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Stability Analysis of a Deterministic Epidemic Model in Metapopulation Setting

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Abstract

We present in this article an epidemic model with saturated in metapopulation setting. We develop the mathematical modelling of HIV transmission among adults in Metapopulation setting. We discussed the positivity of the system. We calculated the reproduction number, If $R_{0j} \le 1$ for $j = 1,2,3,4$, then each infectious individual in Sub-Population j infects on average less than one other person and the disease is likely to die out. Otherwise, if $R_{0j} > 1$ for $j = 1,2,3,4$, then each infectious individual in Sub-Population j infects on average more than one other person; the infection could therefore establish itself in the population and become endemic. An epidemic model, where the presence or absence of an epidemic wave is characterized by the value of R_{0j} both ideas of the inner equilibrium point of stability properties are discussed.

Keywords

Basic Reproduction Ratio, Lyapunov Function, Meta-Population, Disease-Free and the Endemic Equilibrium

1. Introduction

Numerous mathematical models have been developed in order to understand disease transmissions and behavior of epidemics. One of the earliest of these models as discussed by Kermack [1], by considering the total population into three classes, namely, susceptible (S) individuals, infected (I) individuals, and recovered (R) individuals which is known to us as SIR epidemic model. This SIR or SI epidemic model is very significant in todays analysis of diseases. SIR Model: The SIR model labels these three compartments S = number susceptible, I = number infectious, and R = number recovered. This is a good and simple model for many infectious diseases. Birth→S→I→R→Death and The SI model labels

these two compartments S = number susceptible and I = number infectious. This is a good and simple model for many infectious diseases. Birth→S→I→Death.

In the mathematical epidemiology area an key concept is associated to the basic reproduction number (R_0). This is defined as the second expected number produced from just a one individual in a susceptible population. For any infectious disease, one of the most key concerns is its capacity to invade a population, as studied by various authors [2]. This can be expressed by a threshold parameter: if the disease free equilibrium is locally asymptotically stable, then the disease cannot invade the population and $R_0 < 1$, whereas if the number of infected individuals grows, the disease can invade the population and $R_0 > 1$, as studied by various authors [3].

2. Compartmental Model and Differential Equations

In this section, we approached this study by using SI deterministic model.

In our model system the recruitment into the susceptible human population is only by births (λ). The size of the human population is decreased by natural deaths (μ), infected and awareness/education. Uneducated and educated infected female youth move to the classes S_{YUF}, S_{YEF} respectively at the rate β_1 whereas uneducated and educated infected males youth move to the classes I_{YUM}, I_{YEM} respectively at the rate β_2 resulting in an increase in the youth infectious classes. The infectious classes are all decreased by natural deaths (μ) and disease induced deaths (δ, δ_1). S_{YEF} and I_{YEM} is decreased further as a result of the infected educated, II vertical transmission and tested youths going through the Antiretroviral therapy thus moving to the treatment class T_Y at the rate α. We assume that once a person becomes infected with HIV they do not fully recover as there is no immunity to HIV and that only the educated and tested persons qualify the antiretroviral therapy.

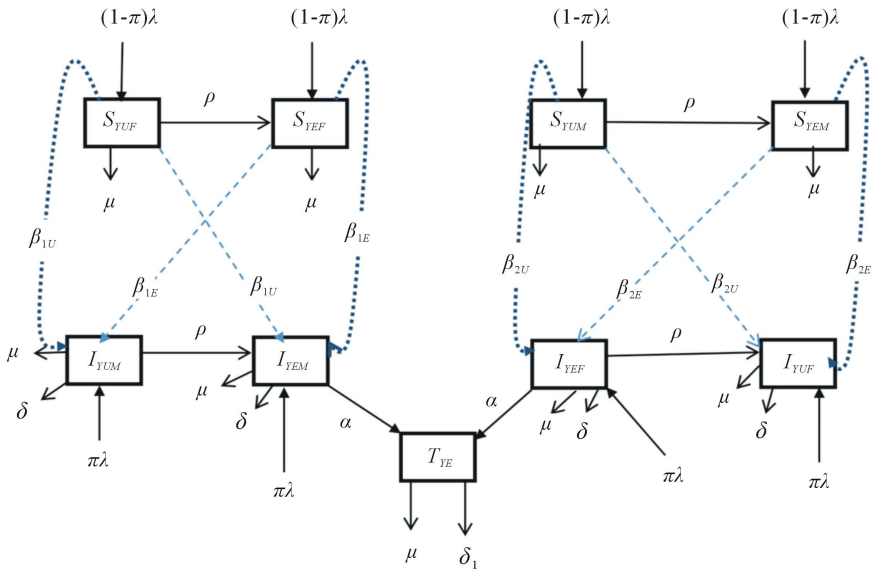

Figure 1. Proposed schematics of the compartmental model.

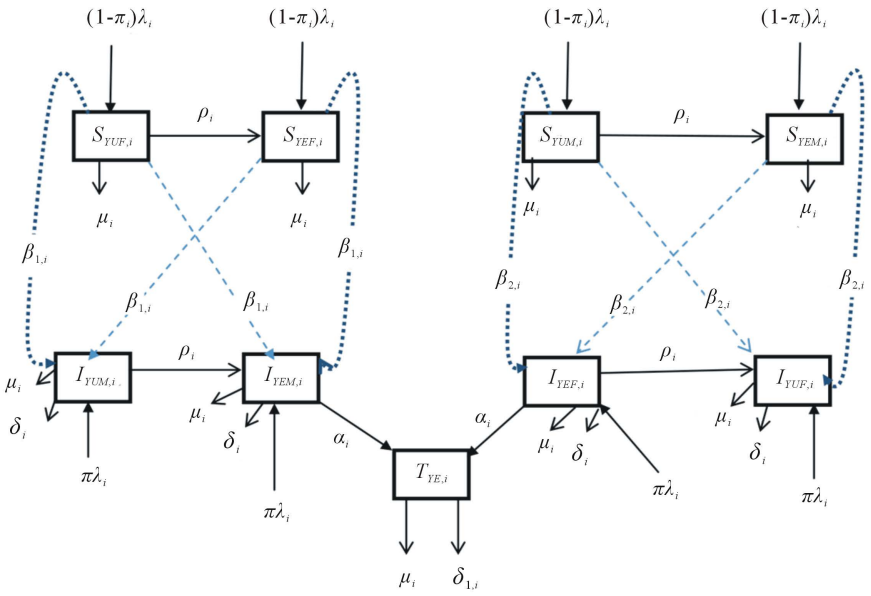

Figure 2. Schematics of the metapopulation model.

