I_n^-, j \in I_n^+. \tag{103}$$

From (103) we obtain the inequalities

$$f_j^n \leq 1 + \frac{1-f_i^n}{-d_i^n} d_j^n, d_i^n < 0, d_j^n > 0, i \in I_n^-, j \in I_n^+. \tag{104}$$

Two cases are possible: 1) for all $i \in I_n^-$, $f_i^n \leq 1$; 2) there exists $i \in I_n^-$ such that $f_i^n > 1$. First, let us consider the case a).

Since the inequalities (104) are valid for every $\frac{1-f_i^n}{-d_i^n}$, as $d_i^n < 0$, and $f_i^n \leq 1, i \in I_n^-$, then if to denote

$$\alpha_n = \inf_{\{i, d_i^n < 0\}} \frac{1-f_i^n}{-d_i^n}, \tag{105}$$

we have $0 \leq \alpha_n < \infty$ and

$$f_j^n \leq 1 + \alpha_n d_j^n, d_j^n > 0, j \in I_n^+. \tag{106}$$

From the definition of α_n we obtain the inequalities

$$f_i^n \leq 1 + \alpha_n d_i^n, d_i^n < 0, i \in I_n^-. \tag{107}$$

Now, if $d_i^n = 0$ for some $i \in I_n^-$, then in this case $f_i^n \leq 1$. All these inequalities give

$$f_i^n \leq 1 + \alpha_n d_i^n, i \in I_n^- \cup I_n^+. \tag{108}$$

Consider the case b). From the inequality (104), we obtain

$$f_j^n \leq 1 - \frac{1-f_i^n}{d_i^n} d_j^n, d_i^n < 0, d_j^n > 0, i \in I_n^-, j \in I_n^+. \tag{109}$$

The last inequalities give

$$\frac{1-f_i^n}{d_i^n} \leq \min_{\{j, d_j^n > 0\}} \frac{1}{d_j^n} < \infty, d_i^n < 0, i \in I_n^-. \tag{110}$$

Let us define $\alpha_n = \sup_{\{i, d_i^n < 0\}} \frac{1-f_i^n}{d_i^n} < \infty$. Then from (109) we obtain

$$f_j^n \leq 1 - \alpha_n d_j^n, d_j^n > 0, j \in I_n^+. \tag{111}$$

From the definition of α_n , we have

$$f_i^n \leq 1 - \alpha_n d_i^n, d_i^n < 0, i \in I_n^-. \tag{112}$$

The inequalities (111), (112) give

$$f_j^n \leq 1 - \alpha_n d_j^n, j \in I_n^- \cup I_n^+. \tag{113}$$

Multiplying on $\chi_{A_i^n}$ the inequalities (108) and the inequalities (113) on $\chi_{A_j^n}$ and summing over all $i, j \in I_n^- \cup I_n^+$ we obtain the needed inequality. The Lemma 11 is proved. \square

Theorem 11. Suppose that the conditions of Lemma 11 are valid. Then every non negative super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures \mathcal{M} , satisfying the conditions

$$\frac{f_m}{f_{m-1}} \leq C_m < \infty, m = \overline{1, \infty}, \tag{114}$$

is a local regular one, where C_m are constants.

Proof. From the conditions (114) it follows that $\sup_{P \in M} E^P f_m < \infty$. Consider the random value $\xi_n = \frac{f_n}{f_{n-1}}$. Due to Lemma 11

$$\frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq 1 + \alpha_n (m_n - m_{n-1}) = \xi_n^0. \tag{115}$$

It is evident that $E^P \{ \xi_n^0 | \mathcal{F}_{n-1} \} = 1, P \in M, n = \overline{1, \infty}$. Since $\sup_{P \in M} E^P \xi_n \leq 1$, then

$$\frac{f_n}{f_{n-1}} \leq \xi_n^0, n = \overline{1, \infty}. \tag{116}$$

Theorem 7 and the inequalities (116) prove Theorem 11. □

5.3. An arbitrary Space of Elementary Events

In this subsection, we consider an arbitrary space of elementary events and prove the optional decomposition for non negative super-martingales.

Let \mathcal{F} be a certain σ -algebra of subsets of the set of elementary events Ω and let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be an increasing set of the σ -algebras, where $\mathcal{F}_0 = \{ \emptyset, \Omega \}$. Denote M a set of equivalent measures on a measurable space $\{ \Omega, \mathcal{F} \}$. We assume that the σ -algebras $\mathcal{F}_n, n = \overline{1, \infty}$, and \mathcal{F} are complete relative to any measure $P \in M$. Further, we suppose that the set A_0 contains an element $\xi_0 \neq 1$. Let $m_n = E^P \{ \xi_0 | \mathcal{F}_n \}, P \in M, n = \overline{1, \infty}$.

Consider the difference $d^n(\omega) = m_n - m_{n-1}$. We assume that every $\omega \in \Omega$ belongs to the σ -algebra $\mathcal{F}_n, n = \overline{1, \infty}$, and $P(\{ \omega \}) = 0, \omega \in \Omega, P \in M$.

For the random value $d^n(\omega)$ there exists not more then a countable set of the real number d_s^n such that $P(A_s^n) > 0$, where $A_s^n = \{ \omega \in \Omega, d^n(\omega) = d_s^n \}$. It is evident that $A_i^n \cap A_j^n = \emptyset, i \neq j$. Suppose that $P\left(\Omega \setminus \bigcup_{i=1}^{\infty} A_i^n\right) > 0$. Introduce for every n two subsets $I_n^- = \{ \omega \in \Omega, d^n(\omega) \leq 0 \}, I_n^+ = \{ \omega \in \Omega, d^n(\omega) > 0 \}$ of the set $\{ \omega \in \Omega, |d_n(\omega)| < \infty \}$.

Denote M_n the contraction of the set of measures M on the σ -algebra \mathcal{F}_n . Introduce into the set M_n the metrics

$$\rho_n(P_1, P_2) = \sup_B \sum_{s=1}^k |P_1(B_s^n) - P_2(B_s^n)|, P_1, P_2 \in M_n, n = \overline{1, \infty}, \tag{117}$$

where $B = \{ B_1^n, \dots, B_k^n \}$ is a partition of Ω on k subsets, that is, $B_i^n \in \mathcal{F}_n, i = \overline{1, k}, B_i^n \cap B_j^n = \emptyset, i \neq j, \bigcup_{i=1}^k B_i^n = \Omega$. The supremum in the formula (117) is all over the partitions of the set Ω , belonging to the σ -algebra \mathcal{F}_n .

