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Estimate on the Dimension of Global Attractor for Nonlinear Higher-Order Coupled Kirchhoff Type Equations

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Abstract

In this paper, we investigate the finite dimensions of the global attractor for nonlinear higher-order coupled Kirchhoff type equations with strong linear damping in Hilbert spaces E_0 and E_1. Under the appropriate assumptions, we acquire a precise estimate of the upper bound for its Hausdorff and Fractal dimensions.

Keywords

Higher-Order, Kirchhoff-Type Equations, Global Attractor, Hausdorff Dimension, Fractal Dimension

1. Introduction

G. G. Lin and L. J., Hu have studied the existence of a global attractor for coupled Kirchhoff type equations with strongly linear damping in [1]. In this paper, we are concerned with the finite dimensions of the global attractor as mentioned above:

$$u_{tt}+M\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g_1(u,v) = f_1(x),\ \text{in } \Omega\times[0,+\infty), \tag{1.1}$$

$$v_{tt}+M\left(\left\|D^m u\right\|^2+\left\|D^m v\right\|^2\right)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v) = f_2(x),\ \text{in } \Omega\times[0,+\infty), \tag{1.2}$$

$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \ x\in\Omega, \tag{1.3}$$

$$v(x,0)=v_0(x),\ \ v_t(x,0)=v_1(x),\ \ x\in\Omega, \tag{1.4}$$

$$\frac{\partial^i u}{\partial n^i}=0,\quad \frac{\partial^i v}{\partial n^i}=0,\quad i=0,1,2,\cdots,m-1,\quad x\in\partial\Omega,\ t\geq 0, \tag{1.5}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $\beta>0$ is real number, $m\geq 1$ is positive integer. $M(s)$ and $g_j(u,v)(j=1,2)$ are given functions later.

In demonstrating the longtime behavior of evolutional equation, we currently aim to show that the dynamics of the equation is finite dimensional. To be precise, one possible way to express it is to say that the dynamical systems of equation exists a global attractor with finite Hausdorff and Fractal dimensions.

Concerning the wave equation with linear and semi-linear dissipative system, existence of the global attractor with finite Hausdorff and Fractal dimensions is proved in [2], for the nonlinear wave equation, the existence of the global attractor with finite Hausdorff and Fractal dimensions is proved in [3] [4] [5]. When the equation is nonlinear, the process of dimension estimation is more complicated. The method of linearization works very well on it, and meanwhile we take fully consideration of assumptions on the nonlinearities of the equation.

Recently, Z. J, Yang [6] studied the longtime behavior of the Kirchhoff type equation with strong damping on R^N. It showed that the related continuous semi-group $S(t)$ possesses a global attractor which is connected and has finite Fractal and Hausdorff dimensions.

$$u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u-\Delta u_t+u+u_t+g(x,u)=f(x),\ \text{ in } R^N\times R^+. \tag{1.6}$$

At the same time, Z. J. Yang [7] dealt with the global attractors and their Hausdorff dimensions for a class of Kirchhoff models, and got the existence, regularity, and Hausdorff dimensions of global attractors for a class of Kirchhoff models arising in elastoplastic flow.

$$u_{tt}-div\left\{\sigma\left(|\nabla u|^2\right)\nabla u\right\}-\Delta u_t+\Delta^2 u+h(u_t)+g(u)=f(x),\quad \text{in } \Omega\times R^+. \tag{1.7}$$

Furthermore, X. M. Fan and S. F. Zhou [8] proved the existence of compact kernel sections for the process generated by strongly damped wave equations of non-degenerate Kirchhoff type modelling the nonlinear vibrations of an elastic string, and they obtained a precise estimate of upper bound of Hausdorff dimension of kernel sections.

$$u_{tt}-\alpha\Delta u_t-\left(\beta+\gamma\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)^\rho\right)\Delta u+h(u_t)+f(u,t)=g(x,t),\quad x\in\Omega,t>\tau. \tag{1.8}$$

In addition, G. G. Lin and Y. L. Gao [9] studied the longtime behavior of solution to initial boundary value problem for a class of strongly damped higher-order Kirchhoff type equation:

$$u_{tt}+(-\Delta)^m u_t+\left(\alpha+\beta\|\nabla^m u\|^2\right)^q(-\Delta)^m u+g(u)=f(x),\ (x,t)\in\Omega\times[0,+\infty), \tag{1.9}$$

they got the existence and uniqueness of the solution by the Galerkin method and obtained the existence of the global attractor in $H_0^m(\Omega)\times L^2(\Omega)$ according to the attractor theorem, besides, the estimation of the upper bound of Hausdorff dimension for the attractor was established.

The paper is arranged as follows. In Section 2, some preliminaries and main results are stated. In Section 3, in order to acquire the result of the estimation, we show the differentiability of semigroup. Eventually, the Hausdorff and Fractal dimensions of the global attractor for the dynamics system associated with problem (1.1)-(1.5) are discussed in detail.

2. Preliminaries and Main Results

Throughout this paper, we need some notations for convenience. We consider a family of Hilbert spaces $V_a = D\left(A^{a/2}\right), a\in R$, whose inner products and norms are given by $(\cdot,\cdot)_{V_a} = \left(A^{a/2}, A^{a/2}\right)$ and $\|\cdot\|_{V_a} = \left\|A^{a/2}\cdot\right\|$. Obviously

$$V_0 = L^2(\Omega),\quad V_m = H^m\times H_0^1,\quad V_{2m} = H^{2m}\times H_0^1,$$

we denote

$$E_0 = V_m\times V_0\times V_m\times V_0,$$

$$E_1 = V_{2m}\times V_m\times V_{2m}\times V_m,$$

$$E^* = V_m\times V_m.$$

For our purpose, we define a weighted inner product and norm in E_0 by

$$(\varphi,\tilde{\varphi})_{E_0} = (u_1,u_2)+(p_1,p_2)+(v_1,v_2)+(q_1,q_2),$$

$$\|\varphi\|_{E_0}^2 = (\varphi,\varphi)_{E_0} = \left\|A^{\frac{m}{2}}u_1\right\|^2+\|p_1\|^2+\left\|A^{\frac{m}{2}}v_1\right\|^2+\|q_1\|^2,$$

with any $\varphi=(u_1,p_1,v_1,q_1)^{\mathrm{T}}, \tilde{\varphi}=(u_2,p_2,v_2,q_2)^{\mathrm{T}}\in E_0$.

Next, we make the following assumptions for problem (1.1)-(1.5).

(A1) $M(s)\in C^2([0,+\infty),R)$ is not decreasing function and for positive constants m_0, m_1,

1) $0<\beta<m_0\le M(s)\le m_1$,

2) $m^* = \begin{cases} m_0, \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\|D^m u\|^2+\|D^m v\|^2\right)>0. \\ m_1, \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\|D^m u\|^2+\|D^m v\|^2\right)<0. \end{cases}$

(A2) There exists $0<\kappa_1,\kappa_2<1$, and for every R_0, there exist $c_0=c_0(R_0)$, $c_0'=c_0'(R_0)$ such that

$$\|g_{1i}(\tilde{u},\tilde{v})-g_{1i}(u,v)\|\le c_0\|(\tilde{u},\tilde{v})-(u,v)\|_{E^*}^{\kappa_1},$$

$$\|g_{2i}(\tilde{u},\tilde{v})-g_{2i}(u,v)\|\le c_0'\|(\tilde{u},\tilde{v})-(u,v)\|_{E^*}^{\kappa_2},\quad (i=u,v),$$

$$\forall(\tilde{u},\tilde{v}),(u,v)\in E^*,\quad \|(\tilde{u},\tilde{v})\|_{E^*}\le c(R_0),\ \|(u,v)\|_{E^*}\le c(R_0).$$

where $\|(\tilde{u},\tilde{v})-(u,v)\|_{E^*}^{\kappa} = \|D^m(\tilde{u}-u)\|^{\kappa}+\|D^m(\tilde{v}-v)\|^{\kappa}$.

