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The Analyticity for the Product of Analytic Functions on Octonions and Its Applications

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Abstract

Given two left $\mathbf{O}^c$ -analytic functions f, g in some open set Ω of $\mathbf{R}^8$, we obtain some sufficient conditions for fg is also left $\mathbf{O}^c$ -analytic in Ω. Moreover, we prove that $f\lambda$ is a left $\mathbf{O}^c$ -analytic function for any constants $\lambda \in \mathbf{O}^c$ if and only if $\overline{f}$ is a complex Stein-Weiss conjugate harmonic system. Some applications and connections with Cauchy-Kowalewski product are also considered.

Keywords

Octonions, $\mathbf{O}^c$ -Analytic Functions, Stein-Weiss Conjugate Harmonic System, Cauchy-Kowalewski Product

1. Introduction

Let Ω be an open set of $\mathbf{R}^8$. A function f in $C^1(\Omega, \mathbf{O})$ is said to be left (right) $\mathbf{O}$ -analytic in Ω when

$$Df = \sum_{i=0}^{7} e_i \frac{\partial f}{\partial x_i} = 0 \quad \left(fD = \sum_{i=0}^{7} \frac{\partial f}{\partial x_i} e_i = 0 \right),$$

where the Dirac D-operator and its adjoint $\bar{D}$ are the first-order systems of differential operators in $C^1(\Omega, \mathbf{O})$ defined by $D = \sum_0^7 e_i \frac{\partial}{\partial x_i}$ and $\bar{D} = e_0 \frac{\partial}{\partial x_0} - \sum_1^7 e_i \frac{\partial}{\partial x_i}$.

If f is a simultaneously left and right $\mathbf{O}$ -analytic function, then f is called an $\mathbf{O}$ -analytic function. If f is a (left) $\mathbf{O}$ -analytic function in $\mathbf{R}^8$, then f is called a (left) $\mathbf{O}$ -entire function.

Since octonions is non-commutative and non-associative, the product $f(x)g(x)$ of two left $\mathbf{O}$-analytic functions $f(x)$ and $g(x)$ is generally no longer a left $\mathbf{O}$-analytic function. Furthermore, if $g(x)\equiv\lambda$ becomes an octonionic constant function, the product $f(x)\lambda$ is also probably not a left $\mathbf{O}$-analytic function; that is, the collection of left $\mathbf{O}$-analytic functions is not a right module (see [1]).

The purpose of this paper is to study the analyticity for the product of two left $\mathbf{O}^c$-analytic functions in the framework of complexification of $\mathbf{O}$, $\mathbf{O}^c$. Especially, the analyticity for the product of left $\mathbf{O}^c$-analytic functions and $\mathbf{O}^c$ constants will be consider more by us.

The rest of this paper is organized as follows. Section 2 is an overview of some basic facts concerning octonions and octonionic analysis. Section 3 we give some sufficient conditions for the product $f(x)g(x)$ of two left $\mathbf{O}^c$-analytic functions $f(x)$ and $g(x)$ is also a left $\mathbf{O}^c$-analytic function. In Section 3, we prove that, $f(x)\lambda$ is a left $\mathbf{O}^c$-analytic function for any constants $\lambda\in\mathbf{O}^c$ if and only if $\overline{f(x)}$ is a complex Stein-Weiss conjugate harmonic system. This gives the solution of the problem in [2]. In the last section we give some applications for our results.

2. Preliminaries: Octonions and Octonionic Analysis

It is well known that there are only four normed division algebras [3] [4] [5]: the real numbers $\mathbf{R}$, complex numbers $\mathbf{C}$, quaternions $\mathbf{H}$ and octonions $\mathbf{O}$, with the relations $\mathbf{R}\subseteq\mathbf{C}\subseteq\mathbf{H}\subseteq\mathbf{O}$. In other words, for any $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)\in\mathbf{R}^n$, if we define a product "xy" such that $xy\in\mathbf{R}^n$ and $|x\cdot y|=|x||y|$, where $|x|=\sqrt{\sum_1^n x_i^2}$, then the only four values of n are 1,2,4,8. Quaternions $\mathbf{H}$ is not commutative and octonions $\mathbf{O}$ is neither commutative nor associative. Unlike $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$, the non-associative octonions can not be embedded into the associative Clifford algebras [6].

Octonions stand at the crossroads of many interesting fields of mathematics, they have close relations with Clifford algebras, spinors, Bott periodicity, Projection and Lorentzian geometry, Jordan algebras, and exceptional Lie groups, and also, they have many applications in quantum logic, special relativity and supersymmetry [3] [4].

Denote the set $\mathscr{W}$ by

$$\mathscr{W}=\{(1,2,3),(1,4,5),(1,7,6),(2,4,6),(2,5,7),(3,4,7),(3,6,5)\}.$$

Then the multiplication rules between the basis $e_0,e_1,\cdots,e_7$ on octonions are given by [3] [7]:

$$e_0^2=e_0,\ e_ie_0=e_0e_i=e_i,\ e_i^2=-1,\ i=1,2,\cdots,7,$$

and for any triple $(\alpha,\beta,\gamma)\in\mathscr{W}$,

$$e_\alpha e_\beta=e_\gamma=-e_\beta e_\alpha,\quad e_\beta e_\gamma=e_\alpha=-e_\gamma e_\beta,\quad e_\gamma e_\alpha=e_\beta=-e_\alpha e_\gamma.$$

For each $x=\sum_0^7 x_ie_i\in\mathbf{O}(x_i\in\mathbf{R},i=0,1,\cdots,7)$, x_0 is called the scalar part of

x and $\underline{x} = \sum_1^7 x_i e_i$ is termed its vector part. Then the norm of x is $|x| = \left(\sum_0^7 x_i^2\right)^{\frac{1}{2}}$ and its conjugate is defined by $\bar{x} = \sum_0^7 x_i \bar{e}_i = x_0 - \underline{x}$. We have $x\bar{x} = \bar{x}x = \sum_0^7 x_i^2$, $\overline{xy} = \bar{y}\bar{x}\,(x, y \in \mathbf{O})$ Hence, $x^{-1} = \frac{\bar{x}}{|x|^2}$ is the inverse of $x(\neq 0)$.