Differential Equation of the model

$$
\begin{cases}
\dfrac{dS_{YUF,i}}{dt} = (1-\Pi)\lambda - \mu_i\beta_{1,i}^{U}\dfrac{I_{YUM,i}}{N_{YUM,i}}S_{YUF,i} - \beta_{1,i}^{E}\dfrac{I_{YEM,i}}{N_{YEM,i}}S_{YUF,i} - (\rho+\mu)S_{YUF,i} \\
\dfrac{dS_{YEF,i}}{dt} = (1-\Pi)\lambda + \rho S_{YUF,i} - \beta_{1,i}^{U}\dfrac{I_{YUM,i}}{N_{YUM,i}}S_{YEF,i} - \beta_{1,i}^{E}\dfrac{I_{YEM,i}}{N_{YEM,i}}S_{YEF,i} - \mu S_{YEF,i} \\
\dfrac{dS_{YUM,i}}{dt} = (1-\Pi)\lambda - \beta_{2,i}^{U}\dfrac{I_{YUF,i}}{N_{YUF,i}}S_{YUM,i} - \beta_{2,i}^{E}\dfrac{I_{YEF,i}}{N_{YEF,i}}S_{YUM,i} - (\rho+\mu)S_{YUM,i} \\
\dfrac{dS_{YEM,i}}{dt} = (1-\Pi)\lambda + \rho S_{YUM,i} - \beta_{2,i}^{U}\dfrac{I_{YUF,i}}{N_{YUF,i}}S_{YEM,i} - \beta_{2,i}^{E}\dfrac{I_{YEF,i}}{N_{YEF,i}}S_{YEM,i} - \mu S_{YEM,i} \\
dI_{YUM} = \Pi\lambda + \beta_{1,i}^{U}\dfrac{I_{YUM,i}}{N_{YUM,i}}S_{YUF,i} + \beta_{1,i}^{U}\dfrac{I_{YUM,i}}{N_{YUM,i}}S_{YEF,i} - (\rho+\mu+\delta)I_{YUM,i} \\
\dfrac{dI_{YEM}}{dt} = \Pi\lambda + \rho_i I_{YUM,i} + \beta_{1,i}^{E}\dfrac{I_{YEM,i}}{N_{YEM,i}}S_{YUF,i} + \beta_{1,i}^{E}\dfrac{I_{YEM,i}}{N_{YEM,i}}S_{YEF,i} - (\alpha+\mu+\delta)I_{YEM,i} \\
\dfrac{dI_{YEF}}{dt} = \Pi\lambda + \rho_i I_{YU,i}^{F} + \beta_{2,i}^{E}\dfrac{I_{YEF,i}}{N_{YEF,i}}S_{YUM,i} + \beta_{2,i}^{E}\dfrac{I_{YEF,i}}{N_{YEF,i}}S_{YEM,i} - (\alpha+\mu+\delta)I_{YEF,i} \\
\dfrac{dI_{YUF}}{dt} = \Pi\lambda + \beta_{2,i}^{U}\dfrac{I_{YUF,i}}{N_{YUF,i}}S_{YUM,i} + \beta_{2,i}^{U}\dfrac{I_{YUF,i}}{N_{YUF,i}}S_{YEM,i} - (\rho+\mu+\delta)I_{YUF,i} \\
\dfrac{dT_{Y,i}}{dt} = \alpha I_{YEM} + \alpha I_{YEF} - (\mu+\delta_{1,i})T_{Y,i}
\end{cases}
$$

From the proposed schematics of the compartment model shown (see Figure 1), we extracted a metapopulation model for HIV dynamics among the youth coupled with awareness/education *i.e.*, we extended the single patch disease model to include multiple patches. A schematic of the Metapopulation Model (see Figure 2) for HIV transmission in the youths coupled with awareness/education in each patch i, $i = 1, 2, \cdots, n$.

3. Positivity and Boundedness

The theory of ordinary differential equations requires that, for every set of initial conditions

$$\left(S_{YUF,i_0}, S_{YEF,i_0}, S_{YUM,i_0}, S_{YEM,i_0}, I_{YUM,i_0}, I_{YEM,i_0}, I_{YEF,i_0}, I_{YUF,i_0}, T_{i_0}\right)$$

the state variables

$$\left(S_{YUF,i}(t), S_{YEF,i}(t), S_{YUM,i}(t), S_{YEM,i}(t), I_{YUM,i}(t), I_{YEM,i}(t), I_{YEF,i}(t), I_{YUF,i}(t), T_i(t)\right)$$

of the solution must remain non-negative.

Proposition 3.1. Let

$$\left(S_{YUF,i}(t), S_{YEF,i}(t), S_{YUM,i}(t), S_{YEM,i}(t), I_{YUM,i}(t), I_{YEM,i}(t), I_{YEF,i}(t), I_{YUF,i}(t), T_i(t)\right)$$

be the solution of the system (2.0).

1) Given the initial condition

$$\left(S_{YUF,i_0}, S_{YEF,i_0}, S_{YUM,i_0}, S_{YEM,i_0}, I_{YUM,i_0}, I_{YEM,i_0}, I_{YEF,i_0}, I_{YUF,i_0}, T_{i_0}\right) \in \Omega$$

then there exist a unique positive solution

$$\left(\left(S_{YUF,i}(t), S_{YEF,i}(t), S_{YUM,i}(t), S_{YEM,i}(t), I_{YUM,i}(t), I_{YEM,i}(t), I_{YEF,i}(t), I_{YUF,i}(t), T_i(t)\right)\right)$$

for every $t \geq 0$ such that the solution will remain in Ω with probability of one.