Definition 5. On a measurable space $\{ \Omega, \mathcal{F} \}$ with filtration \mathcal{F}_n on it, a convex set of equivalent measure M we call complete if for every $1 \leq n < \infty$ the

closure in metrics (117) of the set of measures M_n contains the measures

$$P_{\omega_1, \omega_2}^n(A) = \begin{cases} 0, & \omega_1, \omega_2 \in \Omega \setminus A, \\ \frac{d^n(\omega_2)}{-d^n(\omega_1) + d^n(\omega_2)}, & \omega_1 \in A, A \cap \{\omega_2\} = \emptyset, \\ \frac{-d^n(\omega_1)}{-d^n(\omega_1) + d^n(\omega_2)}, & \omega_2 \in \Omega \setminus A, (\Omega \setminus A) \cap \{\omega_1\} = \emptyset \end{cases} \quad (118)$$

for $\omega_1 \in I_n^-$ and $\omega_2 \in I_n^+$.

Lemma 12. Let a convex family of equivalent measures M be a complete one and the set A_0 contains an element $\xi_0 \neq 1$. Then for every non negative bounded \mathcal{F}_n -measurable random value ξ_n there exists a real number α_n such that

$$\frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq 1 + \alpha_n (m_n - m_{n-1}), \quad n = \overline{1, \infty}. \quad (119)$$

Proof. On the set \overline{M}_n , the functional $\varphi(P) = \int \xi_n dP$ is a continuous one relative to the metrics $\rho_n(P_1, P_2)$, where \overline{M}_n is the closure of the set M_n in this metrics. From this it follows that the equality

$$\sup_{P \in M_n} \int \xi_n dP = \sup_{P \in \overline{M}_n} \int \xi_n dP \quad (120)$$

is valid. Denote $f^n(\omega) = \frac{\xi_n(\omega)}{\sup_{P \in M_n} E^P \xi_n(\omega)}$. Then

$$E^P f^n(\omega) \leq 1, \quad P \in \overline{M}_n. \quad (121)$$

The last inequalities can be written in the form

$$\int_{I_n^-} f_n(\omega) dP + \int_{I_n^+} f_n(\omega) dP \leq 1, \quad P \in \overline{M}_n. \quad (122)$$

The inequality (122) for the measures (118) is as follows

$$f^n(\omega_1) \frac{d^n(\omega_2)}{-d^n(\omega_1) + d^n(\omega_2)} + \frac{-d^n(\omega_1)}{-d^n(\omega_1) + d^n(\omega_2)} f^n(\omega_2) \leq 1, \quad \omega_1 \in I_n^-, \omega_2 \in I_n^+. \quad (123)$$

From (123) we obtain the inequalities

$$f^n(\omega_2) \leq 1 + \frac{1 - f^n(\omega_1)}{-d^n(\omega_1)} d^n(\omega_2), \quad (124)$$

$$d^n(\omega_1) < 0, \quad d^n(\omega_2) > 0, \quad \omega_1 \in I_n^-, \quad \omega_2 \in I_n^+. \quad (125)$$

Two cases are possible: 1) for all $\omega_1 \in I_n^-$, $f^n(\omega_1) \leq 1$; 2) there exists $\omega_1 \in I_n^-$ such that $f^n(\omega_1) > 1$. First, let us consider the case a).

Since the inequalities (124) are valid for every $\frac{1 - f^n(\omega_1)}{-d^n(\omega_1)}$, as $d^n(\omega_1) < 0$,

and $f^n(\omega_1) \leq 1, \omega_1 \in I_n^-$, then if to denote

$$\alpha_n = \inf_{\{\omega_1, d^n(\omega_1) < 0\}} \frac{1 - f^n(\omega_1)}{-d^n(\omega_1)}, \tag{126}$$

we have $0 \leq \alpha_n < \infty$ and

$$f^n(\omega_2) \leq 1 + \alpha_n d^n(\omega_2), \quad d^n(\omega_2) > 0, \quad \omega_2 \in I_n^+. \tag{127}$$

From the definition of α_n we obtain the inequalities

$$f^n(\omega_1) \leq 1 + \alpha_n d^n(\omega_1), \quad d^n(\omega_1) < 0, \quad \omega_1 \in I_n^-. \tag{128}$$

Now, if $d^n(\omega_1) = 0$ for some $\omega_1 \in I_n^-$, then in this case $f^n(\omega_1) \leq 1$. All these inequalities give

$$f^n(\omega) \leq 1 + \alpha_n d^n(\omega), \quad \omega \in I_n^- \cup I_n^+. \tag{129}$$

Consider the case b). From the inequality (124), we obtain

$$f^n(\omega_2) \leq 1 - \frac{1 - f^n(\omega_1)}{d^n(\omega_1)} d^n(\omega_2), \tag{130}$$

$$d^n(\omega_1) < 0, \quad d^n(\omega_2) > 0, \quad \omega_1 \in I_n^-, \quad \omega_2 \in I_n^+. \tag{131}$$

The last inequalities give

$$\frac{1 - f^n(\omega_1)}{d^n(\omega_1)} \leq \inf_{\{\omega_2, d^n(\omega_2) > 0\}} \frac{1}{d^n(\omega_2)} < \infty, \quad d^n(\omega_1) < 0, \quad \omega_1 \in I_n^-. \tag{132}$$

Let us define $\alpha_n = \sup_{\{\omega_1, d^n(\omega_1) < 0\}} \frac{1 - f^n(\omega_1)}{d^n(\omega_1)} < \infty$. Then from (130) we obtain

$$f^n(\omega_2) \leq 1 - \alpha_n d^n(\omega_2), \quad d^n(\omega_2) > 0, \quad \omega_2 \in I_n^+. \tag{133}$$

From the definition of α_n we have

$$f^n(\omega_1) \leq 1 - \alpha_n d^n(\omega_1), \quad d^n(\omega_1) < 0, \quad \omega_1 \in I_n^-. \tag{134}$$

The inequalities (133), (134) give

$$f^n(\omega) \leq 1 - \alpha_n d^n(\omega), \quad \omega \in I_n^- \cup I_n^+. \tag{135}$$

Since the set $I_n^- \cup I_n^+$ has probability one, Lemma 12 is proved. \square

Theorem 12. Suppose a convex set of equivalent measures \mathcal{M} is a complete one and the conditions of Lemma 12 are valid. Then every non negative super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures \mathcal{M} , satisfying conditions

$$\frac{f_m}{f_{m-1}} \leq C_m < \infty, \quad m = \overline{1, \infty}, \tag{136}$$

is a local regular one, where $C_m, m = \overline{1, \infty}$ are constants.

