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Study on the Existence of Sign-Changing Solutions of Case Theory Based a Class of Differential Equations Boundary-Value Problems

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Abstract

By using the fixed point theorem under the case structure, we study the existence of sign-changing solutions of A class of second-order differential equations three-point boundary-value problems, and a positive solution and a negative solution are obtained respectively, so as to popularize and improve some results that have been known.

Keywords

Case Theory, Boundary-Value Problems, Fixed Point Theorem, Sign-Changing Solutions

1. Introduction

The existence of nonlinear three-point boundary-value problems has been studied [1]-[6], and the existence of sign-changing solutions is obtained. In the past, most studies were focused on the cone fixed point index theory [7] [8] [9], just a few took use of case theory to study the topological degree of non-cone mapping and the calculation of fixed point index, and the case theory was combined with the topological degree theory to study the sign-changing solutions. Recent study Ref. [10] [11] have given the method of calculating the topological degree under the case structure, and taken use of the fixed point theorem of non-cone mapping to study the existence of nontrivial solutions for the nonlinear Sturm-Liouville problems. Relevant studies as [12] [13] [14].

Inspired by the Ref. [8]-[13] and by using the new fixed point theorem under the case structure, this paper studies three-point boundary-value problems for A class of nonlinear second-order equations

$$\begin{cases} u''(t)+f(u(t))=0, 0\le t\le 1; \\ u'(0)=0, u(1)=\alpha u(\eta), \end{cases} \tag{1}$$

Existence of the sign-changing solution, constant $0<\alpha<1, 0<\eta<1$, $f\in C(R,R)$.

Boundary-value problem (1) is equivalent to Hammerstein nonlinear integral equation hereunder

$$u(t)=\int_0^1 G(t,s)f(u(s))\mathrm{d}s, 0\le t\le 1 \tag{2}$$

Of which $G(t,s)$ is the Green function hereunder

$$G(t,s)=\frac{1}{1-\alpha}\begin{cases} (1-s)-\alpha(\eta-s), 0\le s\le\eta, 0\le t\le s; \\ (1-s), \eta\le s\le 1, 0\le t\le s; \\ (1-\alpha\eta)-t(1-\alpha), 0\le s\le\eta, s\le t\le 1; \\ (1-\alpha y)-t(1-\alpha), \eta\le s\le 1, s\le t\le 1. \end{cases}$$

Defining linear operator K as follow

$$(Ku)(t)=\int_0^1 G(t,s)u(s)\mathrm{d}s, u\in C[0,1]. \tag{3}$$

Let $Fu(t)=f(u(t))$, $t\in[0,1]$, obviously composition operator $A=KF$, *i.e.*

$$(Au)(t)=\int_0^1 G(t,s)f(u(s))\mathrm{d}s, 0\le t\le 1 \tag{4}$$

It's easy to get: $u\in C^2[0,1]$ is the solution of boundary-value problem (1), and $u\in C[0,1]$ is the solution of operator equation $u=Au$.

We note that, in Ref. [9] [10], an abstract result on the existence of sign-changing solutions can be directly applied to problem (1). After the necessary preparation, when the non-linear term f is under certain assumptions, we get the existence of sign-changing solution of such boundary-value problems. Compared with the Ref. [8], we can see that we generalize and improve the non-linear term f, and remove the conditions of strictly increasing function, and the method is different from Ref. [8].

For convenience, we give the following conditions.

(H_1) $f(u):R\to R$ continues, $f(u)u>0$, $\forall u\in R, u\ne 0$, and $f(0)=0$.

(H_2) $\lim_{u\to 0}\frac{f(u)}{u}=\beta$, and $n_0\in N$, make $\lambda_{2n_0}<\beta<\lambda_{2n_0+1}$, of which $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\lambda_{n+1}<\cdots$ is the positive sequence of $\cos\sqrt{x}=\alpha\cos\eta\sqrt{x}$.

(H_3) exists $\varepsilon>0$, make $\limsup_{|u|\to+\infty}\frac{f(u)}{u}\le\lambda_1-\varepsilon$.

2. Knowledge

Provided P is the cone of E in *Banach* space, the semi order in E is exported by cone P. If the constant $N>0$, and $\theta\le x\le y\Rightarrow\|x\|\le N\|y\|$, then P is a normal cone; if P contains internal point, *i.e.* $\operatorname{int} P \neq \varnothing$, then P is a solid cone.