2) The solution

$$\left(\left(S_{YUF,i}, S_{YEF,i}, S_{YUM,i}, S_{YEM,i}, I_{YUM,i}, I_{YEM,i}, I_{YEF,i}, I_{YUF,i}, T_i\right)\right)$$

is defined in the interval $[0,\infty)$ and $\lim_{t\to\infty} \sup N(t) \leq \frac{4\lambda}{\mu}$ where

$$N(t) = S_{YUF}(t) + S_{YEF}(t) + S_{YUM}(t) + S_{YEM}(t) + I_{YUM}(t) + I_{YEM}(t) + I_{YEF}(t) + I_{YUF}(t) + T(t)$$

Proof: In (1) we let

$$\left(S_{YUF,i_0}, S_{YEF,i_0}, S_{YUM,i_0}, S_{YEM,i_0}, I_{YUM,i_0}, I_{YEM,i_0}, I_{YEF,i_0}, I_{YUF,i_0}, T_{i_0}\right) \in \Omega$$

Evidently, the coefficients of system (2.0) are locally Lipschitz continuous. Hence, for any given initial condition

$$\left(S_{YUF,i_0}, S_{YEF,i_0}, S_{YUM,i_0}, S_{YEM,i_0}, I_{YUM,i_0}, I_{YEM,i_0}, I_{YEF,i_0}, I_{YUF,i_0}, T_{i_0}\right) \in \Omega$$

there exist a unique local solution

$$\left(\left(S_{YUF,i}(t), S_{YEF,i}(t), S_{YUM,i}(t), S_{YEM,i}(t), I_{YUM,i}(t), I_{YEM,i}(t), I_{YEF,i}(t), I_{YUF,i}(t), T_i(t)\right)\right)$$

for every $t \in [0,T)$, where T is the final time. Here, it can be deduced that

$$S_{YUF,i}(t) + S_{YEF,i}(t) + S_{YUM,i}(t) + S_{YEM,i}(t) + I_{YUM,i}(t) + I_{YEM,i}(t) + I_{YEF,i}(t) + I_{YUF,i}(t) + T_i(t) \leq \frac{4\lambda}{\mu}$$

for every $t \in [0,T)$. Summing the total population of system (2.0) gives

$\mathrm{d}N(t) \le (4\lambda - \mu N)\mathrm{d}t$. Suppose $x(t)$ is the solution of the differential equation $\mathrm{d}N(t) = (4\lambda - \mathrm{d}x(t))\mathrm{d}t$, $x(0) = N(0)$ where

$$N_i(0) = S_{YUF,i}(0) + S_{YEF,i}(0) + S_{YUM,i}(0) + S_{YEM,i}(0) + I_{YUM,i}(0) + I_{YEM,i}(0) + I_{YEF,i}(0) + I_{YUF,i}(0) + T_i(0)$$

Hence, by comparison theorem; $N(t) \le x(t) \le \frac{4\lambda}{\mu}$ for $t \in [0,T)$ as required. Again, we can verify in (2) that

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} \le 4\lambda_i - \mu_i S_{YUF,i} - \mu S_{YEF,i} - \mu_i S_{YUM,i} - \mu_i S_{YME,i} - \mu_i I_{YUM,i} - \mu_i I_{YEM,i} - \mu_i I_{YEF,i} - \mu_i I_{YUF,i} - \mu_i T_i - \delta_{1,i} T_i$$

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = \frac{\mathrm{d}\left(\sum_{i=1}^{n} N_i\right)}{\mathrm{d}t} = \sum_{i=1}^{n}\left(4\lambda_i - \mu_i N_i - \delta_{1,i} T_i\right) \le 4\lambda_i - \mu_i N_i$$

Integrating inequality (3.0) gives $N(t) \le \frac{4\lambda}{\mu}\left(1 - \mathrm{e}^{-\mu t}\right)$ for every $t \in [0,T]$ which implies $N(t) \le \frac{8\lambda}{\mu}$. It can therefore be verified that the solution $\left(\left(S_{YUF,i}, S_{YEF,i}, S_{YUM,i}, S_{YEM,i}, I_{YUM,i}, I_{YEM,i}, I_{YEF,i}, I_{YUF,i}, T_i\right)\right)$ is bounded within the interval $t \in [0,T]$. This implies $N(t) \le \frac{4\lambda}{\mu}\left(1 - \mathrm{e}^{-\mu t}\right)$ for every $t \in [0,\infty)$. Hence $\lim_{t\to\infty} \sup N(t) \le \frac{4\lambda}{\mu}$. Hence, employing the same intuition used in proving proposition 3.1, we see that system (2.0) with non-negative initial conditions $S_{YUF,i_0} \ge 0$, $S_{YEF,i_0} \ge 0$, $S_{YUM,i_0} \ge 0$, $S_{YEM,i_0} \ge 0$, $I_{YUM,i_0} \ge 0$, $I_{YEM,i_0} \ge 0$, $I_{YEF,i_0} \ge 0$, $I_{YUF,i_0} \ge 0$, $T_{i_0} \ge 0$ has a non-negative solution defined in R and the set

$$\Omega = \left\{ \left(S_{YUF,i}, S_{YEF,i}, S_{YUM,i}, S_{YEM,i}, I_{YUM,i}, I_{YEM,i}, I_{YEF,i}, I_{YUF,i}, T_i\right) / S_{c,i} > 0, \right.$$
$$S_{YUF,i} > 0, S_{YEF,i} > 0, S_{YUM,i} > 0, S_{YEM,i} > 0, I_{YUM,i} > 0, I_{YEM,i} > 0,$$
$$I_{YEF,i} > 0, I_{YUF,i} > 0, T_i > 0 \text{ and } S_{YUF} + S_{YEF} + S_{YUM} + S_{YEM}$$
$$\left. + I_{YUM} + I_{YEM} + I_{YEF} + I_{YUF} + T = \frac{4\lambda}{\mu} \right\}$$

is invariant by system (2.0).

