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The Commutativity of a *-Ring with Generalized Left *-α-Derivation

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Abstract

In this paper, it is defined that left *-α-derivation, generalized left *-α-derivation and *-α-derivation, generalized *-α-derivation of a *-ring where α is a homomorphism. The results which proved for generalized left *-derivation of R in [1] are extended by using generalized left *-α-derivation. The commutativity of a *-ring with generalized left *-α-derivation is investigated and some results are given for generalized *-α-derivation.

Keywords

*-Ring, Prime *-Ring, Generalized Left *-α-Derivation, Generalized *-α-Derivation

1. Introduction

Let R be an associative ring with center $Z(R)$. $xy+yx$ where $x,y\in R$ is denoted by (x,y) and $xy-yx$ where $x,y\in R$ is denoted by $[x,y]$ which holds some properties: $[xy,z]=x[y,z]+[x,z]y$ and $[x,yz]=[x,y]z+y[x,z]$. An additive mapping α which holds $\alpha(xy)=\alpha(x)\alpha(y)$ for all $x,y\in R$ is called a *homomorphism* of *R*. An additive mapping β which holds $\beta(xy)=\beta(y)\beta(x)$ for all $x,y\in R$ is called an *anti-homomorphism* of *R*. A homomorphism of R is called *an epimorphism* if it is surjective. A ring R is called a *prime* if $aRb=(0)$ implies that either $a=0$ or $b=0$ for fixed $a,b\in R$. In private, if $b=a$, it implies that R is a *semiprime ring*. An additive mapping $*:R\to R$ which holds $(xy)^*=y^*x^*$ and $(x^*)^*=x$ for all $x,y\in R$ is called an *involution* of *R*. A ring R which is equipped with an involution * is called a **-ring*. A *-ring R is called a *prime *-ring* (*resp. semiprime *-ring*) if R is prime (resp. semiprime). A ring R is called a **-prime ring* if $aRb=aRb^*=(0)$

implies that either $a=0$ or $b=0$ for fixed $a,b\in R$.

Notations of left *-derivation and generalized left *-derivation were given in abu: Let R be a *-ring. An additive mapping $d:R\to R$ is called a *left *-derivation* if $d(xy)=x^*d(y)+yd(x)$ holds for all $x,y\in R$. An additive mapping $F:R\to R$ is called a *generalized left *-derivation* if there exists a left *-derivation d such that $F(xy)=x^*F(y)+yd(x)$ holds for all $x,y\in R$. An additive mapping $T:R\to R$ is called a *right *-centralizer* if $T(xy)=x^*T(y)$ for all $x,y\in R$. It is clear that a generalized left *-derivation associated with zero mapping is a right *-centralizer on a *-ring.

A *-derivation on a *-ring was defined by Bresar and Vukman in [2] as follows: An additive mapping $d:R\to R$ is said to be a *-derivation if $d(xy)=d(x)y^*+xd(y)$ for all $x,y\in R$.

A generalized *-derivation on a *-ring was defined by Shakir Ali in Shakir: An additive mapping $F:R\to R$ is said to be a *generalized *-derivation* if there exists a *-derivation $d:R\to R$ such that $F(xy)=F(x)y^*+xd(y)$ for all $x,y\in R$.

In this paper, motivated by definition of a left *-derivation and a generalized left *-derivation in [1], it is defined that a left *-α-derivation and a generalized left *-α-derivation are as follows respectively: Let R be a *-ring and α be a homomorphism of R. An additive mapping $d:R\to R$ such that $d(xy)=x^*d(y)+\alpha(y)d(x)$ for all $x,y\in R$ is called a *left *-α-derivation* of R. An additive mapping f is called a *generalized left *-α-derivation* if there exists a left *-α-derivation d such that $f(xy)=x^*f(y)+\alpha(y)d(x)$ for all $x,y\in R$. Similarly, motivated by definition of a *-derivation in [2] and a generalized *-derivation in [3], it is defined that a *-α-derivation and a generalized *-α-derivation are as follows respectively: Let R be a *-ring and α be a homomorphism of R. An additive mapping t which holds $t(xy)=t(x)y^*+\alpha(x)t(y)$ for all $x,y\in R$ is called a **-α-derivation* of R. An additive mapping g is called a *generalized *-α-derivation* if there exists a *-α-derivation t such that $g(xy)=g(x)y^*+\alpha(x)t(y)$ holds for all $x,y\in R$.