E becomes a case when semi order $\leq$, *i.e.* any $x, y \in E$, $\sup\{x, y\}$ and $\inf\{x, y\}$ is existed, for $x \in E$, $x^{+} = \sup\{x, \theta\}$, $x^{-} = \sup\{-x, \theta\}$, we call positive and negative of x respectively, call $|x| = x^{+} + x^{-}$ as the modulus of x. Obviously, $x^{+} \in P$, $x^{-} \in (-P)$, $|x| \in P$, $x = x^{+} - x^{-}$.

For convenience, we use the following signs: $x_{+} = x^{+}$, $x_{-} = -x^{-}$. Such that $x = x_{+} + x_{-}$, $|x| = x_{+} - x_{-}$.

Provided Banach space $E = C[0,1]$, and E's norm as $\|\cdot\|$, *i.e.* $\|u\| = \max\limits_{0 \leq t \leq 1} |u(t)|$. Let $P = \{u \in E \mid u(t) \geq 0, t \in [0,1]\}$, then P is the normal cone of E, and E becomes a case under semi order $\leq$.

Now we give the definitions and theorems

Def 1 [10] provided $D \subset E, A: D \to E$ is an operator (generally a nonlinear). If $Ax = Ax_{+} + Ax_{-}, \forall x \in E$, then A is an additive operator under case structure; if $v^{*} \in E$, and $Ax = Ax_{+} + Ax_{-} + v^{*}, \forall x \in E$, then A is a quasi additive operator.

Def 2 provided x is a fixed point of A, if $x \in (P \setminus \{\theta\})$, then x is a positive fixed point; if $x \in ((-P) \setminus \{\theta\})$, then x is a negative fixed point; if $x \notin (P \cup (-P))$, then x is a sign-changing fixed point.

Lemma 1 [6] $G(t,s)$ is a nonnegative continuous function of $[0,1] \times [0,1]$, and when $t, s \in [0,1]$, $G(t,s) \geq \gamma G(0,s)$, of which $\gamma = \dfrac{\alpha(1-\eta)}{1-\alpha\eta}$.

Lemma 2 $K: P \to P$ is completely continuous operator, and $A: E \to E$ is completely continuous operator.

Lemma 3 A is a quasi additive operator under case structure.

Proof: Similar to the proofs in Lemma 4.3.1 in Ref. [10], get Lemma 3 works.

Lemma 4 [6] the eigenvalues of the linear operator K are $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, \cdots, \dfrac{1}{\lambda_n}, \dfrac{1}{\lambda_{n+1}}, \cdots$. And the sum of algebraic multiplicity of all eigenvalues is 1, of which λ_n is defined by (H_2).

The lemmas hereunder are the main study bases.

Lemma 5 [10] provided E is *Banach* space, P is the normal cone in E, $A: E \to E$ is completely continuous operator, and quasi additive operator under case structure. Provided that

1) There exists positive bounded linear operator B_1, and B_1's $r(B_1) < 1$, and $u^{*} \in P, u_1 \in P$, get

$$-u^{*} \leq Ax \leq B_1 x + u_1, \forall x \in P;$$

2) There exists positive bounded linear operator B_2, B_2's $r(B_2) < 1$, and $u_2 \in P$, get

$$Ax \geq B_2 x - u_2, \forall x \in (-P);$$

3) $A\theta = \theta$, there exists *Frechet* derivative A'_θ of A at θ, 1 is not the eigenvalue of A'_θ, and the sum μ of algebraic multiplicity of A'_θ's all eigenvalues in the range $(1, \infty)$ is a nonzero even number,

$$A\left(P\setminus\{\theta\}\right)\subset\overset{\circ}{P},\ A\left((-P)\setminus\{\theta\}\right)\subset-\overset{\circ}{P}$$

Then A exists three nonzero fixed points at least: one positive fixed point, one negative fixed point and a sign-changing fixed point.

3. Results

Theorem provided (H_1) (H_2) (H_3) works, boundary-value problem (1) exists a sign-changing solution at least, and also a positive solution and a negative solution.