Proof. From the inequalities (136) it follows that $\sup_{P \in \mathcal{M}} E^P f_m < \infty, m = \overline{1, \infty}$.

Consider the random value $\xi_n = \frac{f_n}{f_{n-1}}$. Due to Lemma 12

$$\frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq 1 + \alpha_n (m_n - m_{n-1}) = \xi_n^0. \tag{137}$$

It is evident that $E^P \{ \xi_n^0 | \mathcal{F}_{n-1} \} = 1, P \in M, n = \overline{1, \infty}$. Since $\sup_{P \in M} E^P \xi_n \leq 1$, then

$$\frac{f_n}{f_{n-1}} \leq \xi_n^0, n = \overline{1, \infty}. \tag{138}$$

Theorem 7 and the inequalities (138) prove Theorem 12. □

Consequence 1. If a super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a complete convex set of equivalent measures M satisfy conditions $0 \leq f_m \leq D_m, m = \overline{1, \infty}$, where $D_m < \infty$ are constant, then it is local regular.

Proof. The super-martingale $\{f_m + \varepsilon, \mathcal{F}_m\}_{m=0}^\infty, \varepsilon > 0$ is a nonnegative one and satisfies the conditions

$$\frac{f_m + \varepsilon}{f_{m-1} + \varepsilon} \leq \frac{D_m + \varepsilon}{\varepsilon} = C_m < \infty, m = \overline{1, \infty}. \tag{139}$$

From Theorem 11 it follows the validity of the local regularity for the super-martingale $\{f_m + \varepsilon, \mathcal{F}_m\}_{m=0}^\infty$, therefore, for the super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ the local regularity is also true. □

6. Local Regularity of Majorized Super-Martingales

In this section, we give the elementary proof that a majorized super-martingale relative to the complete set of equivalent measures is local regular one.

Theorem 13. On a measurable space $\{\Omega, \mathcal{F}\}$ with a filtration \mathcal{F}_m on it, let the set M be a complete convex set of equivalent measures on \mathcal{F} and the set A_0 contains an element $\xi_0 \neq 1$. Then every bounded super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to the complete convex set of equivalent measures M is a local regular one.

Proof. From Theorem 13 conditions, there exists a constant $0 < C < \infty$ such that $|f_m| \leq C, m = \overline{1, \infty}$. Consider the super-martingale $\{f_m + C, \mathcal{F}_m\}_{m=0}^\infty$. Then $0 \leq f_m + C \leq 2C$. Due to Consequence 1, for the super-martingale $\{f_m + C, \mathcal{F}_m\}_{m=0}^\infty$ the local regularity is true. So, the same statement is valid for the super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$. Theorem 13 is proved. □

The next Theorem is analogously proved as Theorem 13.

Theorem 14. On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_m on it, let the set M be a complete convex set of equivalent measures on \mathcal{F} and the set A_0 contains an element $\xi_0 \neq 1$. Then a super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to the complete convex set of equivalent measures M satisfying the conditions

$$|f_m| \leq C_1 \xi_0, f_m + C_1 \xi_0 \leq C_2, m = \overline{1, \infty}, \xi_0 \in A_0, \tag{140}$$

for certain constants $0 < C_1, C_2 < \infty$ is a local regular one.

7. Application to Mathematical Finance

Due to Corollary 3, we can give the following definition of the fair price of contingent claim f_N relative to a convex set of equivalent measures M .

Definition 6. Let $f_N, N < \infty$ be a \mathcal{F}_N -measurable integrable random value relative to a convex set of equivalent measures M such that for some $0 \leq \alpha_0 < \infty$ and $\xi_0 \in A_0$

$$P(f_N - \alpha_0 E^P \{\xi_0 | \mathcal{F}_N\} \leq 0) = 1. \tag{141}$$

Denote $G_{\alpha_0} = \{\alpha \in [0, \alpha_0], \exists \xi_\alpha \in A_0, P(f_N - \alpha E^P \{\xi_\alpha | \mathcal{F}_N\} \leq 0) = 1\}$. We call

$$f_0 = \inf_{\alpha \in G_{\alpha_0}} \alpha \tag{142}$$

the fair price of the contingent claim f_N relative to a convex set of equivalent measures M , if there exists $\zeta_0 \in A_0$ and a sequences $\alpha_n \in [0, \alpha_0], \xi_{\alpha_n} \in A_0$, satisfying the conditions: $\alpha_n \rightarrow f_0, \xi_{\alpha_n} \rightarrow \zeta_0$ by probability, as $n \rightarrow \infty$, and such that

$$P(f_N - \alpha_n E^P \{\xi_{\alpha_n} | \mathcal{F}_N\} \leq 0) = 1, n = \overline{1, \infty}. \tag{143}$$

Theorem 15. Let the set A_0 be uniformly integrable one relative to every measure $P \in M$. Suppose that for a nonnegative \mathcal{F}_N -measurable integrable contingent claim $f_N, N < \infty$ relative to every measure $P \in M$ there exist $\alpha_0 < \infty$ and $\xi_0 \in A_0$ such that

$$P(f_N - \alpha_0 E^P \{\xi_0 | \mathcal{F}_N\} \leq 0) = 1, \tag{144}$$

then the fair price f_0 of contingent claim f_N exists. For f_0 the inequality

$$\sup_{P \in M} E^P f_N \leq f_0 \tag{145}$$

is valid. If $f_N \geq 0$ and a super-martingale $\left\{ f_m = \text{ess sup}_{P \in M} E^P \{f_N | \mathcal{F}_m\}, \mathcal{F}_m \right\}_{m=0}^\infty$ is

a local regular one, then $f_0 = \sup_{P \in M} E^P f_N$.

Proof. If $f_0 = \alpha_0$, then Theorem 15 is proved. Suppose that $f_0 < \alpha_0$. Then there exists a sequence $\alpha_n \rightarrow f_0$, and $\xi_{\alpha_n} \in A_0, n \rightarrow \infty$, such that

$$P(f_N - \alpha_n E^P \{\xi_{\alpha_n} | \mathcal{F}_N\} \leq 0) = 1, P \in M. \tag{146}$$

Due to the uniform integrability A_0 we obtain

$$1 = \lim_{n \rightarrow \infty} \int_{\Omega} \xi_{\alpha_n} dP = \int_{\Omega} \zeta_0 dP, P \in M. \tag{147}$$

Using again the uniform integrability of A_0 and going to the limit in (146) we obtain

$$P(f_N - f_0 E^P \{\zeta_0 | \mathcal{F}_N\} \leq 0) = 1, P \in M. \tag{148}$$

From the inequality $f_N - f_0 E^P \{\zeta_0 | \mathcal{F}_N\} \leq 0$ it follows the inequality (145). If

$f_N \geq 0$ and $\left\{ f_m = \text{ess sup}_{P \in M} E^P \{f_N | \mathcal{F}_m\}, \mathcal{F}_m \right\}_{m=0}^N$ is a local regular

super-martingale, then

$$f_m = M_m - g_m, m = \overline{0, N}, g_0 = 0, \tag{149}$$

where a martingale $\{M_m, \mathcal{F}_m\}_{m=0}^N$ is a nonnegative one and $E^P M_m = \sup_{P \in M} E^P f_N$

Introduce into consideration a random value $\xi_0 = \frac{M_N}{\hat{f}_0}$, where $\hat{f}_0 = \sup_{P \in M} E^P f_N$.