Let $x = \sum_0^7 x_i e_i, y = \sum_0^7 y_i e_i \in \mathbf{O}\,(x_i, y_i \in \mathbf{R}, i = 0,1,\cdots,7)$, then

$$xy = x_0 y_0 - \underline{x} \cdot \underline{y} + x_0 \underline{y} + y_0 \underline{x} + \underline{x} \times \underline{y}, \tag{2.1}$$

where $\underline{x} \cdot \underline{y} := \sum_1^7 x_i y_i$ is the inner product of vectors $\underline{x}, \underline{y}$ and

$$\begin{aligned}\underline{x} \times \underline{y} := {} & e_1 (A_{23} + A_{45} - A_{67}) + e_2 (-A_{13} + A_{46} + A_{57}) + e_3 (A_{12} + A_{47} - A_{56}) \\ & + e_4 (-A_{15} - A_{26} - A_{37}) + e_5 (A_{14} - A_{27} + A_{36}) \\ & + e_6 (A_{17} + A_{24} - A_{35}) + e_7 (-A_{16} + A_{25} + A_{34})\end{aligned}$$

is the cross product of vectors $\underline{x}, \underline{y}$, with

$$A_{ij} = \det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}, \quad i, j = 1, 2, \cdots, 7.$$

For any $x, y \in \mathbf{O}$, the inner product and cross product of their vector parts satisfy the following rules [8]:

$$(\underline{x} \times \underline{y}) \cdot \underline{x} = 0, \quad (\underline{x} \times \underline{y}) \cdot \underline{y} = 0, \quad \underline{x} \parallel \underline{y} \Leftrightarrow \underline{x} \times \underline{y} = 0, \quad \underline{x} \times \underline{y} = -\underline{y} \times \underline{x}.$$

We usually utilize associator as an useful tool on ontonions since its non-associativity. Define the associator $[x, y, z]$ of any $x, y, z \in \mathbf{O}$ by $[x, y, z] = (xy)z - x(yz)$.

The octonions obey the following some weakened associative laws.

For any $x, y, z, u, v \in \mathbf{O}$, we have (see [7])

$$[x, y, z] = [y, z, x], \quad [x, z, y] = -[x, y, z], \quad [x, x, y] = 0 = [\bar{x}, x, y] \tag{2.2}$$

and the so-called Moufang identities [5]

$$(uvu)x = u(v(ux)), \quad x(uvu) = ((xu)v)u, \quad u(xy)u = (ux)(uy).$$

Proposition 2.1 ([7]). *For any* $i, j, k \in \{0,1,\cdots,7\}$, $[e_i, e_j, e_k] = 0 \Leftrightarrow ijk = 0$ *or* $(i-j)(j-k)(k-i) = 0$ *or* $(e_i e_j)e_k = \pm 1$.

Proposition 2.2 ([7]). *Let* e_i, e_j, e_k *be three different elements of* $\{e_1, e_2, \cdots, e_7\}$ *and* $(e_i e_j)e_k \neq \pm 1$. *Then* $(e_i e_j)e_k = -e_i(e_j e_k)$.

Since octonions is an alternative algebra (see [3] [9] [10]), we have the following power-associativity of octonions.

Proposition 2.3. *Let* $x_1, x_2, \cdots, x_k \in \mathbf{O}$, $(l_1, \cdots, l_n)$ *be* n *elements out of* $\{1, \cdots, k\}$ *repetitions being allowed and let* $(x_{l_1} x_{l_2} \cdots x_{l_n})_{\otimes_n}$ *be the product of* n *octonions in a fixed associative order* $\otimes_n$. *Then* $\sum_{\pi(l_1,\cdots,l_n)} (x_{l_1} x_{l_2} \cdots x_{l_n})_{\otimes_n}$ *is independent of the associative order* $\otimes_n$, *where the sum runs over all* *distinguishable permutations of* $(l_1,\cdots,l_n)$

Proof. Let $x=\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k$, then $\sum_{\pi(l_1,\cdots,l_n)}(x_{l_1}x_{l_2}\cdots x_{l_n})_{\otimes_n}$ is just the coefficient of $\lambda_{l_1}\lambda_{l_2}\cdots\lambda_{l_n}$ in the product of $x^n=\underbrace{(xx\cdots x)}_{n\ xs}{}_{\otimes_n}$. By induction and (2.2), one can easily prove that $x^n=\underbrace{(xx\cdots x)}_{n\ xs}{}_{\otimes_n}$ is independent of the associative order $\otimes_n$ for any $x\in\mathbf{O}$. Hence $\sum_{\pi(l_1,\cdots,l_n)}(x_{l_1}x_{l_2}\cdots x_{l_n})_{\otimes_n}$ is also independent of the associative order $\otimes_n$. □

$\mu=(\mu_0,\mu_1,\cdots,\mu_n)$ is called a Stein-Weiss conjugate harmonic system if they satisfy the following equations (see [11]):

$$\sum_{i=0}^{n}\frac{\partial\mu_i}{\partial x_i}=0,\quad \frac{\partial\mu_i}{\partial x_j}=\frac{\partial\mu_j}{\partial x_i}\quad (0\le i<j\le n).$$

It is easy to see that if $F(x_0,x_1,\cdots,x_7)=(f_0,f_1,\cdots,f_7)$ is a Stein-Weiss conjugate harmonic system in an open set Ω of $\mathbf{R}^8$, then there exists a real-valued harmonic function Φ in Ω such that F is the gradient of Φ. Thus $\overline{F}=f_0e_0-f_1e_1-\cdots-f_7e_7=\overline{D}\Phi$ is an $\mathbf{O}$-analytic function. But inversely, this is not true [12].

Example. Observe the $\mathbf{O}$-analytic function $g(x)=(x_6^2-x_7^2)e_2-2x_6x_7e_3$. Since

$$\frac{\partial g_2}{\partial x_6}=2x_6\neq 0=\frac{\partial g_6}{\partial x_2},$$

$\overline{g}$ is not a Stein-Weiss conjugate harmonic system.

In [13] Li and Peng proved the octonionic analogue of the classical Taylor theorem. Taking account of Proposition 2.3, we obtain an improving of Taylor type theorem for $\mathbf{O}$-analytic functions (see [14] [15]).