3. The Hausdorff and Fractal Dimensions of the Attractor

In order to obtain the result of the dimension estimation, we should prepare the following lemmas.

Lemma 3.1. ([1]) Suppose that the assumptions of [1] hold, the constants $T>0,\beta>0$ and initial value $(u_0,p_0,v_0,q_0)\in E_0$, then for the problem (1.1)-(1.5), there exists a unique weak solution such that $(u,p,v,q)\in L^\infty((0,+\infty);E_0)$.

Lemma 3.2. ([1]) Suppose that $M(s)$ and $g_i(u,v)(i=1,2)$ satisfy assumptions of [1] respectively. Then for $\beta>0$, the problem (1.1)-(1.5) possesses the global attractor A, which is compact among bounded absorbing set B_0 in E_0, that is

$$A=\omega(B_0)=\bigcap_{s\ge0}\overline{\bigcup_{t\ge s}S(t)B_0},$$

this lemma is result on the existence of a global attractor of system (1.1)-(1.5) generated by semigroup $S(t)$.

The first step will be to prove the differentiability of $S(t)$. Denote $\bar U=(U,P,V,Q)$. In what follows, we put $M(s)=M\left(\|D^mu\|^2+\|D^mv\|^2\right)$ for simplicity. The first variation equations of (1.1)-(1.5) as the following

$$\begin{aligned}&P_t+M(s)A^mU+2M'(s)\left(A^{\frac m2}U,A^{\frac m2}u\right)A^mu+2M'(s)\left(A^{\frac m2}V,A^{\frac m2}v\right)A^mu\\&-\varepsilon\left(\beta A^m-\varepsilon\right)U+\left(\beta A^m-\varepsilon\right)P+g_{1u}(u,v)U+g_{1v}(u,v)V=0,\end{aligned}\tag{3.1}$$

$$\begin{aligned}&Q_t+M(s)A^mV+2M'(s)\left(A^{\frac m2}U,A^{\frac m2}u\right)A^mv+2M'(s)\left(A^{\frac m2}V,A^{\frac m2}v\right)A^mv\\&-\varepsilon\left(\beta A^m-\varepsilon\right)V+\left(\beta A^m-\varepsilon\right)Q+g_{2u}(u,v)U+g_{2v}(u,v)V=0,\end{aligned}\tag{3.2}$$

$$(U,P,V,Q)\big|_{t=0}=(\xi,\zeta,\eta,\sigma),\tag{3.3}$$

$$U(x,t)\big|_{x\in\partial\Omega}=V(x,t)\big|_{x\in\partial\Omega}=0,\tag{3.4}$$

where $U_t+\varepsilon U=P$, $V_t+\varepsilon V=Q$, $(\xi,\zeta,\eta,\sigma)\in E_0$, $(u,p,v,q)=S(t)(u_0,p_0,v_0,q_0)$ is the solution of Equations (1.1)-(1.5) with $(u_0,p_0,v_0,q_0)\in A$.

Given $(u_0,p_0,v_0,q_0)\in A$, the solution $S(t)(u_0,p_0,v_0,q_0)\in E_0$, by standard methods, we can prove that for any $(\xi,\zeta,\eta,\sigma)\in E_0$, the linear initial boundary value problem (3.1)-(3.4) possesses a unique solution $\bar U=(U(t),P(t),V(t),Q(t))\in L^\infty(R,E_0)$.

Lemma 3.3. for any $t>0,R>0$, the mapping $S(t):E_0\to E_0$ is Frechet differentiable on E_0. Its differential at $\rho_0=(u_0,p_0,v_0,q_0)$ is the linear operator on E_0.

$$DS(t)\rho_0:(\xi,\zeta,\eta,\sigma)\to(U,P,V,Q),\tag{3.5}$$

where $\bar U=(U(t),P(t),V(t),Q(t))$ is the solution of (3.1)-(3.4).

Proof. Let $\rho_0=(u_0,p_0,v_0,q_0)\in E_0$, $\tilde\rho_0=(u_0+\xi,p_0+\zeta,v_0+\eta,q_0+\sigma)\in E_0$ with $\|\rho_0\|\le R_0$, $\|\tilde\rho_0\|\le R_0$, we denote $S(t)\rho_0=\rho=(u,p,v,q)$, $S(t)\tilde\rho_0=\tilde\rho=(\tilde u,\tilde p,\tilde v,\tilde q)$. First, we can prove a Lipschitz property of $S(t)$ on the bounded sets on E_0, that is

$$\left\|S(t)\rho_0 - S(t)\tilde{\rho}_0\right\|_{E_0}^2 \le \exp(c't)\left\|(\xi,\zeta,\eta,\sigma)\right\|_{E_0}^2, \quad \forall t \ge 0. \tag{3.6}$$

We now consider the difference $\tilde{\rho}-\rho-\bar{U}=(\tilde{u}-u-U,\tilde{p}-p-P,\tilde{v}-v-V,\tilde{q}-q-Q)=(\omega,\theta,\gamma,\delta)$, with $\bar{U}=(U(t),P(t),V(t),Q(t))$ the solution of (3.1)-(3.4), clearly,

$$\omega(0)=\theta(0)=\gamma(0)=\delta(0)=0, \tag{3.7}$$

$$\begin{aligned}&\theta_t-\varepsilon\theta+\beta A^m\theta+\varepsilon^2\omega-\beta\varepsilon A^m\omega+M(s)A^m w+g_{1u}(u,v)\omega+g_{1v}(u,v)\gamma\\&=h_1+\left(M(s)-M(\tilde{s})\right)A^m\tilde{u}+2M'(s)\left(A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)A^m u\\&+2M'(s)\left(A^{\frac{m}{2}}V,A^{\frac{m}{2}}v\right)A^m u,\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\delta_t-\varepsilon\delta+\beta A^m\delta+\varepsilon^2\gamma-\beta\varepsilon A^m\gamma+M(s)A^m\gamma+g_{2u}(u,v)\omega+g_{2v}(u,v)\gamma\\&=h_2+\left(M(s)-M(\tilde{s})\right)A^m\tilde{v}+2M'(s)\left(A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)A^m v\\&+2M'(s)\left(A^{\frac{m}{2}}V,A^{\frac{m}{2}}v\right)A^m v,\end{aligned} \tag{3.9}$$

where $w_t+\varepsilon w=\theta,\gamma_t+\varepsilon\gamma=\delta$, with

$$\begin{aligned}h_1&=-\left(g_1(\tilde{u},\tilde{v})-g_1(u,v)-g_{1u}(u,v)(\tilde{u}-u)-g_{1v}(u,v)(\tilde{v}-v)\right),\\h_2&=-\left(g_2(\tilde{u},\tilde{v})-g_2(u,v)-g_{2u}(u,v)(\tilde{u}-u)-g_{2v}(u,v)(\tilde{v}-v)\right).\end{aligned} \tag{3.10}$$