Then ξ_0 belongs to the set A_0 and

$$P(f_N - \hat{f}_0 E^P \{\xi_0 | \mathcal{F}_N\} \leq 0) = 1. \tag{150}$$

From this it follows that $f_0 = \sup_{P \in M} E^P f_N$.

Let us prove that f_0 is a fair price for certain evolutions of risk and non risk assets. Suppose that the evolution of risk asset is given by the law $S_m = f_0 M^P \{\zeta_0 | \mathcal{F}_m\}, m = \overline{0, N}$, and the evolution of non risk asset is given by the formula $B_m = 1, m = \overline{0, N}$.

As proved above, for $f_0 = \inf_{\alpha \in G_{a_0}} \alpha$ there exists $\zeta_0 \in A_0$ such that the inequality

$$f_N - f_0 E^P \{\zeta_0 | \mathcal{F}_N\} \leq 0 \tag{151}$$

is valid. Let us put

$$f_m^0 = f_0 E^P \{\zeta_0 | \mathcal{F}_m\}, P \in M, \tag{152}$$

$$\bar{f}_m = \begin{cases} 0, & m < N, \\ f_N - f_0 E^P \{\zeta_0 | \mathcal{F}_m\}, & m = N. \end{cases} \tag{153}$$

It is evident that $\bar{f}_{m-1} - \bar{f}_m \geq 0, m = \overline{0, N}$. Therefore, the super-martingale

$$f_m^0 + \bar{f}_m = \begin{cases} f_0 E^P \{\zeta_0 | \mathcal{F}_m\}, & m < N, \\ f_N, & m = N, \end{cases} \tag{154}$$

is a local regular one. It is evident that

$$f_m^0 + \bar{f}_m = M_m - g_m, m = \overline{0, N}, \tag{155}$$

where

$$M_m = f_0 E^P \{\zeta_0 | \mathcal{F}_m\}, m = \overline{0, N}, \tag{156}$$

$$g_m = 0, m = \overline{0, N-1}, \tag{157}$$

$$g_N = f_0 E^P \{\zeta_0 | \mathcal{F}_N\} - f_N. \tag{158}$$

For the martingale $\{M_m, \mathcal{F}_m\}_{m=0}^N$ the representation

$$M_m = f_0 + \sum_{i=1}^m H_i \Delta S_i, m = \overline{0, N}, \tag{159}$$

is valid, where $H_i = 1, i = \overline{1, N}$. Let us consider the trading strategy $\pi = \{\bar{H}_m^0, \bar{H}_m\}_{m=0}^N$, where

$$\bar{H}_0^0 = f_0, \bar{H}_m^0 = M_m - H_m S_m, m = \overline{1, N}, \tag{160}$$

$$\bar{H}_0 = 0, \bar{H}_m = H_m, m = \overline{1, N}. \tag{161}$$

It is evident that \bar{H}_m^0, \bar{H}_m are \mathcal{F}_{m-1} measurable and the trading strategy π satisfy self-financed condition

$$\Delta \bar{H}_m^0 + \Delta \bar{H}_m S_{m-1} = 0. \tag{162}$$

Moreover, the capital corresponding to the self-financed trading strategy π is given by the formula

$$X_m^\pi = \bar{H}_m^0 + \bar{H}_m S_m = M_m. \tag{163}$$

Here from, $X_0^\pi = f_0$. Further,

$$X_N^\pi = f_N + g_N \geq f_N. \tag{164}$$

The last proves Theorem 15. □

From (148) and Corollary 3 the Theorem 16 follows.

Theorem 16. Suppose that the set A_0 contains only $1 \leq k < \infty$ linear independent elements ξ_1, \dots, ξ_k . If there exist $\xi_0 \in T$ and $\alpha_0 \geq 0$ such that

$$P(f_N - \alpha_0 E^P \{\xi_0 | F_N\} \leq 0) = 1, P \in M, \tag{165}$$

where

$$T = \left\{ \xi \geq 0, \xi = \sum_{i=1}^k \alpha_i \xi_i, \alpha_i \geq 0, i = \overline{1, k}, \sum_{i=1}^k \alpha_i = 1 \right\}, \tag{166}$$

then the fair price f_0 of the contingent claim $f_N \geq 0$ exists, where f_N is F_N measurable and integrable relative to every measure $P \in M$, $N < \infty$.

Proof. The proof is evident, as the set T is a uniformly integrable one relative to every measure from M . □

Corollary 4. On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_m on it, let $\{f_m, \mathcal{F}_m\}_{m=0}^N$ be a non negative local regular super-martingale relative to a convex set of equivalent measures M . If the set A_0 is uniformly integrable relative to every measure $P \in M$, then the fair price of contingent claim f_N exists.

Proof. From the local regularity of super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^N$ we have $f_m = M_m - g_m, m = \overline{0, N}$. Therefore, $P(f_N - \alpha_0 \xi_0 \leq 0) = 1$, where

$$\alpha_0 = E^P M_N, P \in M, \xi_0 = \frac{M_N}{E^P M_N}. \text{ From the last it follows that the conditions of}$$

Theorem 15 are satisfied. Corollary 4 is proved. □

On a probability space $\{\Omega, \mathcal{F}, P\}$, let us consider an evolution of one risk asset given by the law $\{S_m\}_{m=0}^N$, where S_m is a random value taking values in R_+^1 . Suppose that \mathcal{F}_m is a filtration on $\{\Omega, \mathcal{F}, P\}$ and S_m is \mathcal{F}_m -measurable random value. We assume that the non risk asset evolve by the law $B_m^0 = 1, m = \overline{1, N}$. Denote $M^e(S)$ the set of all martingale measures being equivalent to the measure P . We assume that the set $M^e(S)$ of such martingale measures is not empty and the effective market is non complete, see, for example, [16] [17] [18] [19]. So, we have that

$$E^Q \{S_m | \mathcal{F}_{m-1}\} = S_{m-1}, m = \overline{1, N}, Q \in M^e(S). \tag{167}$$

The next Theorem justifies the Definition 6.