Theorem A (Taylor). *If* $f(x)$ *is a left* $\mathbf{O}$*-analytic function in* Ω *which containing the origin, then it can be developed into Taylor series*

$$f(x)=\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)}V_{l_1\cdots l_k}(x)\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}f(0),$$

and if $f(x)$ is a right $\mathbf{O}$-analytic function, then the Taylor series of f at the origin is given by

$$f(x)=\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)}\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}f(0)V_{l_1\cdots l_k}(x),$$

where $(l_1,\cdots,l_k)$ runs over all possible combinations of k elements out of $\{1,\cdots,7\}$ repetitions being allowed.

The polynomials $V_{l_1\cdots l_k}$ of order k in Theorem A is defined by

$$V_{l_1\cdots l_k}(x)=\frac{1}{k!}\sum_{\pi(l_1,\cdots,l_k)}\left(\cdots\left(\left(z_{l_1}z_{l_2}\right)z_{l_3}\right)\cdots\right)z_{l_k},$$

where the sum runs over all distinguishable permutations of $(l_1,\cdots,l_k)$ and

$$z_{l_j} = x_{l_j} e_0 - x_0 e_{l_j},\ j = 1,\cdots,k.$$

We have the following uniqueness theorem for $\mathbf{O}$-analytic functions [7].

Proposition 2.4. *If* f *is left* (*right*) $\mathbf{O}$*-analytic in an open connect set* $\Omega \subset \mathbf{R}^8$ *and vanishes in the open set* $\mathcal{E} \subset \Omega \cap \{x_0 = a_0\} \neq \varnothing$, *then* f *is identically zero in* Ω.

Proof. Without loss of generality, we let $\mathcal{E}$ which containing the origin and let $x_0 = 0$. Then f can be developed into Taylor series

$$f(x) = \sum_{k=0}^{\infty} \sum_{(l_1,\cdots,l_k)} V_{l_1\cdots l_k}(x)\, \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0).$$

Thus we have

$$f(\underline{x}) = \sum_{k=0}^{\infty} \sum_{(l_1,\cdots,l_k)} x_{l_1} x_{l_2} \cdots x_{l_k} \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0) \equiv 0.$$

By the uniqueness of the Taylor series for the real analytic function, we have $\partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0) = 0$ for any $(l_1,\cdots,l_k) \in \{1,2,\cdots,7\}^7$ and $k \in \mathbf{N}$. This shows that f is identically zero in $\mathcal{E}$ and also in Ω. □

For more references about octonions and octonionic analysis, we refer the reader to [7] [13]-[20].

3. Sufficient Conditions

In what follows we consider the complexification of $\mathbf{O}$, it is denoted by $\mathbf{O}^c$. Thus, $\mathbb{z} \in \mathbf{O}^c$ is of the form $\mathbb{z} = \sum_0^7 \mathbb{z}_i e_i, \mathbb{z}_i \in \mathbf{C}$. $\mathbb{z}_0$ and $\underline{\mathbb{z}} = \sum_0^7 \mathbb{z}_i e_i$ are still called the scalar part and vector part, respectively. The norm of $\mathbb{z} \in \mathbf{O}^c$ is $|\mathbb{z}| = \left(\sum_0^7 |\mathbb{z}_i|^2\right)^{\frac{1}{2}}$ and its conjugate is defined by $\overline{\mathbb{z}} = \sum_0^7 \overline{\mathbb{z}}_i \overline{e}_i$, where $\overline{\mathbb{z}}_i$ is of the conjugate in the complex numbers. We can easily show that for any $\mathbb{z}, \mathbb{z}' \in \mathbf{O}^c$, $|\mathbb{z}\mathbb{z}'| \leq \sqrt{2}|\mathbb{z}||\mathbb{z}'|$. For any $\mathbb{z} \in \mathbf{O}^c$, we may rewrite $\mathbb{z}$ as $\mathbb{z} = x + iy$, where $x, y \in \mathbf{O}$. The multiplication rules in $\mathbf{O}^c$ is the same as in (2.1). Note that $\mathbf{O}^c$ is no longer a division algebra. Finally, the properties of associator in (2.2) except that $[\mathbb{z}, \overline{\mathbb{z}}, \mathbb{u}] = 0$ are also true for any $\mathbb{z}, \mathbb{u}, \mathbb{v} \in \mathbf{O}^c$:

$$[\mathbb{z},\mathbb{u},\mathbb{v}] = [\mathbb{u},\mathbb{v},\mathbb{z}], \quad [\mathbb{z},\mathbb{v},\mathbb{u}] = -[\mathbb{z},\mathbb{u},\mathbb{v}], \quad [\mathbb{z},\mathbb{z},\mathbb{u}] = 0. \tag{3.1}$$

Example. Let $\mathbb{z} = e_1 + ie_2, \mathbb{u} = e_4$, then

$$[\mathbb{z},\overline{\mathbb{z}},\mathbb{u}] = [e_1 + ie_2, -e_1 + ie_2, e_4] = i[e_1,e_2,e_4] - i[e_2,e_1,e_4] = 4ie_7 \neq 0.$$

By (3.1) we can get the following lemma, which is useful to deduce our results.

Lemma 3.1. *Let* $\mathbb{z},\mathbb{u},\mathbb{v} \in \mathbf{O}^c$ *and there exists complex numbers* λ *and* $\mu\left(|\lambda| + |\mu| \neq 0\right)$ *such that* $\lambda\underline{\mathbb{z}} + \mu\underline{\mathbb{u}} = 0$ *or* $\lambda\underline{\mathbb{u}} + \mu\underline{\mathbb{v}} = 0$ *or* $\lambda\underline{\mathbb{v}} + \mu\underline{\mathbb{z}} = 0$, *then* $[\mathbb{z},\mathbb{u},\mathbb{v}] = 0$.

For functions, *f*, under study will be defined in an open set Ω of $\mathbf{R}^8$ and take values in $\mathbf{O}^c$, with the form $f(x) = \sum_0^7 f_i(x) e_i$, where $f_i(x)(i = 0,1,\cdots,7)$ are the complex-valued functions.

Hence, we say that, a function $f(x) = g(x) + ih(x)$ is left $\mathbf{O}^c$-analytic in an open set Ω of $\mathbf{R}^8$, if $g(x)$ and $h(x)$ are the left $\mathbf{O}$-analytic functions, since

$$Df = 0 \Leftrightarrow Dg = Dh = 0,$$

where $D = \sum_{i=0}^{7} \frac{\partial}{\partial x_i} e_i$ is the Dirac operator as in Section 1.