In [4], Bell and Kappe proved that if $d:R\to R$ is a derivation holds as a homomorphism or an anti-homomorphism on a nonzero right ideal of R which is a prime ring, then $d=0$. In [5], Rehman proved that if $F:R\to R$ is a nonzero generalized derivation with a nonzero derivation $d:R\to R$ where R is a 2-torsion free prime ring holds as a homomorphism or an anti homomorphism on a nonzero ideal of R, then R is commutative. In [6], Dhara proved some results when a generalized derivation acting as a homomorphism or an anti-homomorphism of a semiprime ring. In [7], Shakir Ali showed that if $G:R\to R$ is a generalized left derivation associated with a Jordan left derivation $\delta:R\to R$ where R is 2-torsion free prime ring and G holds as a homomorphism or an anti-homomorphism on a nonzero ideal of R, then either R is commutative or $G(x)=xq$ for all $x\in R$ and $q\in Q_l(R_C)$. In [1], it is proved that if $F:R\to R$ is a generalized left *-derivation associated with a left

*-derivation on R where R is a prime *-ring holds as a homomorphism or an anti-homomorphism on R, then R is commutative or F is a right *-centralizer on R.

The aim of this paper is to extend the results which proved for generalized left *-derivation of R in [1] and prove the commutativity of a *-ring with generalized left *-α-derivation. Some results are given for generalized *-α-derivation.

The material in this work is a part of first author's Master's Thesis which is supervised by Prof. Dr. Neşet Aydin.

2. Main Results

From now on, R is a prime *-ring where $*: R \to R$ is an involution, α is an epimorphism on R and $f: R \to R$ is a generalized left *-α-derivation associated with a left *-α-derivation d on R.

Theorem 1

1) If f is a homomorphism on R, then either R is commutative or f is a right *-centralizer on R.

2) If f is an anti-homomorphism on R, then either R is commutative or f is a right *-centralizer on R.

Proof. 1) Since f is both a homomorphism and a generalized left *-α-derivation associated with a left *-α-derivation d on R, it holds that for all $x, y, z \in R$

$$\begin{aligned} f(xyz) &= f(x(yz)) = x^* f(yz) + \alpha(yz) d(x) \\ &= x^* f(y) f(z) + \alpha(y) \alpha(z) d(x). \end{aligned}$$

That is, it holds for all $x, y, z \in R$

$$f(xyz) = x^* f(y) f(z) + \alpha(y) \alpha(z) d(x). \quad (1)$$

On the other hand, it holds that for all $x, y, z \in R$

$$f(xyz) = f((xy)z) = f(xy) f(z) = x^* f(y) f(z) + \alpha(y) d(x) f(z).$$

So, it means that for all $x, y, z \in R$

$$f(xyz) = x^* f(y) f(z) + \alpha(y) d(x) f(z). \quad (2)$$

Combining Equation (1) and (2), it is obtained that for all $x, y, z \in R$

$$x^* f(y) f(z) + \alpha(y) \alpha(z) d(x) = x^* f(y) f(z) + \alpha(y) d(x) f(z).$$

