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Asymptotic Theory for a General Second-Order Differential Equation

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Abstract

An asymptotic theory developed for a second-order differential equation. We obtain the form of solutions for some class of the coefficients for large *x*.

Keywords

Asymptotic Form of Solutions, Second-Order

1. Introduction

In this paper, we examine the asymptotic form of two linearly independant solutions of the general second-order differential equation.

$$(py')' + qy' + ry = 0, \tag{1}$$

as $x \to \infty$, where x is the independant variable and the prime denotes $\frac{\mathrm{d}}{\mathrm{d}x}$.

The coefficients p,q and r are nowhere zero in some interval $[a,\infty)$. We shall consider the situation where p and r are small compared to q see (15) to identify the following case:

$$\frac{q'}{q} = \mathrm{o}\left(\frac{r}{q}\right), \ (x \to \infty) \tag{2}$$

and under (2) we shall obtain the forms of the asymptotic solutions for (1) as $x \to \infty$ which is given in Theorem 1.

If $p = 1$, then (1) reduces to the differential equation considered by Walker [1]. We do not investigate the case where $q^2 = \mathrm{o}(pr)$, the analysis for this case is already known for the Sturm-Liouville equation

$$(py')' + ry = 0,$$

see Eastham [2] and Atkinson [3].

We shall use the asymptotic Theorem of Eastham ([3], Section 2), [4] to obtain our main result of (1) in Section 4. The general feature of our method are given in Sections (2) and (3), with some examples in Section (5).

2. The General Method

We write (1) in a standard way [5] as a first-order system:

$$Y' = AY \tag{3}$$

where

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix} \tag{4}$$

and the matrix is given by

$$A = \begin{pmatrix} 0 & p^{-1} \\ -r & -qp^{-1} \end{pmatrix}. \tag{5}$$

As in [6] we express the matrix A in the diagonal form:

$$T^{-1}AT = \Lambda = diag(\lambda_1, \lambda_2) \tag{6}$$

and we therefore require the eigenvalues λ_j and the eigenvectors v_j of A, $j = 1, 2$.

The characteristic equation of is given by:

$$p\lambda^2 + q\lambda + r = 0. \tag{7}$$

An eigenvector v_j corresponding to λ_j is

$$v_j = (1 \quad p\lambda_j)^* \tag{8}$$

where the superscript * denote the transpose.

Now by (7)

$$\lambda_j = -\frac{q}{2p} \pm \frac{(q^2 - 4pr)^{1/2}}{2p} \quad (j = 1, 2) \tag{9}$$

Now we define the matrix T in (6) by

$$T = \begin{bmatrix} 1 & 1 \\ p\lambda_1 & p\lambda_2 \end{bmatrix} \tag{10}$$

Hence by (6), the transformation

$$Y = TZ, \tag{11}$$

takes (3) into

$$Z' = (\Lambda - T^{-1}T')Z \tag{12}$$

Now if we write

$$T^{-1}T' = (t_{jk}), \tag{13}$$

then by (7) and (10)

$$t_{1j} = (\lambda_1 - \lambda_2)^{-1}\left[(p'\lambda_j^2 + q'\lambda_j + r')(2p\lambda_j + q)^{-1} - \frac{p'}{p}\lambda_j\right] \tag{14}$$
$$t_{2j} = -t_{1j} \qquad (j = 1,2).$$

Now we need to work (14) in terms of r, p and q in order to determine (12) and then make progress for (1).

3. The Matrices Λ and $T^{-1}T'$

At this stage we require the following conditions in the coefficients r, p and q as $x \to \infty$.

Condition I. r, p and q are nowhere zero in some interval $[a, \infty)$, and

$$rp = o(q^2), \quad (x \to \infty) \tag{15}$$

we write

$$\delta = \frac{rp}{q^2} \to 0 \quad (x \to \infty) \tag{16}$$

Condition II.

$$\delta\frac{r'}{r}, \delta\frac{p'}{p}, \delta\frac{q'}{q} \text{ are all } L(a, \infty). \tag{17}$$

Now if we let

$$D = \frac{(q^2 - 4pr)^{1/2}}{2p} \tag{18}$$

then (9) gives

$$\lambda_j = -\frac{q}{2p} \pm D \quad (j = 1,2) \tag{19}$$

where by(18) and (16)

$$D = \frac{q}{2p}(1 - 4\delta)^{1/2} \sim \frac{q}{2p} \quad (x \to \infty). \tag{20}$$

Now by (19) and (20)

$$\lambda_1 = -\frac{r}{q}\left[1 + \delta + O(\delta^2)\right], \tag{21}$$

and

$$\lambda_2 = -\frac{q}{p}\left[1 - \delta + O(\delta^2)\right] \tag{22}$$

Now using (14), (21) and (22) we obtain

$$t_{11} = t_{21} = O(\Delta), \tag{23}$$

$$t_{12} = -t_{22} = -\frac{q'}{q} + O(\Delta), \tag{24}$$

where

$$\Delta = \left(\left| \frac{r'}{r}\delta \right| + \left| \frac{p'}{p}\delta \right| + \left| \frac{q'}{q}\delta \right| \right) \tag{25}$$

Hence by (17),

$$\Delta \in L(a,\infty). \tag{26}$$

Therefore, by (23), (24) and (26), we can write (12) as:

$$Z' = (\Lambda + R + S)Z, \tag{27}$$

where

$$R = \begin{bmatrix} 0 & \frac{q'}{q} \\ 0 & -\frac{q'}{q} \end{bmatrix}, \tag{28}$$

and S is $L(a,\infty)$ by (26).