In the case of $\mathbf{O}^c$, we call $f(x) = g(x) + ih(x)$ a complex Stein-Weiss conjugate harmonic system, if $g(x), h(x)$ are the Stein-Weiss conjugate harmonic systems. A left (right) $\mathbf{O}^c$-analytic functions $g(x)$ also have the Taylor expansion as in Theorem A.

Now we consider the product $f(x)g(x)$ of two left $\mathbf{O}^c$-analytic functions $f(x), g(x)$ in Ω. In general, $f(x)g(x)$ is no longer left $\mathbf{O}^c$-analytic in Ω. But, in some particular cases, the product $f(x)g(x)$ can maintain the analyticity for two left $\mathbf{O}^c$-analytic functions $f(x)$ and $g(x)$.

Theorem 3.2. *Let $f(x), g(x)$ be two left $\mathbf{O}^c$-analytic functions in Ω. Then $f(x)g(x)$ is also left $\mathbf{O}$-analytic in Ω if $f(x), g(x)$ satisfy one of the following conditions:*

1) *$\overline{f(x)}$ or $g(x)$ is a complex constant function.*

2) *$\overline{f(x)}$ is a complex Stein-Weiss conjugate harmonic system in Ω and $g(x)$ is an $\mathbf{O}^c$-constant function.*

3) *$f(x)$ is of the form $f(x) = f_0 e_0 + f_i e_i \ (i \in \{1, 2, \cdots, 7\})$ and $f(x), g(x)$ depend only on x_0 and x_i, where f_0, f_i are the complex-valued functions.*

4) *$f(x)$ and $g(x)$ belong to the following class*

$$\mathfrak{S} = \left\{ h(x) \mid Dh(x) = 0, \underline{h(x)} = \sum_{i=1}^{7} h_1(x) e_i, h_1(x) \in C^1(\Omega, \mathbf{C}) \right\}. \tag{3.2}$$

5) *$f(x)$ is of the form $f(x) = f_0 e_0 + f_\alpha e_\alpha + f_\beta e_\beta + f_\gamma e_\gamma$, $g = c_0 e_0 + c_\alpha e_\alpha + c_\beta e_\beta + c_\gamma e_\gamma$ is a constant function, where $(\alpha, \beta, \gamma) \in \mathscr{W}$, $c_0, c_\alpha, c_\beta, c_\gamma \in \mathbf{C}$ and $f(x)$ depends only on $x_0, x_\alpha, x_\beta, x_\gamma$.*

Proof. 1) The proof is trivial.

2) In view of Proposition 2.1 we have $[e_i, e_j, \lambda] = 0$ when $i = 0$ or $j = 0$ or $i = j$ for any $\lambda \in \mathbf{O}^c$. Then we have

$$\begin{aligned} D(f\lambda) &= \sum_{i,j=0}^{7} \frac{\partial f_j}{\partial x_i} e_i (e_j \lambda) \\ &= \sum_{i,j=0}^{7} \frac{\partial f_j}{\partial x_i} (e_i e_j) \lambda - \sum_{i,j=0}^{7} \frac{\partial f_j}{\partial x_i} [e_i, e_j, \lambda] \\ &= (Df)\lambda - \sum_{i,j=0}^{7} \frac{\partial f_j}{\partial x_i} [e_i, e_j, \lambda] \\ &= (Df)\lambda - \sum_{1 \le i \ne j \le 7} \frac{\partial f_j}{\partial x_i} [e_i, e_j, \lambda]. \end{aligned}$$

Since $\overline{f}$ is a complex Stein-Weiss conjugate harmonic system, thus $Df = 0$ and $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$ for $i, j \ge 1, i \ne j$. But $[e_j, e_i, \lambda] = -[e_i, e_j, \lambda]$, therefore

$$D(f\lambda) = -\sum_{1\le i\ne j\le 7} \frac{\partial f_j}{\partial x_i}\left[e_i, e_j, \lambda\right] = \sum_{1\le i<j\le 7}\left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}\right)\left[e_i, e_j, \lambda\right] = 0.$$

3) Since $f(x), g(x)$ are only related to variables x_0 and x_i, we have

$$D(fg) = \left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_i}e_i\right)\left((f_0 + f_i e_i)g\right)$$

$$= \frac{\partial f}{\partial x_0}g + e_i\left(\left(\frac{\partial f_0}{\partial x_i} + \frac{\partial f_i}{\partial x_i}e_i\right)g\right) + f\frac{\partial g}{\partial x_0} + e_i\left((f_0 + f_i e_i)\frac{\partial g}{\partial x_i}\right).$$

By Lemma 3.1 it follows that

$$e_i\left(\left(\frac{\partial f_0}{\partial x_i} + \frac{\partial f_i}{\partial x_i}e_i\right)g\right) = \left(e_i\left(\frac{\partial f_0}{\partial x_i} + \frac{\partial f_i}{\partial x_i}e_i\right)\right)g = \left(e_i\frac{\partial f}{\partial x_i}\right)g$$

and

$$e_i\left((f_0 + f_i e_i)\frac{\partial g}{\partial x_i}\right) = \left(e_i(f_0 + f_i e_i)\right)\frac{\partial g}{\partial x_i} = \left((f_0 + f_i e_i)e_i\right)\frac{\partial g}{\partial x_i} = (f_0 + f_i e_i)\left(e_i\frac{\partial g}{\partial x_i}\right).$$

Thus we get

$$D(fg) = \frac{\partial f}{\partial x_0}g + \left(e_i\frac{\partial f}{\partial x_i}\right)g + f\frac{\partial g}{\partial x_0} + f\left(e_i\frac{\partial g}{\partial x_i}\right) = (Df)g + f(Dg) = 0.$$

4) Let $f(x) = f_0 e_0 + \sum_{i=1}^{7} f_1 e_i$ and $g(x) = g_0 e_0 + \sum_{i=1}^{7} g_1 e_i$, then we have

$$D(f(x)g(x)) = \sum_{j=0}^{7} e_j \frac{\partial}{\partial x_j}\left(\left(f_0 e_0 + f_1\sum_{i=1}^{7} e_i\right)\left(g_0 e_0 + g_1\sum_{i=1}^{7} e_i\right)\right)$$

$$= \sum_{j=0}^{7} e_j\left(\left(\frac{\partial f_0}{\partial x_j}e_0 + \frac{\partial f_1}{\partial x_j}\sum_{i=1}^{7} e_i\right)\left(g_0 e_0 + g_1\sum_{i=1}^{7} e_i\right)\right)$$

$$+ \sum_{j=0}^{7} e_j\left(\left(f_0 e_0 + f_1\sum_{i=1}^{7} e_i\right)\left(\frac{\partial g_0}{\partial x_j}e_0 + \frac{\partial g_1}{\partial x_j}\sum_{i=1}^{7} e_i\right)\right).$$