This yields that for all $x, y, z \in R$

$$\alpha(y)(\alpha(z) d(x) - d(x) f(z)) = 0.$$

Replacing y by yr where $r \in R$ in the last equation, it implies that

$$\alpha(y) \alpha(R)(\alpha(z) d(x) - d(x) f(z)) = (0)$$

for all $x, y, z \in R$. Since α is surjective and R is prime, it follows that for all $x, z \in R$

$$\alpha(z) d(x) = d(x) f(z). \quad (3)$$

Replacing x by xy where $y \in R$ in the last equation, it holds that for all $x, y, z \in R$

$$\alpha(z)x^*d(y)+\alpha(z)\alpha(y)d(x)=x^*d(y)f(z)+\alpha(y)d(x)f(z).$$

Using Equation (3) in the last equation, it implies that for all $x,y,z\in R$

$$\left[\alpha(z),x^*\right]d(y)+\left[\alpha(z),\alpha(y)\right]d(x)=0.$$

Since α is surjective, it holds that for all $x,y,z\in R$

$$\left[z,x^*\right]d(y)+\left[z,\alpha(y)\right]d(x)=0.$$

Replacing z by x^* in the last equation, it follows that for all $x,y\in R$

$$\left[x^*,\alpha(y)\right]d(x)=0.$$

Since α is a surjective, it holds that $\left[x^*,y\right]d(x)=0$ for all $x,y\in R$. Replacing y by yz where $z\in R$ in the last equation, it gets $\left[x^*,y\right]zd(x)=0$ for all $x,y,z\in R$. So, it implies that for all $x,y\in R$

$$\left[x^*,y\right]Rd(x)=(0).$$

Since R is prime, it follows that $\left[x^*,y\right]=0$ or $d(x)=0$ for all $x,y\in R$. Let $A=\left\{x\in R\,|\left[x^*,y\right]=0,\forall y\in R\right\}$ and $B=\left\{x\in R\,|\,d(x)=0\right\}$. Both A and B are additive subgroups of R and R is the union of A and B. But a group can not be set union of its two proper subgroups. Hence, R equals either A or B.

Assume that $A=R$. This means that $\left[x^*,y\right]=0$ for all $x,y\in R$. Replacing x by x^* in the last equation, it gets that $[x,y]=0$ for all $x,y\in R$. Therefore, R is commutative.

Assume that $B=R$. This means that $d(x)=0$ for all $x\in R$. Since f is a generalized left $*$-α-derivation associated with d, it follows that f is a right $*$-centralizer on R.

2) Since f is both an anti-homomorphism and a generalized left $*$-α-derivation associated with a left $*$-α-derivation d on R, it holds that

$$f(xy)=f(y)f(x)=x^*f(y)+\alpha(y)d(x)$$

for all $x,y\in R$. It means that for all $x,y\in R$

$$f(y)f(x)=x^*f(y)+\alpha(y)d(x).$$

Replacing y by xy in the last equation and using that f is an anti-homomorphism, it follows that for all $x,y\in R$

$$x^*f(y)f(x)+\alpha(y)d(x)f(x)=x^*f(y)f(x)+\alpha(x)\alpha(y)d(x)$$

which implies that for all $x,y\in R$

$$\alpha(y)d(x)f(x)=\alpha(x)\alpha(y)d(x). \tag{4}$$

Replacing y by zy where $z\in R$ in the last equation, it holds that for all $x,y,z\in R$

$$\alpha(z)\alpha(y)d(x)f(x)=\alpha(x)\alpha(z)\alpha(y)d(x).$$

Using Equation (4) in the above equation, it gets $\left[\alpha(z),\alpha(x)\right]\alpha(y)d(x)=0$ for all $x,y,z\in R$. Since α is surjective, it holds that $\left[z,\alpha\left(x\right)\right]yd\left(x\right)=0$ for all $x,y,z\in R$. That is, for all $x,z\in R$

$$\left[z,\alpha\left(x\right)\right]Rd\left(x\right)=\left(0\right).$$

Since R is prime, it implies that $\left[z,\alpha\left(x\right)\right]=0$ or $d\left(x\right)=0$ for all $x,z\in R$. Let $K=\left\{x\in R\mid\left[z,\alpha\left(x\right)\right]=0,\forall z\in R\right\}$ and $L=\left\{x\in R\mid d\left(x\right)=0\right\}$. Both K and L are additive subgroups of R and R is the union of K and L. But a group cannot be set union of its two proper subgroups. Hence, R equals either K or L.