Proof provided linear operator $B=\left(\lambda_1-\frac{\varepsilon}{2}\right)K$, Lemma 2 knows $B:C[0,1]\to C[0,1]$ is a positive bounded linear operator. Lemma 4 gets K's $r(K)=\frac{1}{\lambda_1}$, so $r(B)=\left(\lambda_1-\frac{\varepsilon}{2}\right)r(K)=1-\frac{\varepsilon}{2\lambda_1}<1$.

(H_3) knows $m>0$ and gets

$$f(u)\le\left(\lambda_1-\frac{\varepsilon}{2}\right)u+m,\ \forall t\in[0,1],u\ge 0 \tag{5}$$

$$f(u)\ge\left(\lambda_1-\frac{\varepsilon}{2}\right)u-m,\ \forall t\in[0,1],u\le 0 \tag{6}$$

Let $u_0(t)=m\int_0^1 G(t,s)\mathrm{d}s$, obviously, $u_0\in P$. Such that, for any $u(t)\in P$, there

$$\begin{aligned}(Au)(t)&=\int_0^1 G(t,s)f(u(s))\mathrm{d}s\\&\le\int_0^1 G(t,s)\left(\left(\lambda_1-\frac{\varepsilon}{2}\right)u+m\right)\mathrm{d}s\\&\le\left(\lambda_1-\frac{\varepsilon}{2}\right)\int_0^1 G(t,s)u(s)\mathrm{d}s+m\int_0^1 G(t,s)\mathrm{d}s\\&=\left(\lambda_1-\frac{\varepsilon}{2}\right)Ku(t)+u_0(t)\\&=Bu(t)+u_0(t)\end{aligned}$$

And for any $u^*\in P$, from (H_1), obviously gets $(Au)(t)\ge-u^*(t)$.

For any $u(t)\in-P$, there

$$\begin{aligned}(Au)(t)&=\int_0^1 G(t,s)f(u(s))\mathrm{d}s\\&\ge\int_0^1 G(t,s)\left(\left(\lambda_1-\frac{\varepsilon}{2}\right)u-m\right)\mathrm{d}s\\&\ge\left(\lambda_1-\frac{\varepsilon}{2}\right)\int_0^1 G(t,s)u(s)\mathrm{d}s-m\int_0^1 G(t,s)\mathrm{d}s\\&=\left(\lambda_1-\frac{\varepsilon}{2}\right)Ku(t)-u_0(t)\\&=Bu(t)-u_0(t)\end{aligned}$$

Consequently (1) (2) in lemma 5 works.

We note that $f(0)=0$ can get $A\theta=\theta$, from (H_2), we know $\forall\varepsilon>0$, and $\exists r>0$ gets

$$\left|f(u)-\beta u\right|\le\varepsilon u,|u|\le r$$

Then

$$\left|(Fu)(t)-\lambda u(t)\right|=\left|f(u(t))-\beta u(t)\right|\le\varepsilon\|u\|,\forall\|u\|\le r$$

$$\|Au-A\theta-\beta Ku\|=\|K(Fu)-\beta Ku\|\le\varepsilon\|K\|\|u\|,\forall\|u\|\le r$$

Such that

$$\lim_{\|u\|\to 0}\frac{\|Au-A\theta-\beta Ku\|}{\|u\|}=0$$

i.e. $A'_\theta=\beta K$, from lemma 4 we get linear operator *K*'s eigenvalue is $\frac{1}{\lambda_n}$, then A'_θ's eigenvalue is $\frac{\beta}{\lambda_n}$. Because $\lambda_{2n_0}<\beta<\lambda_{2n_0+1}$, let μ be the sum of algebraic multiplicity of A'_θ's all eigenvalues in the range $(1,\infty)$, then $\mu=2n_0$ is an even number.

From (H_1) $f(u)u>0$, $u\in R\setminus\{0\}$, there

$$f(u(t))>0,\forall t\in[0,1],u(t)>0,$$

$$f(u(t))<0,\forall t\in[0,1],u(t)<0.$$

Easy to get

$$F(P\setminus\{\theta\})\subset P\setminus\{\theta\},F((-P)\setminus\{\theta\})\subset(-P)\setminus\{\theta\},$$

Lemma (1) for any $u(t)\in P$,

$(Ku)(t)=\int_0^1 G(t,s)u(s)\mathrm{d}s\ge\gamma\int_0^1 G(0,s)u(s)\mathrm{d}s$,

consequently $K(P\setminus\{\theta\})\subset\overset{\circ}{P}$. Such that

$$A(P\setminus\{\theta\})\subset\overset{\circ}{P},A((-P)\setminus\{\theta\})\subset-\overset{\circ}{P},$$

Such that (3) in lemma 5 works. According to lemma 5, *A* exists three nonzero fixed points at least: one positive fixed point, one negative fixed point and one sign-changing fixed point. Which states that boundary-value problem (1) has three nonzero solutions at least: one positive solution, one negative solution and one sign-changing solution.