We have

$$\begin{aligned}h_1=-\int_0^1\Big[&\left\{g_{1u}\left(u+\theta_1(\tilde{u}-u),v+\theta_1(\tilde{v}-v)\right)-g_{1u}(u,v)\right\}(\tilde{u}-u)\\&+\left\{g_{1v}\left(u+\theta_1(\tilde{u}-u),v+\theta_1(\tilde{v}-v)\right)-g_{1v}(u,v)\right\}(\tilde{v}-v)\Big]\mathrm{d}\theta_1,\\h_2=-\int_0^1\Big[&\left\{g_{2u}\left(u+\theta_2(\tilde{u}-u),v+\theta_2(\tilde{v}-v)\right)-g_{2u}(u,v)\right\}(\tilde{u}-u)\\&+\left\{g_{2v}\left(u+\theta_2(\tilde{u}-u),v+\theta_2(\tilde{v}-v)\right)-g_{2v}(u,v)\right\}(\tilde{v}-v)\Big]\mathrm{d}\theta_2,\end{aligned} \tag{3.11}$$

due to $\forall\theta_1,\theta_2\in[0,1]$,

$$\begin{aligned}&\left\|g_{1u}\left(u+\theta_1(\tilde{u}-u),v+\theta_1(\tilde{v}-v)\right)-g_{1u}(u,v)\right\|\le c_0\theta_1^{\kappa_1}\left\|(\tilde{u},\tilde{v})-(u,v)\right\|_{E^*}^{\kappa_1},\\&\left\|g_{1v}\left(u+\theta_1(\tilde{u}-u),v+\theta_1(\tilde{v}-v)\right)-g_{1v}(u,v)\right\|\le c_0\theta_1^{\kappa_1}\left\|(\tilde{u},\tilde{v})-(u,v)\right\|_{E^*}^{\kappa_1},\\&\left\|g_{2u}\left(u+\theta_2(\tilde{u}-u),v+\theta_2(\tilde{v}-v)\right)-g_{2u}(u,v)\right\|\le c_0'\theta_2^{\kappa_2}\left\|(\tilde{u},\tilde{v})-(u,v)\right\|_{E^*}^{\kappa_2},\\&\left\|g_{2v}\left(u+\theta_2(\tilde{u}-u),v+\theta_2(\tilde{v}-v)\right)-g_{2v}(u,v)\right\|\le c_0'\theta_2^{\kappa_2}\left\|(\tilde{u},\tilde{v})-(u,v)\right\|_{E^*}^{\kappa_2},\end{aligned} \tag{3.12}$$

$$\left\|h_1\right\|\le 2c_1\left\|(\tilde{u},\tilde{v})-(u,v)\right\|_{E^*}^{\kappa_1+1}.$$

Similarly

$$\left\|h_2\right\|\le 2c_1'\left\|(\tilde{u},\tilde{v})-(u,v)\right\|_{E^*}^{\kappa_2+1}. \tag{3.13}$$

Taking the scalar product of each side of (3.8)-(3.9) with θ and δ, and then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\theta\|^2+\|\delta\|^2+\varepsilon^2\left(\|\omega\|^2+\|\gamma\|^2\right)+\left(m^*-\beta\varepsilon\right)\left(\left\|A^{\frac{m}{2}}\omega\right\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2\right)\right\}\\
&+\left(\frac{\beta}{2}\lambda_1^m-\varepsilon\right)\left(\|\theta\|^2+\|\delta\|^2\right)+\frac{\beta}{2}\left(\left\|A^{\frac{m}{2}}\theta\right\|^2+\left\|A^{\frac{m}{2}}\delta\right\|^2\right)\\
&+\varepsilon^3\left(\|\omega\|^2+\|\gamma\|^2\right)+\varepsilon\left(m_0-\beta\varepsilon\right)\left(\left\|A^{\frac{m}{2}}\omega\right\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2\right)\\
&\le\left(h_1-g_{1u}(u,v)\omega-g_{1v}(u,v)\gamma,\theta\right)+\left(h_2-g_{2u}(u,v)\omega-g_{2v}(u,v)\gamma,\delta\right)\\
&\quad+\left(\left(M(s)-M(\tilde{s})\right)A^m\tilde{u},\theta\right)+\left(\left(M(s)-M(\tilde{s})\right)A^m\tilde{v},\delta\right)\\
&\quad+\left(2M'(s)\left(A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)A^mu+2M'(s)\left(A^{\frac{m}{2}}V,A^{\frac{m}{2}}v\right)A^mu,\theta\right)\\
&\quad+\left(2M'(s)\left(A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)A^mv+2M'(s)\left(A^{\frac{m}{2}}V,A^{\frac{m}{2}}v\right)A^mv,\delta\right) \qquad (3.14)\\
&=\left(h_1-g_{1u}(u,v)\omega-g_{1v}(u,v)\gamma,\theta\right)+\left(h_2-g_{2u}(u,v)\omega-g_{2v}(u,v)\gamma,\delta\right)\\
&\quad+\left(J_1,\theta\right)+\left(J_1',\delta\right).
\end{aligned}
$$

Because of

$$
\begin{aligned}
&\left(h_1-g_{1u}(u,v)\omega-g_{1v}(u,v)\gamma,\theta\right)\\
&\le 2c_1\|\theta\|\left\|(\tilde{u},\tilde{v})-(u,v)\right\|_{E^*}^{\kappa_1+1}+c_2\|\omega\|\|\theta\|+c_3\|\gamma\|\|\theta\|,\\
&\left(h_2-g_{2u}(u,v)\omega-g_{2v}(u,v)\gamma,\delta\right) \qquad (3.15)\\
&\le 2c_1'\|\delta\|\left\|(\tilde{u},\tilde{v})-(u,v)\right\|_{E^*}^{\kappa_2+1}+c_2'\|\omega\|\|\delta\|+c_3'\|\gamma\|\|\delta\|.
\end{aligned}
$$