Theorem 17. Let a contingent claim f_N be a \mathcal{F}_N -measurable integrable random value with respect to every measure from $M^e(S)$ and the conditions

of the Theorem 16 are satisfied with $\xi_i = \frac{S_i}{S_0}, i = \overline{0, N}$. Then there exists self-financed trading strategy π the capital evolution $\{X_m^\pi\}_{m=0}^N$ of which is a martingale relative to every measure from $M^e(S)$ satisfying conditions $X_0^\pi = f_0, X_N^\pi \geq f_N$, where f_0 is a fair price of contingent claim f_N .

Proof. Due to Theorems 15, 16, for $f_0 = \inf_{\alpha \in G_{\alpha_0}} \alpha$ there exists $\zeta_0 \in A_0$ such that the inequality

$$f_N - f_0 E^P \{ \zeta_0 | \mathcal{F}_N \} \leq 0 \tag{168}$$

is valid. Let us put

$$f_m^0 = f_0 E^P \{ \zeta_0 | \mathcal{F}_m \}, P \in M^e(S), \tag{169}$$

$$\bar{f}_m = \begin{cases} 0, & m < N, \\ f_N - f_0 E^P \{ \zeta_0 | \mathcal{F}_m \}, & m = N. \end{cases} \tag{170}$$

It is evident that $\bar{f}_{m-1} - \bar{f}_m \geq 0, m = \overline{0, N}$. Therefore, the super-martingale

$$f_m^0 + \bar{f}_m = \begin{cases} f_0 E^P \{ \zeta_0 | \mathcal{F}_m \}, & m < N, \\ f_N, & m = N \end{cases} \tag{171}$$

is a local regular one. It is evident that

$$f_m^0 + \bar{f}_m = M_m - g_m, m = \overline{0, N}, \tag{172}$$

where

$$M_m = f_0 E^P \{ \zeta_0 | \mathcal{F}_m \}, m = \overline{0, N}, \tag{173}$$

$$g_m = 0, m = \overline{0, N-1}, \tag{174}$$

$$g_N = f_0 E^P \{ \zeta_0 | \mathcal{F}_N \} - f_N. \tag{175}$$

Due to Theorem 20, for the martingale $\{M_m\}_{m=0}^N$ the representation

$$M_m = f_0 + \sum_{i=1}^m H_i \Delta S_i, m = \overline{0, N}, \tag{176}$$

is valid. Let us consider the trading strategy $\pi = \{ \bar{H}_m^0, \bar{H}_m \}_{m=0}^N$, where

$$\bar{H}_0^0 = f_0, \bar{H}_m^0 = M_m - H_m S_m, m = \overline{1, N}, \tag{177}$$

$$\bar{H}_0 = 0, \bar{H}_m = H_m, m = \overline{1, N}. \tag{178}$$

It is evident that \bar{H}_m^0, \bar{H}_m are \mathcal{F}_{m-1} -measurable ones and the trading strategy π satisfy the self-financed condition

$$\Delta \bar{H}_m^0 + \Delta \bar{H}_m S_{m-1} = 0. \tag{179}$$

Moreover, a capital corresponding to the self-financed trading strategy π is given by the formula

$$X_m^\pi = \bar{H}_m^0 + \bar{H}_m S_m = M_m. \tag{180}$$

Herefrom, $X_0^\pi = f_0$. Further,

$$X_N^\pi = f_N + g_N. \tag{181}$$

Therefore $X_N^x \geq f_N$. Theorem 17 is proved. \square

In the next Theorem we assume that the evolutions of risk and non risk assets generate incomplete market [16] [17] [18] [19] [20], that is, the set of martingale measures contains more than one element.

Theorem 18. Let an evolution $\{S_m\}_{m=0}^N$ of the risk asset satisfy the conditions $P(D_m^1 \leq S_m \leq D_m^2) = 1$, where the constants D_m^i satisfy the inequalities $D_{m-1}^1 \geq D_m^1 > 0$, $D_{m-1}^2 \leq D_m^2 < \infty$, $m = \overline{1, N}$, and let the non risk asset evolution be deterministic one given by the law $\{B_m\}_{m=0}^N$, $B_m = 1$, $m = \overline{0, N}$. The fair price of Standard European Call Option with the payment function $f_N = (S_N - K)^+$ is given by the formula

$$f_0 = \begin{cases} S_0 \left(1 - \frac{K}{D_N^2} \right), & K \leq D_N^2, \\ 0, & K > D_N^2. \end{cases} \tag{182}$$

The fair price of Standard European Put Option with the payment function $f_N = (K - S_N)^+$ is given by the formula

$$f_0 = \begin{cases} K - D_N^1, & K \geq D_N^1, \\ 0, & K < D_N^1. \end{cases} \tag{183}$$

Proof. In Theorem 18 conditions, the set of equations $E^P \zeta = 1$, $\zeta \geq 0$ has the solutions $\zeta_i = \frac{S_i}{S_0}$, $i = \overline{0, N}$. It is evident that $\alpha_0 = S_0$ and $\zeta_N = \frac{S_N}{S_0}$, since

$$\frac{(S_N - K)^+}{B_N} - \alpha_0 \frac{S_N}{S_0} \leq 0, \quad \omega \in \Omega. \tag{184}$$

Let us prove the needed formula. Consider the inequality

$$(S_N - K) - \alpha \sum_{i=0}^N \gamma_i \frac{S_i}{S_0} \leq 0, \quad \gamma \in V_0, \tag{185}$$

where $V_0 = \left\{ \gamma = \{\gamma_i\}_{i=0}^N, \gamma_i \geq 0, \sum_{i=0}^N \gamma_i = 1 \right\}$. Or,

$$S_N \left(1 - \frac{\alpha \gamma_N}{S_0} \right) - K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{S_i}{S_0} \leq 0. \tag{186}$$

Suppose that α satisfies the inequality

$$1 - \frac{\alpha}{S_0} > 0. \tag{187}$$

If α satisfies additionally the equality

$$D_N^2 \left(1 - \frac{\alpha \gamma_N}{S_0} \right) - K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{D_i^1}{S_0} = 0, \tag{188}$$

then for all $\omega \in \Omega$ (186) is valid. From (188) we obtain for α

$$\alpha = \frac{S_0 (D_N^2 - K)}{\left(D_N^2 \gamma_N + \sum_{i=0}^{N-1} \gamma_i D_i^1 \right)}. \tag{189}$$

If $D_N^2 - K > 0$, then

$$\inf_{\gamma \in V_0} \frac{S_0(D_N^2 - K)}{\left(D_N^2 \gamma_N + \sum_{i=0}^{N-1} \gamma_i D_i^1\right)} = \frac{S_0(D_N^2 - K)}{D_N^2}, \tag{190}$$

since $D_N^2 \geq D_i^1$. From here we obtain

$$f_0 = S_0 \left(1 - \frac{K}{D_N^2}\right). \tag{191}$$

It is evident that $\alpha = f_0$ satisfies the inequality (187).