Assume that $K=R$. This means that $\left[z,\alpha\left(x\right)\right]=0$ for all $x,z\in R$. Since α is surjective, it holds that $\left[z,x\right]=0$ for all $x,z\in R$. It follows that R is commutative.

Assume that $L=R$. Now, required result is obtained by applying similar techniques as used in the last paragraph of the proof of 1).

Lemma 2 *If f is a nonzero homomorphism* (*or an anti-homomorphism*) *and* $f\left(R\right)\subset Z\left(R\right)$ *then R is commutative.*

Proof. Let f be either a nonzero homomorphism or an anti-homomorphism of R. From Theorem 1, it implies that either R is commutative or f is a right *-centralizer on R. Assume that R is noncommutative. In this case, f is a right *-centralizer on R. Since $f\left(R\right)$ is in the center of R, it holds that $\left[f\left(x^{*}y\right),r\right]=0$ for all $x,y,r\in R$. Using that f is a right *-centralizer and $f\left(R\right)\subset Z\left(R\right)$, it yields that for all $x,y,r\in R$

$$0=\left[f\left(x^{*}y\right),r\right]=\left[xf\left(y\right),r\right]=\left[x,r\right]f\left(y\right)$$

which follows that for all $x,y,r\in R$

$$\left[x,r\right]f\left(y\right)=0.$$

Since $f\left(R\right)$ is in the center of R, it is obtained that for all $x,y,r\in R$

$$\left[x,r\right]Rf\left(y\right)=\left(0\right).$$

Using primeness of R, it is implied that either $\left[x,r\right]=0$ or $f\left(y\right)=0$ for all $x,y,r\in R$. Since f is nonzero, it means that R is commutative. This is a contradiction which completes the proof.

Theorem 3 *If f is a nonzero homomorphism* (*or an anti-homomorphism*) *and* $f\left(\left[x,y\right]\right)=0$ *for all* $x,y\in R$ *then R is commutative.*

Proof. Let f be a homomorphism of R. It holds that R is commutative or f is a right *-centralizer on R from Theorem 1. Assume that R is noncommutative. In this case, f is a right *-centralizer on R. From the hypothesis, it gets that $f\left(\left[x,y\right]\right)=0$ for all $x,y\in R$. Since f is a homomorphism, it holds that for all $x,y\in R$

$$0=f\left(\left[x,y\right]\right)=f\left(xy-yx\right)=f\left(x\right)f\left(y\right)-f\left(y\right)f\left(x\right)=\left[f\left(x\right),f\left(y\right)\right]$$

i.e., for all $x,y\in R$

$$\left[f\left(x\right),f\left(y\right)\right]=0.$$

Replacing x by $x^{*}z$ in the last equation, using that f is a right *-centralizer on R and using the last equation, it holds that

$0=\left[f\left(x^{*}z\right),f\left(y\right)\right]=\left[xf\left(z\right),f\left(y\right)\right]=\left[x,f\left(y\right)\right]f\left(z\right)$ for $x,y,z\in R$. So, it follows that for all $x,y,z\in R$

$$\left[x,f\left(y\right)\right]f\left(z\right)=0.$$

Replacing x by xr where $r\in R$ and using the last equation, it holds that $\left[x,f\left(y\right)\right]rf\left(z\right)=0$ for all $x,y,z,r\in R$. This implies that for all $x,y,z\in R$

$$\left[x,f\left(y\right)\right]Rf\left(z\right)=\left(0\right).$$

Using the primeness of R, it is obtained that either $\left[x,f\left(y\right)\right]=0$ or $f\left(z\right)=0$ for all $x,y,z\in R$. Since f is nonzero, it follows that $f\left(R\right)\subset Z\left(R\right)$. Using Lemma 2, it is obtained that R is commutative. This is a contradiction which completes the proof.