$$
\begin{aligned}
J_1&=\left(M(s)-M(\tilde{s})\right)A^m\tilde{u}+2M'(s)\left(\left(A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)+\left(A^{\frac{m}{2}}V,A^{\frac{m}{2}}v\right)\right)A^mu\\
&=-2M'(s)\left(\left(-A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)+\left(-A^{\frac{m}{2}}V,A^{\frac{m}{2}}v\right)\right)A^mu+M'\left(\alpha s+(1-\alpha)\tilde{s}\right)\\
&\quad\times\left[\left(D^m(u-\tilde{u}),D^m(\tilde{u}+u)\right)+\left(D^m(v-\tilde{v}),D^m(\tilde{v}+v)\right)\right]A^m\tilde{u}\\
&=\left(M'\left(\alpha s+(1-\alpha)\tilde{s}\right)-M'(s)\right)\\
&\quad\times\left[\left(D^m(u-\tilde{u}),D^m(\tilde{u}+u)\right)+\left(D^m(v-\tilde{v}),D^m(\tilde{v}+v)\right)\right]A^m\tilde{u}\\
&\quad+M'(s)\left[\left(D^m(u-\tilde{u}),D^m(\tilde{u}+u)\right)+\left(D^m(v-\tilde{v}),D^m(\tilde{v}+v)\right)\right]A^m\tilde{u}\\
&\quad-2M'(s)\left[\left(D^m(u-\tilde{u}),D^mu\right)+\left(D^m(v-\tilde{v}),D^mv\right)\right]A^m\tilde{u}\\
&\quad+2M'(s)\left[\left(D^m(u-\tilde{u}),D^mu\right)+\left(D^m(v-\tilde{v}),D^mv\right)\right]A^m\tilde{u}\\
&\quad-2M'(s)\left[\left(D^m(u-\tilde{u}),D^mu\right)+\left(D^m(v-\tilde{v}),D^mv\right)\right]A^mu\\
&\quad-2M'(s)\left(\left(D^m\omega,D^mu\right)+\left(D^m\gamma,D^mv\right)\right)A^mu
\end{aligned}
$$

$$\begin{aligned}
&\le M''(\xi)\Big[\big(D^m(u-\tilde{u}),D^m(\tilde{u}+u)\big)+\big(D^m(v-\tilde{v}),D^m(\tilde{v}+v)\big)\Big]^2 A^m\tilde{u}\\
&\quad+M'(s)\Big[\big(D^m(u-\tilde{u}),D^m(\tilde{u}-u)\big)+\big(D^m(v-\tilde{v}),D^m(\tilde{v}-v)\big)\Big]A^m\tilde{u}\\
&\quad+2M'(s)\Big[\big(D^m(u-\tilde{u}),D^m u\big)+\big(D^m(v-\tilde{v}),D^m v\big)\Big]A^m(\tilde{u}-u) \qquad (3.16)\\
&\quad-2M'(s)\Big(\big(D^m\omega,D^m u\big)+\big(D^m\gamma,D^m v\big)\Big)A^m u\\
&=I_1+I_2+I_3+I_4.
\end{aligned}$$

Taking the scalar product of right side of (3.16) with θ, and then we obtain

$$\begin{aligned}
&(I_1,\theta)\\
&\le\Big(2M''(\xi)\Big[\big(D^m(u-\tilde{u}),D^m(\tilde{u}+u)\big)^2+\big(D^m(v-\tilde{v}),D^m(\tilde{v}+v)\big)^2\Big]D^m\tilde{u},D^m\theta\Big)\\
&\le 2c_4\left(\frac{4}{\varepsilon^2}\left\|D^m(\tilde{u}-u)\right\|^4+\frac{4}{\varepsilon^2}\left\|D^m(\tilde{v}-v)\right\|^4+\frac{\varepsilon^2\left\|D^m\theta\right\|^2}{8}\right),
\end{aligned}$$

$$\begin{aligned}
&(I_2,\theta)\\
&=\Big(M'(s)\Big[\big(D^m(u-\tilde{u}),D^m(\tilde{u}-u)\big)+\big(D^m(v-\tilde{v}),D^m(\tilde{v}-v)\big)\Big]D^m\tilde{u},D^m\theta\Big)\\
&\le c_5\left(\frac{4}{\varepsilon^2}\left\|D^m(\tilde{u}-u)\right\|^4+\frac{4}{\varepsilon^2}\left\|D^m(\tilde{v}-v)\right\|^4+\frac{\varepsilon^2\left\|D^m\theta\right\|^2}{8}\right),
\end{aligned}$$

$$\begin{aligned}
(I_3,\theta)&\le\Big(2M'(s)\Big[\big(D^m(u-\tilde{u}),D^m u\big)+\big(D^m(v-\tilde{v}),D^m v\big)\Big]D^m(\tilde{u}-u),D^m\theta\Big)\\
&\le 2c_6\left(\frac{8}{\varepsilon^2}\left\|D^m(\tilde{u}-u)\right\|^4+\frac{4}{\varepsilon^2}\left\|D^m(\tilde{v}-v)\right\|^4+\frac{3\varepsilon^2\left\|D^m\theta\right\|^2}{16}\right),
\end{aligned}$$

$$\begin{aligned}
(I_4,\theta)&=\Big(-2M'(s)\Big[\big(D^m\omega,D^m u\big)+\big(D^m\gamma,D^m v\big)\Big]D^m u,D^m\theta\Big)\\
&\le 2c_7\left(\frac{4\left\|D^m\omega\right\|^2}{\varepsilon^2}+\frac{4\left\|D^m\gamma\right\|^2}{\varepsilon^2}+\frac{\varepsilon^2\left\|D^m\theta\right\|^2}{8}\right),
\end{aligned}$$

which implies that

$$\left|(J_1,\theta)\right|\le\frac{c_8}{\varepsilon^2}\Big(\left\|D^m(\tilde{u}-u)\right\|^4+\left\|D^m(\tilde{v}-v)\right\|^4+\left\|D^m\omega\right\|^2+\left\|D^m\gamma\right\|^2\Big)+c_9\varepsilon^2\left\|D^m\theta\right\|^2. \qquad (3.17)$$

Analogously,

$$(I_1',\delta)\le 2c_4'\left(\frac{4}{\varepsilon^2}\left\|D^m(\tilde{u}-u)\right\|^4+\frac{4}{\varepsilon^2}\left\|D^m(\tilde{v}-v)\right\|^4+\frac{\varepsilon^2\left\|D^m\delta\right\|^2}{8}\right),$$

$$(I_2',\delta)\le c_5'\left(\frac{4}{\varepsilon^2}\left\|D^m(\tilde{u}-u)\right\|^4+\frac{4}{\varepsilon^2}\left\|D^m(\tilde{v}-v)\right\|^4+\frac{\varepsilon^2\left\|D^m\delta\right\|^2}{8}\right),$$

$$(I_3',\delta)\le 2c_6'\left(\frac{4}{\varepsilon^2}\left\|D^m(\tilde{u}-u)\right\|^4+\frac{8}{\varepsilon^2}\left\|D^m(\tilde{v}-v)\right\|^4+\frac{3\varepsilon^2\left\|D^m\delta\right\|^2}{16}\right),$$

$$(I_4',\delta)\le 2c_7'\left(\frac{4\|D^m\omega\|^2}{\varepsilon^2}+\frac{4\|D^m\gamma\|^2}{\varepsilon^2}+\frac{\varepsilon^2\|D^m\delta\|^2}{8}\right),$$

which means that

$$\left|(J_1',\delta)\right|\le\frac{c_8'}{\varepsilon^2}\left(\|D^m(\tilde{u}-u)\|^4+\|D^m(\tilde{v}-v)\|^4+\|D^m\omega\|^2+\|D^m\gamma\|^2\right)+c_9'\varepsilon^2\|D^m\delta\|^2. \tag{3.18}$$