4. The Asymptotic Form of Solutions

Theorem 1. Let the coefficients r and p in (1) be $C^1[a,\infty)$ while q to be $C^2[a,\infty)$.

Let (15) and (17) hold.

Let

$$\frac{q'}{q} = o\left(\frac{r}{q}\right) \quad (x \to \infty) \tag{29}$$

$$\left(\frac{q'p}{q^2}\right)', \frac{r^2 p}{q^3} \text{ are } L(a,\infty) \tag{30}$$

Let

$$Re\left[\frac{q}{p} - 2\frac{r}{q} + \frac{q'}{q}\right] \text{ be of one sign in } [a,\infty). \tag{31}$$

Then (1) has solutions y_1 and y_2 such that

$$y_1 \sim \exp\left(-\int_a^x \frac{r}{q}\mathrm{d}t\right), \tag{32}$$

$$y_1' = o\left[qp^{-1}\exp\left(-\int_a^x \frac{r}{q}\mathrm{d}t\right)\right] \tag{33}$$

while

$$y_2 \sim q^{-1}\exp\left(\int_a^x \left[-\frac{q}{p} + \frac{r}{q}\right]\mathrm{d}t\right), \tag{34}$$

$$y_2' \sim p^{-1}\exp\left(\int_a^x \left[-\frac{q}{p} + \frac{r}{q}\right]\mathrm{d}t\right). \tag{35}$$

Proof. As in [6], we apply the Eastham theorem ([3], section 2) to the system (27) provided only that Λ and R, satisfy the required conditions.

We shall use (15), (17), (29), and (31).

We first require that

$$\frac{q'}{q} = o(\lambda_1 - \lambda_2), \tag{36}$$

this being [2] for our system,

$$\lambda_1 - \lambda_2 = \frac{q}{p}(1-4\delta)^{1/2}, \tag{37}$$

Thus (36) holds by (15) and (29).

Second, we need

$$\left[(\lambda_1 - \lambda_2)^{-1}\frac{q'}{q}\right]' \in L(a,\infty). \tag{38}$$

this being [2] for our system. By (38), this requirement is implied by (17) and (30).

Finally we show that the eigenvalues μ_k of $\Lambda + R$ satisfy the dichotomy condition [2].

As in [6] and [7], the dichotomy condition holds if

$$Re(\mu_1 - \mu_2) = f + g, \tag{39}$$

where f has one sign in $[a,\infty)$ and g is $L(a,\infty)$ [2].

Now by (6) and (28):

$$\mu_1(x) = \lambda_1(x),\ \mu_2(x) = \lambda_2(x) - \frac{q'}{q}, \tag{40}$$

then by (21), (22) and (40)

$$Re(\mu_1 - \mu_2) = Re\left(\frac{q}{p} - 2\frac{r}{p} + \frac{q'}{q}\right) + O\left(\frac{r^2 p}{q^3}\right), \tag{41}$$

Thus, by (31) and (30), (39) holds. Since (27) satisfies all the conditions for the asymptotic result [3, section 2], it follows that as $x \to \infty$, (27) has two linearly independant solutions.

$$Z_k(x) = [e_k + o(1)]\exp\left(\int_a^x \mu_k(t)\,dt\right) \tag{42}$$

with e_k the coordinate vector with k-th coponment unity and other coponments zero.

Finally, on transforming back to y via (10), (11), (4) and making use of (40), (21), (22) and (30), we obtain (33), also (32) after adjusing y_1 by a constant multiple, and similary for y_2 and y_2'.□

5. Examples

Example 1. *We consider the cofficients in* (1) *given by*

$$r(x) = c_1 x^{\alpha_1},\ q(x) = c_2 x^{\alpha_2},\ p(x) = c_3 x^{\alpha_3}.$$

α_i and c_i $(1 \le i \le 3)$ are real constants with $c_i \ne 0$. Then (15) and (17) of

Theorem 4.1 hold under the conditions

$$2\alpha_2 - \alpha_1 - \alpha_3 > 0. \tag{43}$$

Also (29) true if

$$\alpha_1 - \alpha_2 + 1 > 0 \tag{44}$$

Now in (30) $\left(\frac{q'p}{q^2}\right)'$ is $L(a,\infty)$ if

$$\alpha_2 - \alpha_3 + 1 > 0 \tag{45}$$

wich is *true* by (43) and (44).

Also, in (30), $\frac{r^2 p}{q^3}$ is $L(a,\infty)$ if

$$3\alpha_2 - 2\alpha_1 - \alpha_3 > 1. \tag{46}$$

So all conditions of theorem 4.1 are true under (43), (44) and (46). For example if we take $\alpha_1 = \alpha_2$.

Then all condition are true if

$$\alpha_2 - \alpha_3 > 1. \tag{47}$$

Example 2. *Let* $r(x) = c_1 x^{\alpha_1} \exp(x^a)$, $p(x) = c_2 x^{\alpha_2} \exp(-4x^b)$, $q(x) = c_3 x^{\alpha_3} \exp(-x^b)$
where $b \ge a > 0$, α_i and c_i $(1 \le i \le 3)$ are real constants with $c_i \ne 0$.

Again it is easy to check that all conditions of Theorem 4.1 are satisfied.

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