By Lemma 3.1 we get

$$e_j\left(\left(\frac{\partial f_0}{\partial x_j}e_0 + \frac{\partial f_1}{\partial x_j}\sum_{i=1}^{7} e_i\right)\left(g_0 e_0 + g_1\sum_{i=1}^{7} e_i\right)\right) = \left(e_j\frac{\partial f}{\partial x_j}\right)g$$

and

$$e_j\left(\left(f_0 e_0 + f_1\sum_{i=1}^{7} e_i\right)\left(\frac{\partial g_0}{\partial x_j}e_0 + \frac{\partial g_1}{\partial x_j}\sum_{i=1}^{7} e_i\right)\right)$$

$$= e_j\left(\left(\frac{\partial g_0}{\partial x_j}e_0 + \frac{\partial g_1}{\partial x_j}\sum_{i=1}^{7} e_i\right)\left(f_0 e_0 + f_1\sum_{i=1}^{7} e_i\right)\right)$$

$$= \left(e_j\frac{\partial g}{\partial x_j}\right)f.$$

Hence we obtain

$$D\left(f(x)g(x)\right)=\sum_{j=0}^{7}\left(\left(e_j\frac{\partial f}{\partial x_j}\right)g+\left(e_j\frac{\partial g}{\partial x_j}\right)f\right)=(Df)g+(Dg)f=0.$$

5) This case is equivalent to a left quaternionic analytic function right-multiplying by a quaternionic constant, the analyticity is obvious since the multiplication of the quaternion is associative.

The proof of Theorem 3.2 is complete. □

From Theorem 3.2(d), if $f(x),g(x)\in\mathfrak{S}$, then $f(x)g(x)\in\mathfrak{S}$; that is, the multiply operation in $\mathfrak{S}$ is closed. Also, the division operation is closed in $\mathfrak{S}$. Actually, let $f(x)=f_0(x)+\sum_{i=1}^{7}f_1(x)e_i\in\mathfrak{S}$, assume $f_0^2+7f_1^2\neq 0$, then

$$\left(f(x)\right)^{-1}=\frac{f_0-f_1\left(e_1+e_2+\cdots+e_7\right)}{f_0^2+7f_1^2}.$$

Thus we have

$$\begin{aligned}
D\left(f(x)\right)^{-1}&=\sum_{i=0}^{7}e_i\frac{\partial\left(f(x)\right)^{-1}}{\partial x_i}\\
&=\sum_{i=0}^{7}e_i\left(\left(f_0^2+7f_1^2\right)^{-1}\left(\frac{\partial f_0}{\partial x_i}-\frac{\partial f_1}{\partial x_i}(e_1+e_2+\cdots+e_7)\right)\right.\\
&\quad\left.-\left(f_0-f_1\left(e_1+e_2+\cdots+e_7\right)\right)\left(2f_0\frac{\partial f_0}{\partial x_i}+14f_1\frac{\partial f_1}{\partial x_i}\right)\left(f_0^2+7f_1^2\right)^{-2}\right)\\
&=\sum_{i=0}^{7}e_i\left(\left(\frac{\partial f_0}{\partial x_i}+\frac{\partial f_1}{\partial x_i}(e_1+\cdots+e_7)\right)\left(7f_1^2-f_0^2+2f_0f_1\left(e_1+\cdots+e_7\right)\right)\left(f_0^2+7f_1^2\right)^{-2}\right)\\
&=\sum_{i=0}^{7}\left(e_i\left(\frac{\partial f_0}{\partial x_i}+\frac{\partial f_1}{\partial x_i}(e_1+\cdots+e_7)\right)\right)\left(7f_1^2-f_0^2+2f_0f_1\left(e_1+\cdots+e_7\right)\right)\left(f_0^2+7f_1^2\right)^{-2}\\
&=\left(Df(x)\right)\left(7f_1^2-f_0^2+2f_0f_1\left(e_1+\cdots+e_7\right)\right)\left(f_0^2+7f_1^2\right)^{-2}\\
&=0.
\end{aligned}$$

An element belongs to $\mathfrak{S}$ is the exponential function:

$$\exp(x)=\mathrm{e}^{x_1+\cdots+x_7}\left(\cos\left(x_0\sqrt{7}\right)e_0+\left(-\frac{1}{\sqrt{7}}\left(e_1+\cdots+e_7\right)\right)\sin\left(x_0\sqrt{7}\right)\right).\tag{3.3}$$

The results in Theorem 3.2 also hold on octonions(no complexification), since $\mathbf{O}^c$ contains $\mathbf{O}$. If one switch the locations of $f(x),g(x)$, and the "left" change into "right" in Theorem 3.2, then this theorem is also true, since left and right is symmetric. These principles also hold in the rest of this paper.

4. Necessary and Sufficient Conditions

If we consider the product of a left $\mathbf{O}^c$-analytic function and an $\mathbf{O}^c$-constant, we can get the necessary and sufficient conditions for the analyticity(these results obtained in this section for $\mathbf{O}$-analytic functions are also described in [19]).

Applying Theorem 3.2(a) and (b), if $f(x)$ is a left $\mathbf{O}^c$-analytic function and λ is a complex constant, or $\overline{f(x)}$ is a complex Stein-Weiss conjugate harmonic system and λ is an $\mathbf{O}^c$-constant, then $f(x)\lambda$ is a left $\mathbf{O}^c$-analytic function. In what follows we will see that these conditions are also necessary in some sense.

Theorem 4.1. *Let* $\lambda \in \mathbf{O}^c$, *then* $f\lambda$ *is a left* $\mathbf{O}^c$*-analytic function for any left* $\mathbf{O}^c$*-analytic functions* f *if and only if* $\lambda \in \mathbf{C}$.