Taking (3.15)-(3.18) into (3.14), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\theta\|^2+\|\delta\|^2+\varepsilon^2\left(\|\omega\|^2+\|\gamma\|^2\right)+\left(m^*-\beta\varepsilon\right)\left(\left\|A^{\frac{m}{2}}\omega\right\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2\right)\right\}\\&+\left(\frac{\beta}{2}\lambda_1^m-\varepsilon\right)\left(\|\theta\|^2+\|\delta\|^2\right)+\varepsilon^3\left(\|\omega\|^2+\|\gamma\|^2\right)+\varepsilon\left(m_0-\beta\varepsilon\right)\left(\left\|A^{\frac{m}{2}}\omega\right\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2\right)\\&\le c_{11}\left(\left\|A^{\frac{m}{2}}\omega\right\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2+\|\theta\|^2+\|\delta\|^2\right)+\frac{c_{11}'}{\varepsilon^2}\left(\|D^m(\tilde{u}-u)\|^4+\|D^m(\tilde{v}-v)\|^4\right)\\&\quad+c_{10}\left(\|D^m(\tilde{u}-u)\|^{2(\kappa_1+1)}+\|D^m(\tilde{v}-v)\|^{2(\kappa_1+1)}\right)\\&\quad+c_{10}'\left(\|D^m(\tilde{u}-u)\|^{2(\kappa_2+1)}+\|D^m(\tilde{v}-v)\|^{2(\kappa_2+1)}\right).\end{aligned} \tag{3.19}$$

Let $k'=\min\left\{\beta\lambda_1^m-2\varepsilon,2\varepsilon,\frac{2\varepsilon(m_0-\beta\varepsilon)}{m^*-\beta\varepsilon}\right\}>0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\theta\|^2+\|\delta\|^2+\|\omega\|^2+\|\gamma\|^2+\left\|A^{\frac{m}{2}}\omega\right\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2\right\}\\&+k'\left(\|\theta\|^2+\|\delta\|^2+\|\omega\|^2+\|\gamma\|^2+\left\|A^{\frac{m}{2}}\omega\right\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2\right)\\&\le c_{11}\left(\left\|A^{\frac{m}{2}}\omega\right\|^2+\|\theta\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2+\|\delta\|^2\right)+\frac{c_{11}'}{\varepsilon^2}\left(\|D^m(\tilde{u}-u)\|^4+\|D^m(\tilde{v}-v)\|^4\right)\\&\quad+c_{10}\left(\|D^m(\tilde{u}-u)\|^{2(\kappa_1+1)}+\|D^m(\tilde{v}-v)\|^{2(\kappa_1+1)}\right)\\&\quad+c_{10}'\left(\|D^m(\tilde{u}-u)\|^{2(\kappa_2+1)}+\|D^m(\tilde{v}-v)\|^{2(\kappa_2+1)}\right),\end{aligned} \tag{3.20}$$

applying the Gronwall inequality and (3.6) we deduce from (3.20) that

$$\begin{aligned}&\left\{\left\|A^{\frac{m}{2}}\omega\right\|^2+\|\theta\|^2+\left\|A^{\frac{m}{2}}\gamma\right\|^2+\|\delta\|^2\right\}\\&\le\frac{c_{12}'}{c_{11}}\exp(c_{11}t)\int_0^t\Big(\|D^m(\tilde{u}-u)\|^4+\|D^m(\tilde{v}-v)\|^4+\|D^m(\tilde{u}-u)\|^{2(\kappa_1+1)}\\&\quad+\|D^m(\tilde{v}-v)\|^{2(\kappa_1+1)}+\|D^m(\tilde{u}-u)\|^{2(\kappa_2+1)}+\|D^m(\tilde{v}-v)\|^{2(\kappa_2+1)}\Big)\mathrm{d}\tau\\&\le c_{13}'\exp(c_{12}t)\left(\left\{\|(\xi,\zeta,\eta,\sigma)\|_{E_0}^2\right\}^2+\left\{\|(\xi,\zeta,\eta,\sigma)\|_{E_0}^2\right\}^{\kappa_1+1}+\left\{\|(\xi,\zeta,\eta,\sigma)\|_{E_0}^2\right\}^{\kappa_2+1}\right).\end{aligned} \tag{3.21}$$

This is equivalent to

$$\begin{aligned}&\left\|S(t)\tilde{\rho}_0-S(t)\rho_0-\left(DS(t)\rho_0\right)(\xi,\zeta,\eta,\sigma)\right\|_{E_0}^2\\&\le c_{13}'\exp(c_{12}t)\left(\left\|(\xi,\zeta,\eta,\sigma)\right\|_{E_0}^4+\left\|(\xi,\zeta,\eta,\sigma)\right\|_{E_0}^{2(\kappa_1+1)}+\left\|(\xi,\zeta,\eta,\sigma)\right\|_{E_0}^{2(\kappa_2+1)}\right),\end{aligned}\tag{3.22}$$

and consequently $\dfrac{\left\|S(t)\tilde{\rho}_0-S(t)\rho_0-\left(DS(t)\rho_0\right)(\xi,\zeta,\eta,\sigma)\right\|_{E_0}^2}{\left\|(\xi,\zeta,\eta,\sigma)\right\|_{E_0}^2}\to 0$ as $(\xi,\zeta,\eta,\sigma)\to 0$ in E_0.

The differentiability of $S(t)$ is proved.

The next step will be used in demonstrating the process of dimension estimation. It seems obvious that the equations (1.1)-(1.2) also can be written as

$$\varphi_t+H(\varphi)=F(\varphi),\tag{3.23}$$

$$H(\varphi)=\begin{pmatrix}\varepsilon u-p\\ A^m u-\varepsilon\left(\beta A^m-\varepsilon\right)u+\left(\beta A^m-\varepsilon\right)p\\ \varepsilon v-q\\ A^m v-\varepsilon\left(\beta A^m-\varepsilon\right)v+\left(\beta A^m-\varepsilon\right)q\end{pmatrix},$$

$$F(\varphi)=\begin{pmatrix}0\\ f_1(x)-g_1(u,v)+\left(1-M\left(\left\|A^{\frac{m}{2}}u\right\|^2+\left\|A^{\frac{m}{2}}v\right\|^2\right)\right)A^m u\\ 0\\ f_2(x)-g_2(u,v)+\left(1-M\left(\left\|A^{\frac{m}{2}}u\right\|^2+\left\|A^{\frac{m}{2}}v\right\|^2\right)\right)A^m v\end{pmatrix}.\tag{3.24}$$

Lemma 3.4. For any $\varphi=(u_1,p_1,v_1,q_1)^{\mathrm{T}}\in E_0$, we have

$$\left(H(\varphi),\varphi\right)\ge\frac{\varepsilon}{2}\|\varphi\|_{E_0}^2+\frac{\beta}{4}\left(\left\|A^{\frac{m}{2}}p\right\|^2+\left\|A^{\frac{m}{2}}q\right\|^2\right).\tag{3.25}$$