4. Calculation of the Basic Reproduction Number

The basic reproduction number (R_0) is defined as an infections originating from an infected individual that invades a population originally of susceptible individuals. R_0 is used to predict whether the epidemic will spread or die out. In the next part, we will analyze the dynamics of $I_{YUM,i}$, $I_{YEM,i}$, $I_{YEF,i}$ and $I_{YUF,i}$ so as to be able to obtain R_0. Here, the functions (F) and (V) denote the matrix of the infection rates and the matrix of the transition rates respectively. Let us thus look at the following system of differential equations (The reduce

model from 2.0).

$$\begin{cases} \dfrac{\mathrm{d}I_{YUM,i}}{\mathrm{d}t} = \Pi\lambda + \beta_{1,i}^{U}\dfrac{I_{YUM,i}}{N_{YUM,i}}\left(N_{YUF,i} - I_{YUF,i}\right) \\ \qquad + \beta_{1,i}^{U}\dfrac{I_{YUM,i}}{N_{YUM,i}}\left(I_{YEF,i} - I_{YEF,i}\right) - \left(\rho_i + \mu_i + \delta_i\right) I_{YUM,i} \\ \dfrac{\mathrm{d}I_{YEM,i}}{\mathrm{d}t} = \Pi\lambda + \rho_i I_{YUM,i} + \beta_{1,i}^{E}\dfrac{I_{YEM,i}}{N_{YEM,i}}\left(N_{YUF,i} - I_{YUF,i}\right) \\ \qquad + \beta_{1,i}^{E}\dfrac{I_{YEM,i}}{N_{YEM,i}}\left(N_{YEF,i} - I_{YEF,i}\right) - \left(\alpha_i + \mu_i + \delta_i\right) I_{YEM,i} \\ \dfrac{\mathrm{d}I_{YEF,i}}{\mathrm{d}t} = \Pi\lambda + \rho_i I_{YU,i}^{F} + \beta_{2,i}^{E}\dfrac{I_{YEF,i}}{N_{YEF,i}}\left(N_{YUM,i} - I_{YUM,i}\right) \\ \qquad + \beta_{2,i}^{E}\dfrac{I_{YEF,i}}{N_{YEF,i}}\left(N_{YEM,i} - I_{YEM,i}\right) - \left(\alpha_i + \mu_i + \delta_i\right) I_{YEF,i} \\ \dfrac{\mathrm{d}I_{YUF,i}}{\mathrm{d}t} = \Pi\lambda + \beta_{2,i}^{U}\dfrac{I_{YUF,i}}{N_{YUF,i}}\left(N_{YUM,i} - I_{YUM,i}\right) \\ \qquad + \beta_{2,i}^{U}\dfrac{I_{YUF,i}}{N_{YUF,i}}\left(N_{YEM,i} - I_{YEM,i}\right) - \left(\rho_i + \mu_i + \delta_i\right) I_{YUF,i} \\ \dfrac{\mathrm{d}T_{Y,i}}{\mathrm{d}t} = \alpha_i I_{YEM,i} + \alpha_i I_{YEF,i} - \left(\mu_i + \delta_{1,i}\right) T_{Y,i} \end{cases}$$

The above system can be represented in matrix form as $I^{\cdot} = fI + vI$ where f is the matrix of the infection rates and v is the matrix of the transition rates.

The spectral radius of the Metzler Matrix, $\rho\left(-FV^{-1}\right)$, is defined as the largest eigenvalue of the Metzler Matrix [4]. Thus:

$$\rho\left(-FV^{-1}\right) = \left|\left(-FV^{-1}\right) - \lambda I\right| =$$

$$R_1 = \frac{\beta_{2,i}^{U}\left(N_{YEM,i} + N_{YUM,i}\right)}{\left(\rho_i + \mu_i + \delta_i\right) N_{YUF,i}}$$

$$R_2 = \frac{\beta_{1,i}^{E}\left(N_{YEF,i} + N_{YUF,i}\right)}{\left(\alpha_i + \mu_i + \delta_i\right) N_{YEM,i}}$$

$$R_3 = \frac{\beta_{2,i}^{E}\left(N_{YEM,i} + N_{YUM,i}\right)}{\left(\alpha_i + \mu_i + \delta_i\right) N_{YEF,i}}$$

$$R_4 = \frac{\beta_{1,i}^{U}\left(N_{YEF,i} + N_{YUF,i}\right)}{\left(\rho_i + \mu_i + \delta_i\right) N_{YUM,i}}$$

If $R_{0j} \le 1$ for $j = 1,2,3,4$, then each infectious individual in Sub-Population j infects on average less than one other person and the disease is likely to die out Otherwise, If $R_{0j} > 1$ for $j = 1,2,3,4$, then each infectious individual in Sub-Population j infects on average more than one other person; the infection could therefore establish itself in the population and become endemic. An SIR epidemic model, where the presence or absence of an epidemic wave is characterized by the value of R_{0j}.

5. Stability of the Disease Free Equilibrium Stability

Consider the differential equation $\dot{x} = f(t;x), x \in R^n$ then a point x is Liaponouv stable if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x-y| < \delta$ then if $|f(x,t) - f(y,t)| < \epsilon$ for all $t \geq 0$. A point x is quasi-asymptotically stable if there exists $\delta > 0$ such that if $|x-y| < \delta$ then if $|\varphi(x,t) - \varphi(y,t)| \to 0$ as $t \to \infty$. A point x is asymptotically stable if it is both liaponouv stable and quasi-asymptotically stable [5].

Local Asymptotic Stability

A point x^* is an equilibrium point of the system if $f(x^*) = 0$. x^* is locally stable if all solutions which start near x^* (meaning that the initial conditions are in a neighborhood of x^*) remain near x^* for all time. The equilibrium point x^* is said to be locally asymptotically stable if x^* is locally stable and, furthermore, all solutions starting near x^* tend towards x^* as $t \to \infty$ [5].

Global Asymptotic Stability

The system $x^* = f(t;x)$ is globally asymptotically stable if for every trajectory $x(t)$, we have $x(t) \to x^*$ as $t \to \infty$ (implies x^* is the unique equilibrium point) [5].

Liapunov stability

An important technique in stability theory for differential equations is one known as the direct method of Liapunov. A Liapunov function is constructed to prove stability or asymptotic stability of an equilibrium in a given region.

Definition 5.1. A positive-definite function V in an open neighborhood of the origin is said to be a Liapunov function for the autonomous differential system, $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, if $\dot{V}(x,y) \leq 0$ for all $(x,y) \in U(0,0)$. If $\dot{V}(x,y) < 0$ for all $(x,y) \in U(0,0)$, the function V is called a *strict Liapunov* function.