If $D_N^2 - K \leq 0$, then $S_N - K \leq 0$ and from (185) we can put $\alpha = 0$. Then, the formula (186) is valid for all $\omega \in \Omega$.

Let us prove the formula (183) for Standard European Put Option. If $S_N \leq K$ it is evident that $\alpha_0 = K$, and $\zeta_0 = 1$, since

$$(K - S_N) - \alpha_0 \leq 0, \omega \in \Omega. \tag{192}$$

Let us prove the needed formula. Consider the inequality

$$(K - S_N)^+ - \alpha \sum_{i=0}^N \gamma_i \frac{S_i}{S_0} \leq 0, \gamma \in V_0. \tag{193}$$

Or, for $S_N \leq K$

$$-S_N \left(1 + \frac{\alpha \gamma_N}{S_0}\right) + K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{S_i}{S_0} \leq 0. \tag{194}$$

If α is a solution of the equality

$$-D_N^1 \left(1 + \frac{\alpha \gamma_N}{S_0}\right) + K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{D_i^1}{S_0} = 0, \tag{195}$$

then for all $\omega \in \Omega$ (194) is valid. From (195) we obtain for α

$$\alpha = \frac{S_0(K - D_N^1)}{\sum_{i=0}^N \gamma_i D_i^1}. \tag{196}$$

Therefore,

$$\inf_{\gamma \in V_0} \frac{S_0(K - D_N^1)}{\sum_{i=0}^N \gamma_i D_i^1} = K - D_N^1, \tag{197}$$

since $D_i^1 \leq S_0, i = \overline{1, N}, D_0^1 = S_0$. From here we obtain

$$f_0 = K - D_N^1. \tag{198}$$

If $D_N^1 - K > 0$, then $S_N - K > 0$ and from (193) we can put $\alpha = 0$. Then, (194) is valid for all $\omega \in \Omega$. The Theorem 18 is proved. \square

8. Some Auxiliary Results

On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_n on it, let us consider a convex set of equivalent measures M . Suppose that ξ_1, \dots, ξ_d is a set of random

values belonging to the set A_0 . Introduce d martingales relative to a set of measures $M = \{S_n^i, \mathcal{F}_n\}_{n=0}^\infty, i = \overline{1, d}$, where $S_n^i = E^P \{\xi_i | \mathcal{F}_n\}, i = \overline{1, d}, P \in M$. Denote by $M^e(S)$ a set of all martingale measures equivalent to a measure $P \in M$, that is, $Q \in M^e(S)$ if

$$E^Q \{S_n | \mathcal{F}_{n-1}\} = S_{n-1}, E^Q |S_n| < \infty, Q \in M^e(S), n = \overline{1, \infty}. \tag{199}$$

It is evident that $M \subseteq M^e(S)$ and $M^e(S)$ is a convex set. Denote P_0 a certain fixed measure from $M^e(S)$ and let $L^0(R^d)$ be a set of finite valued random values on a probability space $\{\Omega, \mathcal{F}, P_0\}$, taking values in R^d .

Let H^0 be a set of finite valued predictable processes $H = \{H_n\}_{n=1}^N$, where $H_n = \{H_n^i\}_{i=1}^d$ takes values in R^d and H_n is \mathcal{F}_{n-1} -measurable random vector. Introduce into consideration a set of random values

$$K_N^1 = \left\{ \xi \in L^0(R^1), \xi = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle, H \in H^0 \right\}, N < \infty, \tag{200}$$

$$\Delta S_k = S_k - S_{k-1}, \langle H_k, \Delta S_k \rangle = \sum_{s=1}^d H_k^s (S_k^s - S_{k-1}^s). \tag{201}$$

Lemma 13. The set of random values K_N^1 is a closed subset in the set of finite valued random values $L^0(R^1)$ relative to the convergence by measure $P \in M$.

The proof of the Lemma 13 see, for example, [17].

Introduce into consideration a subset

$$V^0 = \left\{ H \in H^0, \|H_n\| < \infty, n = \overline{1, N} \right\} \tag{202}$$

of the set H^0 , where $\|H_n\| = \sup_{\omega \in \Omega} \sum_{i=1}^d |H_n^i|$. Let K_N be a subset of the set K_N^1

$$K_N = \left\{ \xi \in L^0(R^1), \xi = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle, H \in V^0 \right\}. \tag{203}$$

Denote also a set

$$C = \left\{ k - f, k \in K_N, f \in L_+^\infty(\Omega, \mathcal{F}, P_0) \right\}, \tag{204}$$

where $L_+^\infty(\Omega, \mathcal{F}, P_0)$ is a set of bounded nonnegative random values. Let \bar{C} be the closure of C in $L^1(\Omega, \mathcal{F}, P_0)$ metrics.

Lemma 14. If $\zeta \in \bar{C}$ and such that $E^{P_0} \zeta = 0$, then for ζ the representation

$$\zeta = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle \tag{205}$$

is valid for a certain finite valued predictable process $H = \{H_n\}_{n=1}^N$.

Proof. If $\zeta \in K_N$, then Lemma 14 is proved. Suppose that $\zeta \in \bar{C}$, then there exists a sequence $k_n - f_n, k_n \in K_N, f_n \in L_+^\infty(\Omega, \mathcal{F}, P_0)$ such that

$$\|k_n - f_n - \zeta\|_{R_0} \rightarrow 0, n \rightarrow \infty, \text{ where } \|g\|_{R_0} = E^{P_0} |g|. \text{ Since}$$

$$|E^{P_0}(k_n - f_n - \zeta)| \leq \|k_n - f_n - \zeta\|_{R_0}, \text{ we have } E^{P_0} f_n \leq \|k_n - f_n - \zeta\|_{R_0}. \text{ From here}$$

we obtain $\|k_n - \zeta\|_{P_0} \leq 2\|k_n - f_n - \zeta\|_{P_0}$. Therefore, $k_n \rightarrow \zeta$ by measure P_0 . On the basis of Lemma 13, a set

$$K_N^1 = \left\{ \xi \in L^0(R^1), \xi = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle, H \in H^0 \right\}, \tag{206}$$

$$\langle H_k, \Delta S_k \rangle = \sum_{i=1}^d H_k^i (S_k^i - S_{k-1}^i) \tag{207}$$

is a closed subset of $L^0(R^1)$ relative to the convergence by measure P_0 . From this fact, we obtain the proof of Lemma 14, since there exists the finite valued predictable process $H \in H^0$ such that for ζ the representation

$$\zeta = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle \tag{208}$$

is valid. □

Theorem 19 Let $E^Q|\zeta| < \infty, Q \in M^e(S)$. If for every $Q \in M^e(S), E^Q\zeta = 0$, then there exists finite valued predictable process H such that for ζ the representation

$$\zeta = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle \tag{209}$$

is valid.