Proof. For any $\varphi=(u_1,p_1,v_1,q_1)^{\mathrm{T}}\in E_0$, through the above definition, we get

$$\begin{aligned}\left(H(\varphi),\varphi\right)_{E_0}&=\left(\varepsilon A^{\frac{m}{2}}u-A^{\frac{m}{2}}p,A^{\frac{m}{2}}u\right)+\left(\varepsilon A^{\frac{m}{2}}v-A^{\frac{m}{2}}q,A^{\frac{m}{2}}v\right)\\&\quad+\left(A^m u-\varepsilon\left(\beta A^m-\varepsilon\right)u+\left(\beta A^m-\varepsilon\right)p,p\right)\\&\quad+\left(A^m v-\varepsilon\left(\beta A^m-\varepsilon\right)v+\left(\beta A^m-\varepsilon\right)q,q\right)\\&=\varepsilon\left(\left\|A^{\frac{m}{2}}u\right\|^2+\left\|A^{\frac{m}{2}}v\right\|^2\right)+\beta\left(\left\|A^{\frac{m}{2}}p\right\|^2+\left\|A^{\frac{m}{2}}q\right\|^2\right)-\varepsilon\left(\|p\|^2+\|q\|^2\right)\\&\quad-\beta\varepsilon\left(\left(A^m u,p\right)+\left(A^m v,q\right)\right)+\varepsilon^2\left(\left(u,p\right)+\left(v,q\right)\right).\end{aligned}\tag{3.26}$$

By applying the Holder inequality, Young's inequality and Poincare inequality, we deal with the terms in (3.26) by as follows

$$-\beta\varepsilon\left(\left(A^m u,p\right)+\left(A^m v,q\right)\right)\geq-\frac{\beta^2\varepsilon^2}{2}\left(\left\|A^{\frac{m}{2}}u\right\|^2+\left\|A^{\frac{m}{2}}v\right\|^2+\left\|A^{\frac{m}{2}}p\right\|^2+\left\|A^{\frac{m}{2}}q\right\|^2\right), \quad (3.27)$$

$$\varepsilon^2\left(\left(u,p\right)+\left(v,q\right)\right)\geq-\frac{\varepsilon^2}{2\lambda_1^m}\left(\left\|A^{\frac{m}{2}}u\right\|^2+\left\|A^{\frac{m}{2}}v\right\|^2\right)-\frac{\varepsilon^2}{2}\left(\|p\|^2+\|q\|^2\right), \quad (3.28)$$

with $0<\varepsilon<\min\left\{\sqrt{\frac{3}{2\beta}},\frac{\lambda_1^m}{1+\beta^2\lambda_1^m},\frac{-3+\sqrt{9+6\left(1+\beta^2\lambda_1^m\right)\beta\lambda_1^m}}{2\left(1+\beta^2\lambda_1^m\right)}\right\}$, and substituting (3.27)-(3.28) into (3.26), we obtain

$$\begin{aligned}\left(H(\varphi),\varphi\right)&\geq\left(\varepsilon-\frac{\beta^2\varepsilon^2}{2}-\frac{\varepsilon^2}{2\lambda_1^m}\right)\left(\left\|A^{\frac{m}{2}}u\right\|^2+\left\|A^{\frac{m}{2}}v\right\|^2\right)\\&\quad+\left(\beta-\frac{\beta^2\varepsilon^2}{2}\right)\left(\left\|A^{\frac{m}{2}}p\right\|^2+\left\|A^{\frac{m}{2}}q\right\|^2\right)+\left(-\varepsilon-\frac{\varepsilon^2}{2}\right)\left(\|p\|^2+\|q\|^2\right)\\&\geq\frac{\varepsilon}{2}\left(\left\|A^{\frac{m}{2}}u\right\|^2+\left\|A^{\frac{m}{2}}v\right\|^2+\|p\|^2+\|q\|^2\right)+\frac{\beta}{4}\left(\left\|A^{\frac{m}{2}}p\right\|^2+\left\|A^{\frac{m}{2}}q\right\|^2\right)\\&\geq\frac{\varepsilon}{2}\|\varphi\|_{E_0}^2+\frac{\beta}{4}\left(\left\|A^{\frac{m}{2}}p\right\|^2+\left\|A^{\frac{m}{2}}q\right\|^2\right).\end{aligned} \quad (3.29)$$

The proof of lemma 3.4 is completed.

Consider the first variation equation of (3.23)

$$\Psi'+P(\varphi)\Psi=\Gamma_1(\varphi)\Psi+\Gamma_2(\varphi)\Psi \quad (3.30)$$

where $\Psi=(U,P,V,Q)^{\mathrm{T}}\in E_0$, $P=U_t+\varepsilon U$, $Q=V_t+\varepsilon V$ and $\varphi=(u,p,v,q)^{\mathrm{T}}\in E_0$ is a solution of (3.23), $\Psi(0)=\{\xi,\zeta,\eta,\sigma\}\in E_0, t>0$.

$$P(\varphi)=\begin{pmatrix}\varepsilon I & -I & 0 & 0\\ (1-\beta\varepsilon)A^m+\varepsilon^2 I & \beta A^m-\varepsilon I & 0 & 0\\ 0 & 0 & \varepsilon I & -I\\ 0 & 0 & (1-\beta\varepsilon)A^m+\varepsilon^2 I & \beta A^m-\varepsilon I\end{pmatrix}, \quad (3.31)$$

$$\Gamma_1(\varphi)=\begin{pmatrix}0 & 0 & 0 & 0\\ -g_{1u}(u,v) & 0 & -g_{1v}(u,v) & 0\\ 0 & 0 & 0 & 0\\ -g_{2u}(u,v) & 0 & -g_{2v}(u,v) & 0\end{pmatrix}, \quad (3.32)$$

$$\Gamma_2(\varphi)=\begin{pmatrix}0\\ (1-M(s))A^m U-2M'(s)\left(A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)A^m u-2M'(s)\left(A^{\frac{m}{2}}V,A^{\frac{m}{2}}v\right)A^m u\\ 0\\ (1-M(s))A^m V-2M'(s)\left(A^{\frac{m}{2}}U,A^{\frac{m}{2}}u\right)A^m v-2M'(s)\left(A^{\frac{m}{2}}V,A^{\frac{m}{2}}v\right)A^m v\end{pmatrix}. \quad (3.33)$$

Lemma 3.5. ([2]) Let there be given V,H, and A as above. Then for any $s, 0\le s<1$, and for any orthonormal family of elements of $V\times H$, $\{\xi_i,\zeta_i\}, i=1,\cdots,m$, we have

$$\sum_i^m \left|A^{s/2}\xi_i\right|^2 \le \sum_i^m \lambda_i^{s-1}, \tag{3.34}$$

where $\{\lambda_i\}_{i\in N}$ is the eigenvalue of A^m.

Proof. This is a direct consequence for Lemma 6.3 of [2].