Let f be an anti-homomorphism of R. This holds that R is commutative or f is a right *-centralizer on R from Theorem 1. Assume that R is noncommutative. In this case, f is a right *-centralizer on R. From the hypothesis, it gets that $f([x,y])=0$ for all $x,y\in R$. Since f is an anti-homomorphism, it holds that for all $x,y\in R$

$$0=f\left(\left[x,y\right]\right)=f\left(xy-yx\right)=f\left(y\right)f\left(x\right)-f\left(x\right)f\left(y\right)=-\left[f\left(x\right),f\left(y\right)\right]$$

i.e., for all $x,y\in R$

$$\left[f\left(x\right),f\left(y\right)\right]=0.$$

After here, the proof is done by the similarly way in the first case and same result is obtained.

Theorem 4 *If f is a nonzero homomorphism* (*or an anti-homomorphism*), $a\in R$ *and* $\left[f\left(x\right),a\right]=0$ *for all* $x\in R$ *then* $a\in Z\left(R\right)$ *or R is commutative.*

Proof. Let f be either a homomorphism or an anti-homomorphism of R. It holds that R is commutative or f is a right *-centralizer on R from Theorem 1. Assume that R is noncommutative. In this case, f is a right *-centralizer on R. From the hypothesis, it yields that for all $x,y\in R$

$$0=\left[f\left(x^{*}y\right),a\right]=\left[xf\left(y\right),a\right]=x\left[f\left(y\right),a\right]+\left[x,a\right]f\left(y\right)=\left[x,a\right]f\left(y\right)$$

i.e., for all $x,y\in R$

$$\left[x,a\right]f\left(y\right)=0.$$

Replacing x by xr where $r\in R$, it holds that $\left[x,a\right]rf\left(y\right)=0$ for all $x,y,r\in R$. This implies that $\left[x,a\right]Rf\left(y\right)=\left(0\right)$ for all $x,y\in R$. Using the primeness of R, it implies that $\left[x,a\right]=0$ or $f\left(y\right)=0$ for all $x,y\in R$. Since f is nonzero, it follows that $a\in Z\left(R\right)$. That is, it is obtained that either $a\in Z\left(R\right)$ or R is commutative.

Theorem 5 *If f is a nonzero homomorphism* (*or an anti-homomorphism*) *and* $f\left(\left[x,y\right]\right)\in Z\left(R\right)$ *for all* $x,y\in R$ *then R is commutative.*

Proof. Let f be a nonzero homomorphism of R. It implies that either R is commutative or f is a right *-centralizer on R from Theorem 1. Assume that R is noncommutative. In this case, f is a right *-centralizer on R. Since f is a homomorphism and $f([x,y]) \in Z(R)$ for all $x,y \in R$, it holds that for all $x,y \in R$

$$f([x,y]) = f(xy-yx) = f(xy) - f(yx)$$
$$= f(x)f(y) - f(y)f(x) = [f(x), f(y)]$$

i.e., for all $x,y \in R$

$$[f(x), f(y)] \in Z(R).$$

It means that $[[f(x), f(y)], r] = 0$ for all $x,y,r \in R$. Replacing x by x^*z where $z \in R$ in the last equation, it holds that for all $x,y,z,r \in R$

$$0 = [f(x^*z), f(y)], r] = [[xf(z), f(y)], r]$$
$$= [x,r][f(z), f(y)] + [[x, f(y)], r] f(z) + [x, f(y)][f(z), r]$$

which implies that for all $x,y,z,r \in R$

$$[x,r][f(z), f(y)] + [[x, f(y)], r] f(z) + [x, f(y)][f(z), r] = 0.$$

Replacing x by $f(y)$ and r by $f(z)$, it is obtained that for all $x,y,z \in R$

$$[f(y), f(z)][f(z), f(y)] = 0.$$

The last equation multiplies by r from right and using that $[f(x), f(y)] \in Z(R)$ for all $x,y \in R$, it follows that for all $x,y,z,r \in R$

$$[f(y), f(z)] r [f(z), f(y)] = 0$$

i.e., for all $x,y,z,r \in R$.

$$[f(z), f(y)] R [f(z), f(y)] = (0).$$

Using primeness of *R*, it is implied that for all $y,z \in R$

$$[f(z), f(y)] = 0.$$

From Theorem 4, it holds that either $f(y) \in Z(R)$ for all $y \in R$ or R is commutative. By using Lemma 2, it follows that R is commutative. This is a contradiction which completes the proof.