Theorem 5.1. (Liapunovs Stability Theorem [6].) Let $(0,0)$ be an equilibrium of the autonomous system $\dot{x} = f(x,y)$ and let V be a positive definite C^1 function in a neighborhood U of the origin.

1) If $\dot{V}(x,y) \leq 0$ for all $(x,y) \in U(0,0)$, then $(0,0)$ is stable.

2) If $\dot{V}(x,y) < 0$ for all $(x,y) \in U(0,0)$, then $(0,0)$ is asymptotically stable.

3) If $\dot{V}(x,y) < 0$ for some $(x,y) \in U(0,0)$, then $(0,0)$ is unstable.

We note that in case 1 the function V is a Liapunov function and in case (2) V is a strict Liapunov function.

Here, we investigate the local stability of the disease free equilibrium point E_0, by employing the method described in [7] [8] to linearize the model system (2.0) by computing its Jacobian matrix. The Jacobian matrix is computed at disease free equilibrium point by differentiating each equation in the system with respect to the state variables $S_{YUF}, S_{YEF}, S_{YUM}, S_{YEM}, I_{YUM}, I_{YEM}, I_{YEF}, I_{YUF}$ and $T(ART)$.

The Jacobian corresponding to 3.0 is given by

Let

$$A=-\beta_{1,i}^{U}\frac{I_{YU,i}^{M}}{N_{YU,i}^{M}}-\beta_{1,i}^{E}\frac{I_{YE,i}^{M}}{N_{YE,i}^{M}}-(\rho+\mu),$$

$$B=-\beta_{1,i}^{U}\frac{I_{YU,i}^{M}}{N_{YU,i}^{M}}-\beta_{1,i}^{E}\frac{I_{YE,i}^{M}}{N_{YE,i}^{M}}-\mu,$$

$$C=-\beta_{2,i}^{U}\frac{I_{YU,i}^{F}}{N_{YU,i}^{F}}-\beta_{2,i}^{E}\frac{I_{YE,i}^{F}}{N_{YE,i}^{F}}-(\rho+\mu),$$

$$D=-\beta_{2,i}^{U}\frac{I_{YU,i}^{F}}{N_{YU,i}^{F}}-\beta_{2,i}^{E}\frac{I_{YE,i}^{F}}{N_{YE,i}^{F}}-\mu,$$

$$E=\beta_{1,i}^{U}\frac{S_{YU,i}^{F}}{N_{YU,i}^{M}}+\beta_{1,i}^{U}\frac{S_{YE,i}^{F}}{N_{YU,i}^{M}}-(\rho+\mu+\delta),$$

$$F=\beta_{1,i}^{E}\frac{S_{YU,i}^{F}}{N_{YE,i}^{M}}+\beta_{1,i}^{E}\frac{S_{YE,i}^{F}}{N_{YE,i}^{M}}-(\alpha+\mu+\delta),$$

$$G=\beta_{2,i}^{E}\frac{S_{YU,i}^{M}}{N_{YE,i}^{F}}+\beta_{2,i}^{E}\frac{S_{YE,i}^{M}}{N_{YE,i}^{F}}-(\alpha+\mu+\delta),$$

$$H=\beta_{2,i}^{U}\frac{S_{YU,i}^{M}}{N_{YU,i}^{F}}+\beta_{2,i}^{U}\frac{S_{YE,i}^{M}}{N_{YU,i}^{F}}-(\rho+\mu+\delta),$$

$$I=-(\mu+\delta_{1,i}),$$

$$K=-\beta_{2,i}^{U}\frac{S_{YU,i}^{M}}{N_{YU,i}^{F}},$$

$$L=-\beta_{2,i}^{U}\frac{S_{YE,i}^{M}}{N_{YU,i}^{F}}$$

$$J=\begin{pmatrix}
A & 0 & 0 & 0 & -\beta_{1,i}^{U}\frac{S_{YU,i}^{F}}{N_{YU,i}^{M}} & -\beta_{1,i}^{E}\frac{S_{YU,i}^{F}}{N_{YE,i}^{M}} & 0 & 0 & 0\\
\rho & B & 0 & 0 & -\beta_{1,i}^{U}\frac{S_{YE,i}^{F}}{N_{YU,i}^{M}} & -\beta_{1,i}^{E}\frac{S_{YE,i}^{F}}{N_{YE,i}^{M}} & 0 & 0 & 0\\
0 & 0 & C & 0 & 0 & 0 & -\beta_{2,i}^{E}\frac{S_{YU,i}^{M}}{N_{YE,i}^{F}} & K & 0\\
0 & 0 & \rho & D & 0 & 0 & -\beta_{2,i}^{E}\frac{S_{YE,i}^{M}}{N_{YE,i}^{F}} & L & 0\\
\beta_{1,i}^{U}\frac{I_{YU,i}^{M}}{N_{YU,i}^{M}} & \beta_{1,i}^{U}\frac{I_{YU,i}^{M}}{N_{YU,i}^{M}} & 0 & 0 & E & 0 & 0 & 0 & 0\\
\beta_{1,i}^{E}\frac{I_{YE,i}^{M}}{N_{YE,i}^{M}} & \beta_{1,i}^{E}\frac{I_{YE,i}^{M}}{N_{YE,i}^{M}} & 0 & 0 & \rho & F & 0 & 0 & 0\\
0 & 0 & \beta_{2,i}^{E}\frac{I_{YE,i}^{F}}{N_{YE,i}^{F}} & \beta_{2,i}^{E}\frac{I_{YE,i}^{F}}{N_{YE,i}^{F}} & 0 & 0 & G & \rho & 0\\
0 & 0 & \beta_{2,i}^{U}\frac{I_{YU,i}^{F}}{N_{YU,i}^{F}} & \beta_{2,i}^{U}\frac{I_{YU,i}^{F}}{N_{YU,i}^{F}} & 0 & 0 & 0 & H & 0\\
0 & 0 & 0 & 0 & 0 & \alpha & \alpha & 0 & I
\end{pmatrix}$$