4. Conclusion

Provided that all conditions of the theorem are satisfied, and $f(u)$ is an odd function, then boundary-value problem (1) has four nonzero solutions at least: one positive solution, one negative solution and two sign-changing solutions.

Note

By using case theory to study the topological degree of non-cone mapping and the calculation of fixed point index, it's an attempt to combine case theory and topological degree theory, the author thinks it's an up-and-coming topic and expects to have further progress on that.

References

[1] Ma, Y.H. and Ma, R.Y. (2002) Positive Downward Convex Positive Solutions of a Class of Second-Order Three-Point Boundary-Value Problems. (*Natural Science Edition*) *Journal of Northwest Normal University*, **38**, 8-13.

[2] Sun, J.P. and Zhang, X.L. (2012) Existence of Positive Solutions of Nonlinear Third-Order Three-Point Boundary-Value Problems. (*Natural Science Edition*) *Journal of Northwest Normal University*, **48**, 1-5.

[3] Zhou, S.L., Wu, H.P. and Han, X.L. (2014) Existence of Positive Solutions of a Class of Four-Order Three-Point Boundary-Value Problems. (*Natural Science Edition*) *Journal of Sichuan University*, **51**, 11-15.

[4] Ma, Y.H. and Ma, R.Y. (2003) Positive Solutions of a Class of Singular Nonlinear Three-Point Boundary-Value Problems. *Journal of Mathematical Physics*, **23A**, 583-588.

[5] Shen, W.G. and Song, L.A. (2007) Existence of Positive Solutions of Singular Second-Order Ordinary Differential Equations Three-Point Boundary Value Problems under Superlinear Condition. (*Science Edition*) *Journal of Shandong University*, **42**, 91-94.

[6] Xie, C. and Luo, Z.G. (2009) Existence of Multiple Positive Solutions of a Class of Second-Order Ordinary Differential Equations Multi-Point Boundary-Value Problems. (*Natural Science Edition*) *Journal of Inner Mongolia Normal University*, **38**, 647-652.

[7] Liu, J.S. and Qiao, J. (2007) Existence of Sign-Changing Solutions of a Class of Second-Order Three-Point Boundary-Value Problems. *Journal of Taiyuan University of Technology*, **38**, 374-376.

[8] Xu, X.A. (2006) Sign-Changing Solutions of Impulsive Three-Point Boundary-Value Problems. *Applied Mathematics*, **19**, 606-612.

[9] Cui, Y.J., Zou, Y.M. and Li, H.Y. (2009) Sign-Changing Solutions and Applications of Nonlinear Operator Equations. *Systems Science and Mathematics*, **29**, 1094-1101.

[10] Sun, J.X. and Liu, X.Y. (2008) Computation of Topological Degree for Nonlinear Operators and Application. *Nonlinear Analysis*, **69**, 4121-4130. https://doi.org/10.1016/j.na.2007.10.042

[11] Sun, J.X. (2008) Nonlinear Functional Analysis and Its Applications. Science Press—CSPM, Beijing.

[12] Wang, Y.P. and Zhao, Z.Q. (2014) Existence of Sign-Changing Solutions of a Class of Nonlinear Third-Order Differential Equations Two-Point Boundary-Value Problems. *Journal of Qufu Normal University*, **40**, 29-32.

[13] Zhang, X. and Sun, J. (2010) On Multiple Sign-Changing Solutions for Some Second-Order Integral Boundary Value Problems. *Electronic Journal of Qualitative Theory of Differential Equations*, No. 44, 1-15. https://doi.org/10.14232/ejqtde.2010.1.44

[14] Li, H. and Liu, Y. (2011) On Sign-Changing Solutions for a Second-Order Integral Boundary Value Problem. *Computers* & *Mathematics with Applications*, No. 62, 651-656. https://doi.org/10.1016/j.camwa.2011.05.046