Proof. We only prove the necessity. Taking a left $\mathbf{O}^c$-analytic function $f = x_1e_2 - x_0e_3$, then

$$\begin{aligned}D(f\lambda) &= -\sum_{i,j,k=0}^{7}\frac{\partial f_j}{\partial x_i}\lambda_k\left[e_i,e_j,e_k\right] = \sum_{k=1}^{7}\frac{\partial f_2}{\partial x_1}\lambda_k\left[e_2,e_1,e_k\right] = \sum_{k=4}^{7}\lambda_k\left[e_2,e_1,e_k\right] \\ &= \lambda_4\left[e_2,e_1,e_4\right] + \lambda_5\left[e_2,e_1,e_5\right] + \lambda_6\left[e_2,e_1,e_6\right] + \lambda_7\left[e_2,e_1,e_7\right] \\ &= -2\lambda_4e_7 + 2\lambda_5e_6 - 2\lambda_6e_5 + 2\lambda_7e_4.\end{aligned}$$

Thus $\lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$. A similar technique yields $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence $\lambda \in \mathbf{C}$. □

Theorem 4.2. *Let* $f \in C^1(\Omega, \mathbf{O}^c)$. *Then* $D(f\lambda) = 0$ *for any* $\lambda \in \mathbf{O}^c$ *if and only if* f *is a complex Stein-Weiss conjugate harmonic system in* Ω.

Now we postpone the proof of Theorem 4.2 and consider a problem under certain conditions weaker than Theorem 4.2. In [2] the authors proposed an open problem as follows:

Find the necessary and sufficient conditions for an $\mathbf{O}^c$*-valued function* f, *such that the equality* $[\lambda, f(x), D] = 0$ *holds for any constant* $\lambda \in \mathbf{O}^c$.

Note that this problem is of no meaning for an associative system, but octonions is a non-associative algebra, therefore we usually encounter some difficulties while disposing some problems in octonionic analysis. In [2] the authors added the condition $[\lambda, f(x), D] = 0$ for $f(x)$ to study the Cauchy integrals on Lipschitz surfaces in octonions and then prove the analogue of Calderón's conjecture in octonionic space.

Next we give the answer to the Open Problem as follows.

Theorem 4.3. *Let* $f \in C^1(\Omega, \mathbf{O}^c)$. *Then* $[D, f, \lambda] = 0\ ([\lambda, f, D] = 0)$ *for any* $\lambda \in \mathbf{O}^c$ *if and only if*

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad i, j = 1, 2, \cdots, 7. \tag{4.1}$$

Proof. By Proposition 2.1, we have

$$[D, f, \lambda] = \sum_{i,j=0}^{7}\frac{\partial f_j}{\partial x_i}\left[e_i,e_j,\lambda\right] = \sum_{1\le i<j\le 7}\left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right)\left[e_i,e_j,\lambda\right].$$

If f satisfies (4.1), then $[D, f, \lambda] = 0$.

Inversely, let $(\alpha,\beta,\gamma) \in \mathscr{W}$, $\{1,2,\cdots,7\}\setminus\{\alpha,\beta,\gamma\} = \{t_1,t_2,t_3,t_4\}$ and

$$e_{t_1}e_{t_2} = e_\gamma = -e_{t_2}e_{t_1},\ e_{t_3}e_{t_4} = e_\gamma = -e_{t_4}e_{t_3}.$$

From Propositions 2.1 and 2.2 we have $\left[e_\alpha, e_\beta, e_t\right] = 0$ and $\left[e_\alpha, e_\beta, e_t\right] = 2\left(e_\alpha e_\beta\right)e_t = 2e_\gamma e_t$ when $t = \alpha,\beta,\gamma$ and $t = t_1,t_2,t_3,t_4$, respectively.

Hence, taking $\lambda = e_{t_1}$ it follows that

$$\begin{aligned}
&\left[D, f, e_{t_1}\right] \\
&= \sum_{1\le i<j\le 7}\left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right)\left[e_i, e_j, e_{t_1}\right] \\
&= \left(\frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta}\right)\left[e_\alpha, e_\beta, e_{t_1}\right] + \left(\frac{\partial f_{t_4}}{\partial x_{t_3}} - \frac{\partial f_{t_3}}{\partial x_{t_4}}\right)\left[e_{t_3}, e_{t_4}, e_{t_1}\right] + \sum_{s\ne t_2} g_s e_s \\
&= 2\left(\frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} + \frac{\partial f_{t_4}}{\partial x_{t_3}} - \frac{\partial f_{t_3}}{\partial x_{t_4}}\right)e_{t_2} + \sum_{s\ne t_2} g_s e_s.
\end{aligned} \tag{4.2}$$

Similarly, we take $\lambda = e_{t_3}$, then

$$\left[D, f, e_{t_3}\right] = 2\left(\frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} + \frac{\partial f_{t_2}}{\partial x_{t_1}} - \frac{\partial f_{t_1}}{\partial x_{t_2}}\right)e_{t_4} + \sum_{s\ne t_4} h_s e_s, \tag{4.3}$$

Also we can get

$$\left[D, f, e_\alpha\right] = 2\left(\frac{\partial f_{t_2}}{\partial x_{t_1}} - \frac{\partial f_{t_1}}{\partial x_{t_2}} + \frac{\partial f_{t_4}}{\partial x_{t_3}} - \frac{\partial f_{t_3}}{\partial x_{t_4}}\right)e_\beta + \sum_{s\ne \beta} y_s e_s. \tag{4.4}$$

If we require $[D, f, \lambda] = 0$ for any constants $\lambda \in \mathbf{O}^c$, from (4.2), (4.3) and (4.4) we obtain

$$\begin{cases}
\dfrac{\partial f_\beta}{\partial x_\alpha} - \dfrac{\partial f_\alpha}{\partial x_\beta} + \dfrac{\partial f_{t_4}}{\partial x_{t_3}} - \dfrac{\partial f_{t_3}}{\partial x_{t_4}} = 0, \\[2ex]
\dfrac{\partial f_\beta}{\partial x_\alpha} - \dfrac{\partial f_\alpha}{\partial x_\beta} + \dfrac{\partial f_{t_2}}{\partial x_{t_1}} - \dfrac{\partial f_{t_1}}{\partial x_{t_2}} = 0, \\[2ex]
\dfrac{\partial f_{t_2}}{\partial x_{t_1}} - \dfrac{\partial f_{t_1}}{\partial x_{t_2}} + \dfrac{\partial f_{t_4}}{\partial x_{t_3}} - \dfrac{\partial f_{t_3}}{\partial x_{t_4}} = 0.
\end{cases}$$