Proof. If $\zeta \in \bar{C}$, then (209) follows from Lemma 14. So, let ζ does not belong to \bar{C} . As in Lemma 14, \bar{C} is a closure of C in $L^1(\Omega, \mathcal{F}, P_0)$ metrics for the fixed measure P_0 . The set \bar{C} is a closed convex set in $L^1(\Omega, \mathcal{F}, P_0)$. Consider the other convex closed set that consists from one element ζ . Due to Han-Banach Theorem, there exists a linear continuous functional l_1 , which belongs to $L^\infty(\Omega, \mathcal{F}, P_0)$, and real numbers $\alpha > \beta$ such that

$$l_1(\xi) = \int_{\Omega} \xi(\omega) q(\omega) dP_0, q(\omega) \in L^\infty(\Omega, \mathcal{F}, P_0), \tag{210}$$

and the inequalities $l_1(\zeta) > \alpha, l_1(\xi) \leq \beta, \xi \in \bar{C}$ are valid. Since \bar{C} is a convex cone we can put $\beta = 0$. From the condition $l_1(\xi) \leq 0, \xi \in \bar{C}$ we have $l_1(\xi) = 0, \xi \in K_N^1 \cap L^1(\Omega, \mathcal{F}, P_0)$. From (210) and the inclusions $\bar{C} \supset C \supset -L^\infty(\Omega, \mathcal{F}, P_0)$ we have $q(\omega) \geq 0$. Introduce a measure

$$Q^*(A) = \int_A q(\omega) dP_0 \left[\int_{\Omega} q(\omega) dP_0 \right]^{-1}. \tag{211}$$

Then, we have

$$\int_{\Omega} \xi(\omega) dQ^* = 0, \xi \in K_N^1 \cap L^1(\Omega, \mathcal{F}, P_0). \tag{212}$$

Let us choose $\xi = \chi_A(\omega)(S_i^j - S_{i-1}^j), A \in \mathcal{F}_{i-1}$, where $\chi_A(\omega)$ is an indicator of a set A . We obtain

$$\int_A (S_i^j - S_{i-1}^j) dQ^* = 0, A \in \mathcal{F}_{i-1}. \tag{213}$$

So, Q^* is a martingale measure that belongs to the set $M^a(S)$, which is a

set of absolutely continuous martingale measures. Let us choose $Q \in M^e(S)$ and consider a measure $Q_1 = (1-\gamma)Q + \gamma Q^*, 0 < \gamma < 1$. A measure $Q_1 \in M^e(S)$ and, moreover, $E^{Q_1} \zeta = \gamma E^{Q^*} \zeta > 0$. We come to the contradiction with the conditions of Theorem 19, since for $Q \in M^e(S), E^Q \zeta = 0$. So, $\zeta \in \bar{C}$, and in accordance with Lemma 14, for ζ the declared representation in Theorem 19 is valid. \square

Theorem 20. For every martingale $\{M_n, \mathcal{F}_n\}_{n=0}^\infty$ relative to the set of measures $M^e(S)$, there exists a predictable random process H such that for $M_n, n = \overline{0, \infty}$, the representation

$$M_n = M_0 + \sum_{i=1}^n \langle H_i, \Delta S_i \rangle, \quad n = \overline{1, \infty}, \tag{214}$$

is valid.

Proof. For fixed natural $N \geq 1$, let us consider the random value $M_N - M_0 = \zeta$. Since

$$E^Q |\zeta| < \infty, \quad E^Q \zeta = 0, \quad Q \in M^e(S), \tag{215}$$

then ζ satisfies the conditions of Theorem 19 and, therefore, belongs to \bar{C} , so,

there exists a sequence $k_n = \sum_{i=1}^N \langle H_i^n, \Delta S_i \rangle \in K_N$ such that

$$\int_{\Omega} |k_n - \zeta| dP_0 \rightarrow 0, \quad n \rightarrow \infty. \tag{216}$$

From here, we obtain

$$\int_{\Omega} |E^{P_0} \{(k_n - \zeta) | \mathcal{F}_m\}| dP_0 \leq \int_{\Omega} |k_n - \zeta| dP_0 \rightarrow 0, \quad n \rightarrow \infty. \tag{217}$$

But $E^{P_0} \{k_n | \mathcal{F}_m\} = \sum_{i=1}^m \langle H_i^n, \Delta S_i \rangle$. Hence, we obtain that both $\sum_{i=1}^m \langle H_i^n, \Delta S_i \rangle$ and $\sum_{i=1}^N \langle H_i^n, \Delta S_i \rangle$ converges by measure P_0 to $E^{P_0} \{\zeta | \mathcal{F}_m\}$ and ζ , correspondingly. There exists a subsequence n_k such that H^{n_k} converges everywhere to predictable process H . From here, we have $\zeta = \sum_{i=1}^N \langle H_i, \Delta S_i \rangle$ and

$E^{P_0} \{\zeta | \mathcal{F}_m\} = \sum_{i=1}^m \langle H_i, \Delta S_i \rangle$. It proves that for all $m < N$

$$M_m = M_0 + \sum_{i=1}^m \langle H_i, \Delta S_i \rangle. \tag{218}$$

Theorem 20 is proved. \square

9. Conclusions

In the paper, we generalize Doob decomposition for super-martingales relative to one measure onto the case of super-martingales relative to a convex set of equivalent measures. For super-martingales relative to one measure for continuous time Doob's result was generalized in papers [21] [22]. Section 2 contains the definition of local regular super-martingales. Theorem 1 gives the necessary

and sufficient conditions of the local regularity of super-martingale. In spite of its simplicity, the Theorem 1 appeared very useful for the description of the local regular super-martingales.

For this purpose we investigate the structure of super-martingales of special types relative to the convex set of equivalent measures, generated by a certain finite set of equivalent measures. The main result of Section 3 is Lemma 6, which allowed proving Lemma 8, giving the sufficient conditions of the existence of a martingale with respect to a convex set of equivalent measures generated by finite set of equivalent measures.