Theorem 3.1. Let A be the global attractor of problem (1.1)-(1.5), then the Hausdorff dimension of global attractor A is less than or equal to n_0 and its fractal dimension is less than or equal to $2n_0$.

$$d_H(A)\le \min\left\{n_0 \left| n_0\in N, \frac{1}{n_0}\sum_{j=1}^{n_0}\lambda_j^{\upsilon-1} < \frac{\beta\varepsilon}{16k^2}\right.\right\},$$

$$d_F(A)\le 2n_0. \tag{3.35}$$

$$\upsilon=\begin{cases}\dfrac{(n-2m)(r-1)-2m}{2m}, \dfrac{n}{n-2m}\le r\le \dfrac{n+2m}{n-2m}, n\ge 2m,\\ 0,\ \ n<2m \text{ or } 0\le r\le \dfrac{n}{n-2m}, n\ge 2m.\end{cases} (\text{or } r') \tag{3.36}$$

Proof. Let $n_0\in N$ be settled. Consider n_0 solutions $\Psi_1,\Psi_2,\cdots,\Psi_{n_0}$ of (3.30), and we memorize that

$$\begin{aligned}&\left|\Psi_1(t)\wedge\cdots\wedge\Psi_{n_0}(t)\right|_{\wedge^{n_0}E_0}\\ &=\left|\Psi_1(0)\wedge\cdots\wedge\Psi_{n_0}(0)\right|_{\wedge^{n_0}E_0}\exp\int_0^t \operatorname{Tr}F'(\varphi(\tau))\circ Q_{n_0}(\tau)\mathrm{d}\tau,\end{aligned}$$

we see that $\varphi(\tau)=(u(\tau),p(\tau),v(\tau),q(\tau))$, $Q_{n_0}(\tau)=Q_{n_0}(\tau,\varphi_0;\Psi_1(0),\cdots,\Psi_{n_0}(0))$ is the orthogonal projection in E_0 onto the space spanned by $\Psi_1(\tau),\cdots,\Psi_m(\tau)$. At a given time τ, let $h_j(\tau)=\{\xi_j,\zeta_j,\eta_j,\sigma_j\}, j=1,\cdots,n_0$, denote an orthonormal basis of

$$Q_{n_0}(\tau)E_0=Span\{\Psi_1(\tau),\Psi_2(\tau),\cdots,\Psi_{n_0}(\tau)\},$$

$$\begin{aligned}\operatorname{Tr}F'(\varphi(\tau))\circ Q_{n_0}(\tau)&=\sum_{j=1}^{+\infty}\left(\Gamma'(\varphi(\tau))\circ Q_{n_0}(\tau)h_j(\tau),h_j(\tau)\right)_{E_0}\\ &=\sum_{j=1}^{n_0}\left(\Gamma'(\varphi(\tau))h_j(\tau),h_j(\tau)\right)_{E_0}.\end{aligned}$$

With respect to the scalar product $(\cdot,\cdot)_{E_0}$ and norm $\|\cdot\|_{E_0}$, we omit for the moment variable τ,

$$\left(h_j,\tilde h_j\right)_{E_0}=\left(\xi_j,\tilde\xi_j\right)+\left(\zeta_j,\tilde\zeta_j\right)+\left(\eta_j,\tilde\eta_j\right)+\left(\sigma_j,\tilde\sigma_j\right),$$

$$\left\|h_j\right\|_{E_0}^2=\left(h_j,h_j\right)_{E_0}=\left\|A^{\frac{m}{2}}\xi_j\right\|^2+\left\|\zeta_j\right\|^2+\left\|A^{\frac{m}{2}}\eta_j\right\|^2+\left\|\sigma_j\right\|^2=1.$$

By the Lemma 3.4, we have

$$-\left(P(\varphi)h_j,h_j\right)\le-\frac{\varepsilon}{2}-\frac{\beta}{4}\left(\left\|A^{\frac{m}{2}}\zeta_j\right\|^2+\left\|A^{\frac{m}{2}}\sigma_j\right\|^2\right), \tag{3.37}$$

$$\begin{aligned}\left(\Gamma_1(\varphi)h_j,h_j\right)_{E_0} &= \left(-g_{1u}(u,v)\xi_j,\zeta_j\right)+\left(-g_{1v}(u,v)\eta_j,\zeta_j\right)\\ &\quad+\left(-g_{2u}(u,v)\xi_j,\sigma_j\right)+\left(-g_{2v}(u,v)\eta_j,\sigma_j\right)\\ &\le \left\|A^{-\frac{m}{2}}g_{1u}(u,v)\xi_j\right\|\left\|A^{\frac{m}{2}}\zeta_j\right\|+\left\|A^{-\frac{m}{2}}g_{1v}(u,v)\eta_j\right\|\left\|A^{\frac{m}{2}}\zeta_j\right\|\\ &\le \left\|A^{-\frac{m}{2}}g_{2u}(u,v)\xi_j\right\|\left\|A^{\frac{m}{2}}\sigma_j\right\|+\left\|A^{-\frac{m}{2}}g_{2v}(u,v)\eta_j\right\|\left\|A^{\frac{m}{2}}\sigma_j\right\|.\end{aligned} \tag{3.38}$$

By the assumption (H3) in [1], the mean value theorem and the Sobolev embedding theorem

$$H_0^{ms}(\Omega)\subset D\left(A^{\frac{ms}{2}}\right)\subset H^{ms}(\Omega)\subset L^q(\Omega)\subset L^2(\Omega)\subset L^{q'}(\Omega)\subset H^{ms}(\Omega)$$

where

$$\frac{1}{q}=\frac{1}{2}-\frac{ms}{n},\quad \frac{1}{q}+\frac{1}{q'}=1,\quad ms\in[0,1].$$

For $n=1$, $H_0^m(\Omega)\subset L^\infty(\Omega)\subset L^1(\Omega)\subset H^{-m}(\Omega)\subset\left(H_0^m(\Omega)\right)'$, we can easily obtain that

$$\left\|A^{-\frac{m}{2}}g_{1u}(u,v)\xi_j\right\|\le c\left\|g_{1u}(u,v)\xi_j\right\|_{L^1}\le c_{13}(R_0)\left\|\xi_j\right\|,$$

$$\left\|A^{-\frac{m}{2}}g_{1v}(u,v)\eta_j\right\|\le c\left\|g_{1v}(u,v)\eta_j\right\|_{L^1}\le c_{14}(R_0)\left\|\eta_j\right\|,$$

$$\left\|A^{-\frac{m}{2}}g_{2u}(u,v)\xi_j\right\|\le c\left\|g_{2u}(u,v)\xi_j\right\|_{L^1}\le c_{15}(R_0)\left\|\xi_j\right\|,$$

$$\left\|A^{-\frac{m}{2}}g_{2v}(u,v)\eta_j\right\|\le c\left\|g_{2v}(u,v)\eta_j\right\|_{L^1}\le c_{16}(R_0)\left\|\eta_j\right\|. \tag{3.39}$$

For $1<n<2m$, $H_0^m(\Omega)\subset L^q(\Omega)\subset H^{-m}(\Omega)\subset\left(H_0^m(\Omega)\right)', q>0$, we have

$$\left\|A^{-\frac{m}{2}}g_{1u}(u,v)\xi_j\right\|\le c\left\|g_{1u}(u,v)\xi_j\right\|_{L^{\frac{3}{2}}}\le c_{17}(R_0)\left\|\xi_j\right\|,$$

$$\left\|A^{-\frac{m}{2}}g_{1v}(u,v)\eta_j\right\|\le c\left\|g_{1v}(u,v)\eta_j\right\|_{L^{\frac{3}{2}}}\le c_{18}(R_0)\left\|\eta_j\right\|,$$

$$\left\|A^{-\frac{m}{2}}g_{2u}(u,v)\xi_j\right\|\le c\left\|g_{2u}(u,v)\xi_j\right\|_{L^{\frac{3}{2}}}\le c_{19}(R_0)\left\|\xi_j\right\|,$$

$$\left\|A^{-\frac{m}{2}}g_{2v}(u,v)\eta_j\right\|\le c\left\|g_{2v}(u,v)\eta_j\right\|_{L^{\frac{3}{2}}}\le c_{20}(R_0)\left\|\eta_j\right\|. \tag{3.40}$$