Combining above three equations with the randomicity of (α, β, γ) we have (4.1) holds. □

Proof of Theorem 4.2. The sufficient from Theorem 3.2(b). Inversely, if we take $\lambda = 1$ in $D(f\lambda) = 0$ it follows that f is a left $\mathbf{O}^c$-analytic function. Thus for any $\lambda \in \mathbf{O}^c$, we have

$$D(f\lambda) = (Df)\lambda - [D, f, \lambda] = -[D, f, \lambda] = 0.$$

By Theorem 4.3 we get that f satisfies (4.1). On the other hand,

$$Df = \left(\frac{\partial}{\partial x_0} + \nabla\right)\left(f_0 + \underline{f}\right) = \frac{\partial f_0}{\partial x_0} - \nabla\cdot\underline{f} + \frac{\partial \underline{f}}{\partial x_0} + \nabla f_0 + \nabla\times\underline{f} = 0. \tag{4.5}$$

From (4.1) it easily to get $\nabla\times\underline{f} = 0$, again by (4.5) it follows that

$$\frac{\partial f_0}{\partial x_0} - \nabla\cdot\underline{f} + \frac{\partial \underline{f}}{\partial x_0} + \nabla f_0 = 0,$$

namely

$$\frac{\partial f_0}{\partial x_0} - \nabla\cdot\underline{f} = 0, \quad \frac{\partial \underline{f}}{\partial x_0} + \nabla f_0 = 0.$$

Combining this with (4.1) it shows that $\overline{f}$ is a complex Stein-Weiss conjugate harmonic system in Ω. □

5. Some Applications and Relations with the C-K Products

From Theorem A we can see that $V_{l_1\cdots l_k}(x)$ are the basic components for (left) $\mathbf{O}$-analytic functions. It is proved in [13] that the polynomials $V_{l_1\cdots l_k}(x)$ are all $\mathbf{O}$-analytic functions, therefore they are the suitable substitutions of the polynomial z^k in $\mathbf{C}$.

Again from Theorem A, since $V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k}$ is an item in the Taylor expansion of a left $\mathbf{O}$-analytic function, $V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k}$ should be also a left $\mathbf{O}$ analytic function. Applying Theorem 4.2, the conjugate of $V_{l_1\cdots l_k}(x)$ is probably a Stein-Weiss conjugate harmonic system. The following theorem prove this is true.

Theorem 5.1. *For any combination* $(l_1,\cdots,l_k)$ *of* k *elements out of* $\{1,\cdots,7\}$ *repetitions being allowed,* $\overline{V}_{l_1\cdots l_k}(x)$ *is a Stein-Weiss conjugate harmonic system in* $\mathbf{R}^8$.

Proof. Let $s_i\,(i=1,\cdots,7)$ be the appearing times of i in $(l_1,\cdots,l_k)$. Hence the following equality

$$V_{l_1\cdots l_k}(x)=\overline{D}\Phi_{s_1\cdots s_7}(x) \tag{5.1}$$

shows that $\overline{V}_{l_1\cdots l_k}(x)$ is a Stein-Weiss conjugate harmonic system in $\mathbf{R}^8$, where

$$\Phi_{s_1\cdots s_7}(x)=\sum_{\substack{\kappa_i=0\\ i=1,\cdots,7}}^{\left[\frac{s_i}{2}\right]}\left\{\frac{(-1)^{\kappa}\kappa!\,x_0^{2\kappa+1}}{(2\kappa+1)!}\prod_{j=1}^{7}\frac{x_j^{s_j-2\kappa_j}}{\kappa_j!\,(s_j-2\kappa_j)!}\right\}$$

is a real-valued harmonic function of order $(s_1+s_2+\cdots+s_7+1)$ with $\kappa=\sum_{i=1}^{7}\kappa_i$.

Actually, put $x_0=0$, the both sides of (5.1) equal to $\frac{1}{s_1!s_2!\cdots s_7!}x_1^{s_1}\cdots x_7^{s_7}$. On the other hand, $V_{l_1\cdots l_k}(x)$ is left $\mathbf{O}$-analytic in $\mathbf{R}^8$. Thus by Proposition 2.4 we have (5.1) holds. □

Combining Theorem 3.2(b) and Theorem 5.1 it really shows that all the $V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k}$ are left $\mathbf{O}^c$-analytic functions for any $\lambda_{l_1\cdots l_k}\in\mathbf{O}^c$. Hence the following series

$$\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)}V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k} \tag{5.2}$$

is a left $\mathbf{O}^c$-analytic function in some open neighborhood Λ of the origin if $\{\lambda_{l_1\cdots l_k}\}$ satisfies certain bounded conditions.

Theorem 5.2. For any combination $(l_1,\cdots,l_k)$ of k elements out of $\{1,\cdots,7\}$ repetitions being allowed, let $\lambda_{l_1\cdots l_k}\in\mathbf{O}^c, k\in\mathbf{N}$. If $\overline{\lim_{k\to\infty}}\frac{7^k}{k!}\sup_{(l_1\cdots l_k)}\left|\lambda_{l_1\cdots l_k}\right|=\gamma<\infty$, then the series (5.2) converges to a left $\mathbf{O}^c$-analytic function $f(x)$ in the following region

$$\Lambda_\gamma = \left\{ x \in \mathbf{R}^8 : \sqrt{x_0^2 + x_i^2} < \frac{1}{\gamma}, i = 1, 2, \cdots, 7 \right\}.$$

More over, $\lambda_{l_1 \cdots l_k} = \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0)$ Particularly, if $\sup_{\substack{(l_1 \cdots l_k) \\ k \in \mathbf{N}}} |\lambda_{l_1 \cdots l_k}| \le C < \infty$, then f will be a left $\mathbf{O}^c$-entire function.