Therefore the Jacobian J_0 at the disease free equilibrium

$$E_0 = (S_{YUF0}, S_{YEF0}, S_{YUM0}, S_{YEM0}, I_{YUM0}, I_{YEM0}, I_{YEF0}, I_{YUF0}, T_{E0})$$

when

$$E_0 = \left(\frac{\lambda}{\rho+\mu}, \frac{\lambda+\rho N_{YUF}}{\mu}, \frac{\lambda}{\rho+\mu}, \frac{\lambda+\rho N_{YUM}}{\mu}, 0,0,0,0,0 \right)$$

Let

$$S = \beta_{1,i}^{U} \frac{N_{YU,i}^{F}}{N_{YU,i}^{M}} + \beta_{1,i}^{U} \frac{N_{YE,i}^{F}}{N_{YU,i}^{M}} - (\rho+\mu+\delta),$$

$$P = \beta_{1,i}^{E} \frac{N_{YU,i}^{F}}{N_{YE,i}^{M}} + \beta_{1,i}^{E} \frac{N_{YE,i}^{F}}{N_{YE,i}^{M}} - (\alpha+\mu+\delta),$$

$$R = \beta_{2,i}^{E} \frac{N_{YU,i}^{M}}{N_{YE,i}^{F}} + \beta_{2,i}^{E} \frac{N_{YE,i}^{M}}{N_{YE,i}^{F}} - (\alpha+\mu+\delta),$$

$$Q = \beta_{2,i}^{U} \frac{N_{YU,i}^{M}}{N_{YU,i}^{F}} + \beta_{2,i}^{U} \frac{N_{YE,i}^{M}}{N_{YU,i}^{F}} - (\rho+\mu+\delta),$$

$$U = -(\mu+\delta_{1,i})$$

$$J_0 = \begin{pmatrix}
-(\rho+\mu) & 0 & 0 & 0 & -\beta_{1,i}^{U}\frac{N_{YU,i}^{F}}{N_{YU,i}^{M}} & -\beta_{1,i}^{E}\frac{N_{YU,i}^{F}}{N_{YE,i}^{M}} & 0 & 0 & 0 \\
\rho & -\mu & 0 & 0 & -\beta_{1,i}^{U}\frac{N_{YE,i}^{F}}{N_{YU,i}^{M}} & -\beta_{1,i}^{E}\frac{N_{YE,i}^{F}}{N_{YE,i}^{M}} & 0 & 0 & 0 \\
0 & 0 & -(\rho+\mu) & 0 & 0 & 0 & -\beta_{2,i}^{E}\frac{N_{YU,i}^{M}}{N_{YE,i}^{F}} & -\beta_{2,i}^{U}\frac{N_{YU,i}^{M}}{N_{YU,i}^{F}} & 0 \\
0 & 0 & \rho & -\mu & 0 & 0 & -\beta_{2,i}^{E}\frac{N_{YE,i}^{M}}{N_{YE,i}^{F}} & -\beta_{2,i}^{U}\frac{N_{YE,i}^{M}}{N_{YU,i}^{F}} & 0 \\
0 & 0 & 0 & 0 & S & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho & P & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & R & \rho & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha & \alpha & 0 & U
\end{pmatrix}$$

The characteristics equation corresponding to the above matrix

$$\begin{aligned}
&(-(\rho+\mu)-\lambda_1)(-\mu-\lambda_2)(-(\rho+\mu)-\lambda_3)(-\mu-\lambda_4) \\
&\times\left(\left(\beta_{1,i}^{U}\frac{S_{YU,i}^{F}}{N_{YU,i}^{M}}+\beta_{1,i}^{U}\frac{S_{YE,i}^{F}}{N_{YU,i}^{M}}-(\rho+\mu+\delta)\right)-\lambda_5\right) \\
&\times\left(\left(\beta_{1,i}^{E}\frac{S_{YU,i}^{F}}{N_{YE,i}^{M}}+\beta_{1,i}^{E}\frac{S_{YE,i}^{F}}{N_{YE,i}^{M}}-(\alpha+\mu+\delta)\right)-\lambda_6\right) \\
&\times\left(\beta_{2,i}^{E}\frac{S_{YU,i}^{M}}{N_{YE,i}^{F}}+\beta_{2,i}^{E}\frac{S_{YE,i}^{M}}{N_{YE,i}^{F}}-(\alpha+\mu+\delta)-\lambda_7\right) \\
&\times\left(\beta_{2,i}^{U}\frac{S_{YU,i}^{M}}{N_{YU,i}^{F}}+\beta_{2,i}^{U}\frac{S_{YE,i}^{M}}{N_{YU,i}^{F}}-(\rho+\mu+\delta)-\lambda_8\right)\left(-(\mu+\delta_{1,i})-\lambda_9\right)=0
\end{aligned}$$

For E_0 to be asymptotically stable, all eigenvalues $i<0$, (i = 1, 2, 3, 4, 5, 6, 7, 8, 9) of J_0 must be negative. From (5.0.), it is clear that $\lambda_1=-(\rho+\mu)$, $\lambda_2=-\mu$, $\lambda_3=-(\rho+\mu)$, $\lambda_4=-\mu$ and $\lambda_9=-\left(\mu+\delta_{1,i}\right)$ is negative and therefore if

$$\lambda_5=\beta_{1,i}^{U}\frac{S_{YU,i}^{F}}{N_{YU,i}^{M}}+\beta_{1,i}^{U}\frac{S_{YE,i}^{F}}{N_{YU,i}^{M}}-(\rho+\mu+\delta)<0,$$

$$\lambda_6=\beta_{1,i}^{E}\frac{S_{YU,i}^{F}}{N_{YE,i}^{M}}+\beta_{1,i}^{E}\frac{S_{YE,i}^{F}}{N_{YE,i}^{M}}-(\alpha+\mu+\delta)<0,$$

$$\lambda_7=\beta_{2,i}^{E}\frac{S_{YU,i}^{M}}{N_{YE,i}^{F}}+\beta_{2,i}^{E}\frac{S_{YE,i}^{M}}{N_{YE,i}^{F}}-(\alpha+\mu+\delta)<0$$

and

$$\lambda_8=\beta_{2,i}^{U}\frac{S_{YU,i}^{M}}{N_{YU,i}^{F}}+\beta_{2,i}^{U}\frac{S_{YE,i}^{M}}{N_{YU,i}^{F}}-(\rho+\mu+\delta)<0$$