Theorem 2 describes all local regular non-negative super-martingales of the special type (30) relative to the convex set of equivalent measures, generated by the finite set of equivalent measures.

In the Theorem 3, we give the sufficient conditions of the existence of the local regular martingale relative to an arbitrary set of equivalent measures and arbitrary filtration. After that, we present in Theorem 4 the important construction of the local regular super-martingales which we sum up in Corollary 2. Theorem 6 proves that every majorized super-martingale belongs to the described class (53) of the local regular super-martingales.

Theorem 7 gives a variant of the necessary and sufficient conditions of local regularity of non-negative super-martingale relative to a convex set of equivalent measures. Definition 3 determines a class of the complete set of equivalent measures. Lemma 10 guarantees a bound (77) for all non-negative random values allowing us to prove Theorem 8, stating that for every super-martingale the optional decomposition is valid. We extend the results obtained from the finite space of elementary events onto the case as a space of elementary events is a countable one. At last, the subsection 5.3 contains the generalization of the result obtained in subsection 5.2 onto the case of arbitrary space of elementary events. In Section 6, we prove Theorems 13 and 14, stating that for every majorized super-martingale the optional decomposition is valid.

Corollary 3 contains the important construction of the local regular super-martingales playing the important role in the definition of the fair price of contingent claim relative to a convex set of equivalent measures. The Definition 6 is a fundamental one for the evaluation of risks in incomplete markets. Theorem 15 gives the sufficient conditions of the existence of the fair price of contingent claim relative to a convex set of equivalent measures. It also gives the sufficient conditions, when the defined fair price coincides with the classical value. In Theorem 16 the simple conditions of the existence of the fair price of contingent claim are given. In Theorem 17 we prove the existence of the self-financed trading strategy confirming the Definition 6 of the fair price as the parity between the long and short positions in contracts. As an application of the results obtained we prove Theorem 18, where the formulas for the Standard European Call and Put Options in an incomplete market we present. Section 8 contains auxiliary results needed for previous sections.

References

- [1] Kramkov, D.O. (1996) Optional Decomposition of Super-Martingales and Hedging in Incomplete Security Markets. *Probability Theory and Related Fields*, **105**, 459-479. <https://doi.org/10.1007/BF01191909>
- [2] Follmer, H. and Kramkov, D.O. (1997) Optional Decomposition Theorem under Constraints. *Probability Theory and Related Fields*, **109**, 1-25. <https://doi.org/10.1007/s004400050122>
- [3] Follmer, H. and Kabanov, Yu.M. (1996) Optional Decomposition Theorems in Discrete Time. *Atti del convegno in onore di Oliviero Lessi*, Padova, 25-26 marzo 1996, 47-68.
- [4] Follmer, H. and Kabanov, Yu.M. (1998) Optional Decomposition and Lagrange Multipliers. *Finance and Stochastics*, **2**, 69-81.
- [5] El Karoui, N. and Quenez, M.C. (1995) Dynamic Programming and Pricing of Contingent Claims in an Incomplete Market. *SIAM Journal on Control and Optimization*, **33**, 27-66. <https://doi.org/10.1137/S0363012992232579>
- [6] Bouchard, B. and Nutz, M. (2015) Arbitrage and Duality in Nondominated Discrete-Time Models. *The Annals of Applied Probability*, **25**, 823-859. <https://doi.org/10.1214/14-AAP1011>
- [7] Gonchar, N.S. (2008) Mathematical Foundations of Information Economics. Bogolyubov Institute for Theoretical Physics, Kiev.
- [8] Gonchar, N.S. (2015) Mathematical Model of Banking Operation. *Cybernetics and System Analysis*, **51**, 378-399. <https://doi.org/10.1007/s10559-015-9730-0>
- [9] Gonchar, N.S. and Terentieva, L.S. (2008) Default Risk Valuation of the Firm with the Special Process of Internal Yield. *Journal of Automation and Information Sciences*, **40**, 57-71. <https://doi.org/10.1615/JAutomatInfScien.v40.i8.60>
- [10] Gonchar, N.S. (2017) Banking and Risk Assessment, Chapter 8. In: *Banking: Services, Opportunity and Risks*, Nova Science Publisher, Inc., New York.
- [11] Gonchar, N.S. (2001) Stock Market and Economic Growth. Oberehy, Kiev. (In Ukrainian)
- [12] Kallianpur, G. (1980) Stochastic Filtering Theory. Springer, New York. <https://doi.org/10.1007/978-1-4757-6592-2>
- [13] Chow, Y.S., Robbins, H. and Siegmund, D. (1971) Great Expectations: The Theory of Optimal Stopping. Houghton Mifflin Company, Boston.
- [14] Kelley, J. (1955) General Topology. Van Nostrand, New York.
- [15] Delbaen, F. and Schachermayer, W. (2006) The Mathematics and Arbitrage. Springer, Berlin.
- [16] Delbaen, F. and Schachermayer, W. (1994) A General Version of Fundamental Theorem of Asset Pricing. *Mathematische Annalen*, **300**, 463-520. <https://doi.org/10.1007/BF01450498>
- [17] Dalang, R.C., Morton, A. and Willinger, W. (1990) Equivalent Martingale Measures and No-Arbitrage in Stochastic Securities Market Model. *Stochastics and Stochastic Reports*, **29**, 185-201. <https://doi.org/10.1080/17442509008833613>
- [18] Kreps, D.M. (1981) Arbitrage and Equilibrium in Economics with Infinitely Many Commodities. *Journal of Mathematical Economics*, **8**, 15-35. [https://doi.org/10.1016/0304-4068\(81\)90010-0](https://doi.org/10.1016/0304-4068(81)90010-0)
- [19] Harrison, J.M. and Kreps, D.M. (1979) Martingales and Arbitrage in Multiperiod Securities Markets. *Journal of Economic Theory*, **20**, 381-408.

[https://doi.org/10.1016/0022-0531\(79\)90043-7](https://doi.org/10.1016/0022-0531(79)90043-7)

- [20] Harrison, J.M. and Pliska, S.R. (1981) Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and their Applications*, **11**, 215-260. [https://doi.org/10.1016/0304-4149\(81\)90026-0](https://doi.org/10.1016/0304-4149(81)90026-0)
- [21] Meyer, P.A. (1963) A Decomposition Theorem for Super-Martingales. *Illinois Journal of Mathematics*, **7**, 1-17.
- [22] Meyer, P.A. (1972) A Decomposition for Super-Martingales: The Uniqueness Theorem. *Illinois Journal of Mathematics*, **6**, 193-205.