Let f be a nonzero anti-homomorphism of *R*. It implies that either R is commutative or f is a right *-centralizer on R from Theorem 1. Assume that R is noncommutative. In this case, f is a right *-centralizer on *R*. From the hypothesis, it gets that $f([x,y]) \in Z(R)$ for all $x,y \in R$. Since f is an anti-homomorphism, it is obtained that for all $x,y \in R$

$$f([x,y]) = f(xy-yx) = f(y)f(x) - f(x)f(y) = -[f(x), f(y)]$$

i.e., for all $x,y \in R$

$$[f(x), f(y)] \in Z(R).$$

After here, the proof is done by the similar way in the first case and same result is obtained.

Theorem 6 *If f is a nonzero homomorphism* (*or an anti-homomorphism*) *and* $f((x,y)) = 0$ *for all* $x,y \in R$ *then R is commutative.*

Proof. Let f be a homomorphism of R. It holds that R is commutative or f is a right *-centralizer on R from Theorem 1. Assume that R is noncommutative. In this case, f is a right *-centralizer on R. So, it gets that for all $x, y \in R$

$$0 = f((x,y)) = f(xy+yx) = f(xy) + f(yx) = f(x)f(y) + f(y)f(x).$$

It means that for all $x, y \in R$

$$f(x)f(y) + f(y)f(x) = 0.$$

Replacing x by x^*z where $z \in R$ in the above equation and using that f is a right * the last equation, it is obtained that

$$0 = f(x^*z)f(y) + f(y)f(x^*z) = xf(z)f(y) + f(y)xf(z).$$

Using that $f(x)f(y) = -f(y)f(x)$ for all $x, y \in R$ in the last equation

$$\begin{aligned} 0 &= xf(z)f(y) + f(y)xf(z) = -xf(y)f(z) + f(y)xf(z) \\ &= [f(y), x]f(z) \end{aligned}$$

i.e. for all $x, y, z \in R$

$$[f(y), x]f(z) = 0.$$

Replacing x by xr, it follows that $[f(y), x]Rf(z) = (0)$ for all $x, y, z \in R$. Using primeness of R, it holds that either $[f(y), x] = 0$ or $f(z) = 0$ for all $x, y, z \in R$. Since f is nonzero, it implies that $f(R) \subset Z(R)$. Using Lemma 2, it yields that R is commutative. This is a contradiction which completes the proof.

Let f be an anti-homomorphism of R. It holds that R is commutative or f is a right *-centralizer on R from Theorem 1. Assume that R is noncommutative. In this case f is a right *-centralizer on R. Using hypothesis, it gets that for all $x, y \in R$

$$0 = f((x,y)) = f(xy+yx) = f(xy) + f(yx) = f(y)f(x) + f(x)f(y)$$

i.e., for all $x, y \in R$

$$f(y)f(x) + f(x)f(y) = 0.$$

After here, the proof is done by the similar way in the first case and same result is obtained.

Now, $g: R \to R$ is a generalized *-α-derivation associated with a *-α-derivation t on R.

Theorem 7 *Let R be a *-prime ring where * be an involution, α be a homomorphism of R and* $g: R \to R$ *be a generalized *-α-derivation associated with a *-α-derivation t on R. If g is nonzero then R is commutative.*