then both eigenvalues are negative. The condition $\lambda_5<0$, $\lambda_6<0$, $\lambda_7<0$ and $\lambda_8<0$ implies that

$$\beta_{1,i}^{U}\frac{S_{YU,i}^{F}}{N_{YU,i}^{M}}+\beta_{1,i}^{U}\frac{S_{YE,i}^{F}}{N_{YU,i}^{M}}<\rho+\mu+\delta,$$

$$\beta_{1,i}^{E}\frac{S_{YU,i}^{F}}{N_{YE,i}^{M}}+\beta_{1,i}^{E}\frac{S_{YE,i}^{F}}{N_{YE,i}^{M}}<\alpha+\mu+\delta,$$

$$\beta_{2,i}^{E}\frac{S_{YU,i}^{M}}{N_{YE,i}^{F}}+\beta_{2,i}^{E}\frac{S_{YE,i}^{M}}{N_{YE,i}^{F}}<\alpha+\mu+\delta,$$

$$\beta_{2,i}^{U}\frac{S_{YU,i}^{M}}{N_{YU,i}^{F}}+\beta_{2,i}^{U}\frac{S_{YE,i}^{M}}{N_{YU,i}^{F}}<\rho+\mu+\delta$$

respectively. Hence the disease-free equilibrium is locally asymptotically stable if the basic reproduction number,

$$R_{01}=\frac{\beta_{1,i}^{U}\left(S_{YU,i}^{F}+S_{YE,i}^{F}\right)}{N_{YU,i}^{M}(\rho+\mu+\delta)}<1,$$

$$R_{02}=\frac{\beta_{1,i}^{E}\left(S_{YU,i}^{F}+S_{YE,i}^{F}\right)}{N_{YE,i}^{M}(\alpha+\mu+\delta)}<1,$$

$$R_{03}=\frac{\beta_{2,i}^{E}\left(S_{YU,i}^{M}+S_{YE,i}^{M}\right)}{N_{YE,i}^{F}(\alpha+\mu+\delta)}<1$$

and

$$R_{04}=\frac{\beta_{2,i}^{U}\left(S_{YU,i}^{M}+S_{YE,i}^{M}\right)}{N_{YU,i}^{F}(\rho+\mu+\delta)}<1$$

so that the infection does not persist in the metapopulation and under this condition the endemic equilibrium point does not exist. The DFE is unstable for

R_{0j}, $j=1,2,3,4$, and then the endemic equilibrium point exists and the infection persists in the mepopulation.

Theorem 5.2. (see Van den Driessche and Watmough [9]). The disease free equilibrium of system (2.0), E_0, is locally asymptotically stable if $R_0<1$

6. Global Stability of the Disease-Free Equilibrium

In this section, we prove that E_0 is actually globally asymptotically stable when $R_{0j}\le 1$. Therefore, the model (2.0) demonstrates global threshold dynamics. We shall achieve our goal by constructing an appropriate Lyapunov functional.

Theorem 6.1. The disease-free equilibrium

$$E_0=\left(\frac{\lambda}{\rho+\mu},\frac{\lambda+\rho N_{YUF0}}{\mu},\frac{\lambda}{\rho+\mu},\frac{\lambda+\rho N_{YUM0}}{\mu},0,0,0,0,0\right)$$

is globally asymptotically stable in R_+^9 whenever $R_{0j}\le 1$.

Proof: We consider the Lyapunov function

$$\begin{aligned}L(t)=&w_1S_{YUF}+w_2S_{YEF}+w_3S_{YUM}+w_4S_{YEM}+w_5I_{YUM}\\&+w_6I_{YEM}+w_7I_{YEF}+w_8I_{YUF}+w_9T_E\end{aligned}$$

where $w_i, i=1,2,\cdots,9$ are constants that would be chosen in the course of the proof. Hence, calculating the rate of change of Lalong the solution of (2.0) gives,