Proof. Let

$$S_N(x) = \sum_{k=0}^{N} \sum_{(l_1, \cdots, l_k)} V_{l_1 \cdots l_k}(x) \lambda_{l_1 \cdots l_k}, N \in \mathbf{N}.$$

For any $x = \sum_0^7 x_i e_i \in \Lambda_\gamma$, there exists $\gamma' > \gamma$ such that $\sqrt{x_0^2 + x_i^2} < \frac{1}{\gamma'}, i = 1, 2, \cdots, 7$. Thus

$$\begin{aligned} &\sup_{x \in \Lambda_{\gamma'}} |S_N(x) - S_M(x)| \\ &\le \sup_{x \in \Lambda_{\gamma'}} \sum_{k=M}^{N} \sum_{(l_1, \cdots, l_k)} |V_{l_1 \cdots l_k}(x)| |\lambda_{l_1 \cdots l_k}| \\ &\le \sup_{x \in \Lambda_{\gamma'}} \sum_{k=M}^{N} \frac{1}{k!} \sum_{l_1, \cdots, l_k = 1}^{7} |z_{l_1}| \cdots |z_{l_k}| |\lambda_{l_1 \cdots l_k}| \\ &\le \sum_{k=M}^{N} \frac{7^k}{k!} \frac{1}{\gamma'^k} |\lambda_{l_1 \cdots l_k}| \to 0 \quad (\inf(M, N) \to \infty). \end{aligned}$$

From Weierstrass Theorem on octonions [13] and the analyticity of $V_{l_1 \cdots l_k}(x) \lambda_{l_1 \cdots l_k}$, then there exists a left $\mathbf{O}^c$-analytic function f in Λ_γ such that

$$f(x) = \lim_{N \to \infty} S_N(x) = \sum_{k=0}^{\infty} \sum_{(l_1, \cdots, l_k)} V_{l_1 \cdots l_k}(x) \lambda_{l_1 \cdots l_k},$$

and the series uniformly converges to $f(x)$ in each compact subset $K \subset \Lambda_\gamma$. Again from the expansion of $f(x)$ we easily get that $\lambda_{l_1 \cdots l_k} = \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0)$.

If $\sup_{\substack{(l_1 \cdots l_k) \\ k \in \mathbf{N}}} |\lambda_{l_1 \cdots l_k}| \le C < \infty$, then $\Lambda_\gamma = \mathbf{R}^8$, since $\lim_{k \to \infty} \frac{7^k}{k!} = 0$. Therefore f is a left $\mathbf{O}^c$-entire function. □

Example. Taking $\lambda_{l_1 \cdots l_k} \equiv 1$ for all $k \in \mathbf{N}$ in (5.2), then

$$\sum_{k=0}^{\infty} \sum_{(l_1, \cdots, l_k)} V_{l_1 \cdots l_k}(x) \tag{5.3}$$

is an $\mathbf{O}$-entire function. In fact, (5.3) is the Taylor expansion of the exponential function $\exp(x)$ as in (3.3). From (3.3) we can find $\exp(x)$ satisfies

$$\exp(0) = 1, \quad \exp(x + y) = \exp(x) \cdot \exp(y) = \exp(y) \cdot \exp(x).$$

Corollary 5.3. *For any left* $\mathbf{O}^c$*-analytic function* f, *if the coefficients in its Taylor series about the origin satisfy*

$$\begin{cases} \partial_{x_i^k} f(0) \in \mathbf{C} + e_i \mathbf{C}, & k \in \mathbf{N},\ i = 1, 2, \cdots, 7 \\ \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0) \in \mathbf{C}, & \text{otherwise.} \end{cases} \tag{5.4}$$

Then $\overline{f}$ is a complex Stein-Weiss conjugate harmonic system.

Proof. From (5.4), we easily obtain that all the conjugates of $V_{l_1\cdots l_k}(x)\partial_{x_{l_1}}\cdots\partial_{x_{l_k}} f(0)$ are complex Stein-Weiss conjugate harmonic systems. Hence by Weierstrass Theorem, $\overline{f}$ also is a complex Stein-Weiss conjugate harmonic system in its convergent area. □

Combining Theorem 3.2(b), Theorems 5.1 and 5.2, by an analogous method in [6] we can define the Cauchy-Kowalewski product for any two left $\mathbf{O}^c$ analytic functions *f* and *g* in Ω which containing origin. We let their Taylor expansions be

$$f(x)=\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)} V_{l_1\cdots l_k}(x)\partial_{x_{l_1}}\cdots\partial_{x_{l_k}} f(0)$$

and

$$g(x)=\sum_{t=0}^{\infty}\sum_{(s_1,\cdots,s_t)} V_{s_1\cdots s_t}(x)\partial_{x_{s_1}}\cdots\partial_{x_{s_t}} g(0).$$

Then the (left) Cauchy-Kowalewski product of *f* and *g* is defined by

$$f\odot_L g(x)$$
$$=\sum_{k,t=0}^{\infty}\sum_{\substack{(l_1,\cdots,l_k)\\(s_1,\cdots,s_t)}}\left(\prod_{i=1}^{7}\frac{(n_i+n_i')!}{n_i!n_i'!}\right)V_{l_1\cdots l_k s_1\cdots s_t}(x)\left(\partial_{x_{l_1}}\cdots\partial_{x_{l_k}} f(0)\cdot\partial_{x_{s_1}}\cdots\partial_{x_{s_t}} g(0)\right),$$

where n_i and n_i' are the appearing times of *i* in $(l_1,\cdots,l_k)$ and $(s_1,\cdots,s_t)$, respectively.

We have the following relation for the product and the left Cauchy-Kowalewski product between two left $\mathbf{O}^c$ -analytic functions.

Theorem 5.4. *Let* $f(x), g(x)$ *be two left* $\mathbf{O}^c$ *-analytic functions in* Ω *which containing origin. If* $D(f(x)g(x))=0$ *then*

$$f(x)g(x)=f\odot_L g(x).$$

Proof. It is easy to see that $f(\underline{x})g(\underline{x})=f\odot_L g(\underline{x})$, then by Proposition 2.4 and the analyticity of $f(x)g(x)$ and $f\odot_L g(x)$ we get $f(x)g(x)=f\odot_L g(x)$. □

Remark. In this paper we study the analyticity of the product of two left $\mathbf{O}^c$ -analytic functions. Theorem 3.2 give some sufficient conditions for the product of two left $\mathbf{O}^c$ -analytic functions is also a left $\mathbf{O}^c$ -analytic function. From Theorem 5.4 we can see that $D(f(x)g(x))=0$ for two left $\mathbf{O}^c$ -analytic functions $f(x), g(x)$ if and only if this product is just equal to their left Cauchy-Kowalewski product. Since $\mathbf{H}\subseteq\mathbf{O}$, our result is also true for quaternionic cases.

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