Proof. Since g is a generalized *-α-derivation associated with a *-α-derivation t on R, it holds that $g(xy) = g(x)y^* + \alpha(x)t(y)$ for all $x, y \in R$. So it yields that for all $x, y, z \in R$

$$\begin{aligned} g(xyz) &= g((xy)z) = g(xy)z^* + \alpha(xy)t(z) \\ &= (g(x)y^* + \alpha(x)t(y))z^* + \alpha(x)\alpha(y)t(z) \\ &= g(x)y^*z^* + \alpha(x)t(y)z^* + \alpha(x)\alpha(y)t(z) \end{aligned}$$

that is, it holds that for all $x, y, z \in R$

$$g(xyz) = g(x)y^*z^* + \alpha(x)t(y)z^* + \alpha(x)\alpha(y)t(z). \quad (5)$$

On the other hand, it implies that for all $x, y, z \in R$

$$\begin{aligned} g(xyz) = g(x(yz)) &= g(x)(yz)^* + \alpha(x)t(yz) \\ &= g(x)z^*y^* + \alpha(x)(t(y)z^* + \alpha(y)t(z)) \\ &= g(x)z^*y^* + \alpha(x)t(y)z^* + \alpha(x)\alpha(y)t(z) \end{aligned}$$

so, it gets that for all $x, y, z \in R$

$$g(xyz) = g(x)z^*y^* + \alpha(x)t(y)z^* + \alpha(x)\alpha(y)t(z). \quad (6)$$

Now, combining the Equations (5) and (6), it holds that for all $x, y, z \in R$

$$\begin{aligned} &g(x)y^*z^* + \alpha(x)t(y)z^* + \alpha(x)\alpha(y)t(z) \\ &= g(x)z^*y^* + \alpha(x)t(y)z^* + \alpha(x)\alpha(y)t(z) \end{aligned}$$

which follows that

$$g(x)[y^*, z^*] = 0$$

for all $x, y, z \in R$. Replacing y by y^* and z by z^*, it holds that for all $x, y, z \in R$

$$g(x)[y, z] = 0.$$

Replacing y by ry where $r \in R$ in the last equation, it yields that for all $x, y, z, r \in R$

$$0 = g(x)[ry, z] = g(x)r[y, z] + g(x)[r, z]y.$$

Using $g(x)[y, z] = 0$ for all $x, y, z \in R$ in above equation, it is obtained that for all $x, y, z, r \in R$

$$g(x)r[y, z] = 0 \quad (7)$$

i.e., for all $x, y, z \in R$

$$g(x)R[y, z] = (0). \quad (8)$$

Replacing y by y^* and z by $-z^*$, it follows that for all $x, y, z \in R$

$$g(x)R([y, z])^* = (0). \quad (9)$$

Now, combining the Equations (8) and (9),

$$g(x)R[y, z] = g(x)R([y, z])^* = (0)$$

is obtained for all $x, y, z \in R$. Using *-primeness of *R*, it follows that $g(x) = 0$ or $[y, z] = 0$ for all $x, y, z \in R$. Since g is nonzero, R is commutative.

Theorem 8 *Let R be a semiprime *-ring where * be an involution, α be an homomorphism of R and* $g: R \to R$ *be a nonzero generalized *-α-derivation associated with a *-α-derivation t on R then* $g(R) \subset Z(R)$.

Proof. Equation (7) multiplies by s from left, it gets that for all $x, y, z, r, s \in R$

$$sg(x)r[y, z] = 0. \quad (10)$$

Replacing r by sr in the Equation (7), it holds that for all $x, y, z, r, s \in R$

$$g(x)sr[y,z]=0. \tag{11}$$

Now, combining the Equation (10) and (11),

$$sg(x)r[y,z]=g(x)sr[y,z]$$

is obtained for all $x, y, z, r, s \in R$. It follows that for all $x, y, z, r, s \in R$

$$[s,g(x)]r[y,z]=0.$$

This implies that

$$[s,g(x)]R[y,z]=(0)$$

for all $x, y, z, s \in R$. Replacing s by y and z by $g(x)$ in the last equation, it yields that

$$[y,g(x)]R[y,g(x)]=(0)$$

for all $x, y \in R$. Using semiprimeness of *R*, it is implied that for all $x, y \in R$

$$[y,g(x)]=0.$$

That is,

$$g(R)\subset Z(R)$$

which completes the proof.

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