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t}=&\frac{\partial L}{\partial S_{YUF}}\cdot\frac{\mathrm{d}S_{YUF}}{\mathrm{d}t}+\frac{\partial L}{\partial S_{YEF}}\cdot\frac{\mathrm{d}S_{YEF}}{\mathrm{d}t}+\frac{\partial L}{\partial S_{YUM}}\cdot\frac{\mathrm{d}S_{YUM}}{\mathrm{d}t}+\frac{\partial L}{\partial S_{YEM}}\cdot\frac{\mathrm{d}S_{YEM}}{\mathrm{d}t}\\&+\frac{\partial L}{\partial I_{YUM}}\cdot\frac{\mathrm{d}I_{YUM}}{\mathrm{d}t}+\frac{\partial L}{\partial I_{YEM}}\cdot\frac{\mathrm{d}I_{YEM}}{\mathrm{d}t}+\frac{\partial L}{\partial I_{YEF}}\cdot\frac{\mathrm{d}I_{YEF}}{\mathrm{d}t}+\frac{\partial L}{\partial I_{YUF}}\cdot\frac{\mathrm{d}I_{YUF}}{\mathrm{d}t}+\frac{\partial L}{\partial T}\cdot\frac{\mathrm{d}T}{\mathrm{d}t}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t}=&w_1\left((1-\Pi)\lambda-\beta_{1,i}^U\frac{I_{YUM,i}}{N_{YUM,i}}S_{YUF,i}-\beta_{1,i}^E\frac{I_{YEM,i}}{N_{YEM,i}}S_{YUF,i}-(\rho_i+\mu_i)S_{YUF,i}\right)\\&+w_2\left((1-\Pi)\lambda+\rho_iS_{YU,i}^F-\beta_{1,i}^U\frac{I_{YUM,i}}{N_{YUM,i}}S_{YEF,i}-\beta_{1,i}^E\frac{I_{YEM,i}}{N_{YEM,i}}S_{YEF,i}-\mu_iS_{YEF,i}\right)\\&+w_3\left((1-\Pi)\lambda-\beta_{2,i}^U\frac{I_{YUF,i}}{N_{YUF,i}}S_{YUM,i}-\beta_{2,i}^E\frac{I_{YEF,i}}{N_{YEF,i}}S_{YUM,i}-(\rho_i+\mu_i)S_{YUM,i}\right)\\&+w_4\left((1-\Pi)\lambda+\rho S_{YUM,i}-\beta_{2,i}^U\frac{I_{YUF,i}}{N_{YUF,i}}S_{YEM,i}-\beta_{2,i}^E\frac{I_{YEF,i}}{N_{YEF,i}}S_{YEM,i}-\mu_iS_{YEM,i}\right)\\&+w_5\left(\Pi\lambda+\beta_{1,i}^U\frac{I_{YUM,i}}{N_{YUM,i}}S_{YUF,i}+\beta_{1,i}^U\frac{I_{YUM,i}}{N_{YUM,i}}S_{YEF,i}-(\rho_i+\mu_i+\delta_i)I_{YUM,i}\right)\\&+w_6\left(\Pi\lambda+\rho_iI_{YUM,i}+\beta_{1,i}^E\frac{I_{YEM,i}}{N_{YEM,i}}S_{YUF,i}+\beta_{1,i}^E\frac{I_{YEM,i}}{N_{YEM,i}}S_{YEF,i}-(\alpha_i+\mu_i+\delta_i)I_{YEM,i}\right)\\&+w_7\left(\Pi\lambda+\rho_iI_{YUF,i}+\beta_{2,i}^E\frac{I_{YEF,i}}{N_{YEF,i}}S_{YUM,i}+\beta_{2,i}^E\frac{I_{YEF,i}}{N_{YEF,i}}S_{YEM,i}-(\alpha_i+\mu_i+\delta_i)I_{YEF,i}\right)\\&+w_8\left(\Pi\lambda+\beta_{2,i}^U\frac{I_{YUF,i}}{N_{YUF,i}}S_{YUM,i}+\beta_{2,i}^U\frac{I_{YUF,i}}{N_{YUF,i}}S_{YEM,i}-(\rho_i+\mu_i+\delta_i)I_{YUF,i}\right)\\&+w_9\left(\alpha_iI_{YEM,i}+\alpha_iI_{YEF,i}-(\mu_i+\delta_{1,i})T_{Y,i}\right)\end{aligned}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = (w_5 - w_1)\Pi_i\lambda_i + (w_7 - w_3)\Pi_i\lambda_i + (w_8 - w_4)\Pi_i\lambda_i + (w_5 - w_1)\beta_{1,i}^{U}\frac{I_{YUM,i}}{N_{YUM,i}}S_{YUF,i}$$
$$+(w_6 - w_1)\beta_{1,i}^{E}\frac{I_{YEM,i}}{N_{YEM,i}}S_{YUF,i} + (w_2 - w_1)\rho_i S_{YUF,i} - \mu_i w_1 S_{YUF,i}$$
$$+(w_5 - w_2)\beta_{1,i}^{U}\frac{I_{YUM,i}}{N_{YUM,i}}S_{YEF,i} + (w_6 - w_2)\beta_{1,i}^{E}\frac{I_{YEM,i}}{N_{YEM,i}}S_{YEF,i} - \mu_i w_2 S_{YEF,i}$$
$$+(w_8 - w_3)\beta_{2,i}^{U}\frac{I_{YUF,i}}{N_{YUF,i}}S_{YUM,i} + (w_7 - w_3)\beta_{2,i}^{E}\frac{I_{YEF,i}}{N_{YEF,i}}S_{YUM,i}$$
$$+(w_4 - w_3)\rho S_{YUM,i} - \mu_i w_3 S_{YUM,i} + (w_8 - w_4)\beta_{2,i}^{U}\frac{I_{YUF,i}}{N_{YUF,i}}S_{YEM,i}$$
$$+(w_7 - w_4)\beta_{2,i}^{E}\frac{I_{YEF,i}}{N_{YEF,i}}S_{YEM,i} - \mu_i w_4 S_{YEM,i} + (w_6 - w_5)\rho I_{YUM,i}$$
$$-(\mu + \delta)w_5 I_{YUM,i} + (w_9 - w_6)\alpha I_{YEM,i} + (w_5 - w_6)\rho_i I_{YUF,i}$$
$$+(w_7 - w_5)\alpha I_{YEF,i} - (\mu_i + \delta_i)w_5 I_{YEF,i} - (\mu_i + \delta_i)w_6 I_{YUF} - (\mu_i + \delta_{1,i})w_7 T_{Y,i}$$

Choosing $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = w_9$ gives the following

$$-\mu w_1 S_{YUF} - \mu w_2 S_{YEF} - \mu w_3 S_{YUM} - \mu w_4 S_{YEM} - (\mu + \delta)w_5 I_{YUM} - (\mu + \delta)w_6 I_{YEM}$$
$$-(\mu + \delta)w_7 I_{YEF} - (\mu + \delta)w_8 I_{YUF} - (\mu + \delta_{1,i})w_9 T_{Y,i}$$

It follows that $L$is positive definite and $\frac{\mathrm{d}L}{\mathrm{d}t}$ is negative definite. It can therefore be ascertained that the function is a Lyapunov function for system (2.0). Hence by Lyapunov asymptotic stability theorem [10], the equilibrium E_0 is globally asymptotically stable.

7. Conclusion

In this study, we approached using deterministic model. We developed a mathematical model of HIV transmission among adults in Meta-population setting in Ethiopia. Our model captures the disease induced deaths in transmission as HIV is known to cause deaths in transmission. Mathematical analysis was done and it was established that in the absence of the disease a disease free equilibrium will always exist if $R_{0j} \le 1$ for $j = 1,2,3,4$. We also established that the endemic equilibrium exists in the presence of the disease that is when $R_{0j} > 1$ for $j = 1,2,3,4$, with the infectious population greater than zero. Reducing the infection in the vector population reduces R_{0j} for $j = 1,2,3,4$, greatly. Thus the best methods of controlling HIV transmission is to target the Infected uneducated female youth, Infected educated female youth, Infected uneducated male youth, Infected educated male youth. R_{0j} is a threshold that completely determines the global dynamics of disease transmission.

Conflict of Interest

The author(s) declare(s) that there is no conflict of interest regarding the

publication of this paper.

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