For $n\ge 2m$, there exist $c_{21}(R_0),c_{22}(R_0),c_{23}(R_0),c_{24}(R_0)>0$, such that

$$\left\|A^{-\frac{m}{2}}g_{1u}(u,v)\xi_j\right\|\le c\left\|g_{1u}(u,v)\xi_j\right\|_{L^{\frac{2n}{n+2m}}}\le c_{21}(R_0)\left\|A^{\frac{m}{2}\upsilon}\xi_j\right\|,$$

$$\left\| A^{-\frac{m}{2}} g_{1v}(u,v)\eta_j \right\| \le c \left\| g_{1v}(u,v)\eta_j \right\|_{L^{\frac{2n}{n+2m}}} \le c_{22}(R_0) \left\| A^{\frac{m}{2}\upsilon} \eta_j \right\|,$$

$$\left\| A^{-\frac{m}{2}} g_{2u}(u,v)\xi_j \right\| \le c \left\| g_{2u}(u,v)\xi_j \right\|_{L^{\frac{2n}{n+2m}}} \le c_{23}(R_0) \left\| A^{\frac{m}{2}\upsilon} \xi_j \right\|,$$

$$\left\| A^{-\frac{m}{2}} g_{2v}(u,v)\eta_j \right\| \le c \left\| g_{2v}(u,v)\eta_j \right\|_{L^{\frac{2n}{n+2m}}} \le c_{24}(R_0) \left\| A^{\frac{m}{2}\upsilon} \eta_j \right\| \tag{3.41}$$

where υ is as in (3.36), then setting $k = 2\max\{c_i(R_0), i = 13 \sim 24\}$

$$\left(\Gamma_1(\varphi)h_j, h_j\right)_{E_0} \le \frac{k}{2}\left(\left\| A^{\frac{m}{2}\upsilon}\xi_j \right\| + \left\| A^{\frac{m}{2}\upsilon}\eta_j \right\| \right)\left(\left\| A^{\frac{m}{2}}\zeta_j \right\| + \left\| A^{\frac{m}{2}}\sigma_j \right\| \right), \tag{3.42}$$

$$\begin{aligned}
&\left(\Gamma_2(\varphi)h_j, h_j\right)_{E_0} \\
&= \left(1 - M(s)\right)\left(A^m\xi_j, \zeta_j\right) - 2M'(s)\left(A^{\frac{m}{2}}\xi_j, A^{\frac{m}{2}}u \right)\left(A^m u, \zeta_j\right) \\
&\quad - 2M'(s)\left(A^{\frac{m}{2}}\eta_j, A^{\frac{m}{2}}v \right)\left(A^m u, \zeta_j\right) + \left(1 - M(s)\right)\left(A^m\eta_j, \sigma_j\right) \\
&\quad - 2M'(s)\left(A^{\frac{m}{2}}\xi_j, A^{\frac{m}{2}}u \right)\left(A^m v, \sigma_j\right) - 2M'(s)\left(A^{\frac{m}{2}}\eta_j, A^{\frac{m}{2}}v \right)\left(A^m v, \sigma_j\right) \\
&\le \left(1 - m_0\right)\lambda_j^{\frac{m}{2}}\left(\left\| A^{\frac{m}{2}}\xi_j \right\| \left\| \zeta_j \right\| + \left\| A^{\frac{m}{2}}\eta_j \right\| \left\| \sigma_j \right\| \right) \\
&\quad + 2c_{25}\left(\left\| A^{\frac{m}{2}}\xi_j \right\| \left\| \zeta_j \right\| + \left\| A^{\frac{m}{2}}\eta_j \right\| \left\| \zeta_j \right\| \right) + 2c_{25}\left(\left\| A^{\frac{m}{2}}\xi_j \right\| \left\| \sigma_j \right\| + \left\| A^{\frac{m}{2}}\eta_j \right\| \left\| \sigma_j \right\| \right) \\
&\le \frac{\left(1 - m_0\right)}{2}\lambda_j^{\frac{m}{2}}\left(\left\| A^{\frac{m}{2}}\xi_j \right\|^2 + \left\| \zeta_j \right\|^2 + \left\| A^{\frac{m}{2}}\eta_j \right\|^2 + \left\| \sigma_j \right\|^2 \right) \\
&\quad + 2c_{25}\left(\left\| A^{\frac{m}{2}}\xi_j \right\|^2 + \left\| \zeta_j \right\|^2 + \left\| A^{\frac{m}{2}}\eta_j \right\|^2 + \left\| \zeta_j \right\|^2 \right) \\
&\le \frac{\left(1 - m_0\right)}{2}\lambda_j^{\frac{m}{2}} + 2c_{25},
\end{aligned} \tag{3.43}$$

let $0 < \alpha = \left(\left\| \zeta_j \right\|^2 + \left\| \sigma_j \right\|^2 \right) < 1$,

$$\begin{aligned}
&\left(\Gamma'(\varphi)h_j, h_j\right)_{E_0} \\
&= \left(\left(-P(\varphi) + \Gamma_1(\varphi) + \Gamma_2(\varphi) \right)h_j, h_j \right) \\
&\le \left(-\frac{\varepsilon}{2} - \frac{\alpha\beta}{8}\lambda_j^m + \frac{\left(1 - m_0\right)}{2}\lambda_j^{\frac{m}{2}} + 2c_{25} \right) + \frac{2k^2}{\beta}\left(\left\| A^{\frac{m}{2}\upsilon}\xi_j \right\|^2 + \left\| A^{\frac{m}{2}\upsilon}\eta_j \right\|^2 \right).
\end{aligned} \tag{3.44}$$

If $-\frac{\alpha\beta}{8}\lambda_j^m + \frac{\left(1 - m_0\right)}{2}\lambda_j^{\frac{m}{2}} < \frac{\varepsilon}{4}$,

$$\sum_{j=1}^{n_0}\left(\Gamma'(\varphi)h_j,h_j\right)_{E_0}=\sum_{j=1}^{n_0}\left(\left(-P(\varphi)+\Gamma_1(\varphi)+\Gamma_2(\varphi)\right)h_j,h_j\right)$$
$$=-\frac{\varepsilon}{4}n_0+\frac{4k^2}{\beta}\sum_{j}^{n_0}\lambda_j^{\upsilon-1}, \tag{3.45}$$

and if $\frac{1}{n_0}\sum_{j=1}^{n_0}\lambda_j^{\upsilon-1}<\frac{\beta\varepsilon}{16k^2}$,

$$q_{n_0}(t)=\sup_{\varphi_0\in\mathcal{A}}\sup_{\substack{\Psi_j(0)\in E_0\\ \|\Psi_j(0)\|_{E_0}\le 1\\ j=1,\cdots,n_0}}\left\{\frac{1}{t}\int_0^t \operatorname{Tr}F'(\varphi(\tau))\circ Q_{n_0}(\tau)\,\mathrm{d}\tau\right\}$$
$$\le -\frac{k^2 n_0}{4}\left(\frac{\varepsilon}{k^2}-\frac{16}{\beta n_0}\sum_{j=1}^{n_0}\lambda_j^{\upsilon-1}\right) \tag{3.46}$$
$$q_{n_0}=\limsup_{t\to\infty}q_{n_0}(t)<0.$$

The proof of theorem 3.1 is completed.

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