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A Note on Kuratowski's Theorem and Its Related Topics

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Abstract

In point set topology, it is well known that the Kuratowski 14-set problem is one of the most interesting results. In this note, we first give a brief survey of the Kuratowski's theorem. In particular, we will study and investigate the structure of the boundary of a given subset in a topological space. Some new results and topics which are related to the theorem of Kuratowski are presented and discussed. Finally, we pose some open problems of Kuratowski-type.

Keywords

Kuratowski 14-Set, Kuratowski's Theorem

1. Introduction

The problem of Kuratowski 14-sets was first proposed in 1922 by the famous Polish mathematician Kazimierz Kuratowski who discovered that at most 14 distinct sets can be generated by applying the closure operator and the complementation operator on a subset A of a topological space X in any order repeatedly (see [1]). We now call a subset A in a topological space a K-set if A can generate 14 distinct sets by taking closure and complementation on A in any order. To find an example of a K-set in the real line is posed to many mathematics students all over the world, usually in the class of elementary general topology. The students can easily check that the set $A=(0,1)\cup(1,2)\cup\{3\}\cup[4,5]\cap Q$ is a K-set (see [2]). Another interesting example of a K-set in the real line is the set $A=\{1/n \mid n=1,2,3,\cdots\}\cup(3,4)\cup\{4,5\}\cup[6,7]\cup(7,8)\cap Q$ [3] [5].

Now, we denote the complementation of the set A by $c(A)$. As $c(c(A))=A=e(A)$, where e is the identity operator, and so $c^2=e$. We describe this fact by saying that the operator c has the involution property, moreover if $A \subseteq B$, then $c(A) \supseteq c(B)$. Also,we use $f(A)$ to denote the closure of the set *A*. It means the smallest closed set containing *A*. According to Kuratowski, the closure operator f has the following properties:

i) $f(\varnothing) = \varnothing$, (the closure of a void set).

ii) $f(A) \supseteq A$, (the expansion property).

iii) $f(f(A)) = f(A)$, (the idempotent property).

iv) $f(A \cup B) = f(A) \cup f(B)$, (the additivity property).

In point set topology, we usually use A^- to denote the closure of a set *A*. However, for the sake of convenience, in dealing with the algebraic closure of a set *A*, we simply use $f(A)$ to denote the closure of the set *A*. Thus, both A^- and $f(A)$ are used to denote the closure of a set *A*. Now, we also use the notation A° and $i(A)$ to denote the interior of a set *A*. For the sake of simplicity, We write B as the complement of the set *A*.It is clear that $i(A)$ is the complete of $f(B)$. With the applications of the algebraic closure operator, It is well known that the interior of the set *A* is the set $i(A) = cf(B) = A^\circ$. Since B is used to denote $c(A)$, we now list the family of 14 sets generated by *A* in **Table 1**.

Table 1. The family of 14 sets generated by *A*.

$A_1 = A = e(A)$, the identity of A	$B_1 = c(A)$,
$A_2 = f(A) = A^-$,	$B_2 = fc(A) = B^-$,
$A_3 = cf(A) = B^0$,	$B_3 = cfc(A) = A^0$,
$A_4 = fcf(A) = B^{0-}$,	$B_4 = fcfc(A) = A^{0-}$,
$A_5 = cfcf(A) = A^{-0}$,	$B_5 = cfcfc(A) = B^{-0}$,
$A_6 = fcfcf(A) = A^{-0-}$,	$B_6 = fcfcfc(A) = B^{-0-}$,
$A_7 = cfcfcf(A) = A^{0-0}$,	$B_7 = cfcfcfc(A) = B^{0-0}$.

In order to illustrate the inclusion relationship of the above 14 sets, we give the following relationship diagram. As the relationship diagram would look quite bulky if we put all the set inclusion symbols in the diagram. Thus, for the sake of simplicity, we use "→" to mean the inclusion relationship in the diagram. This nice relationship diagram was first given in Kuratowski in [1] (**Diagram 1**).

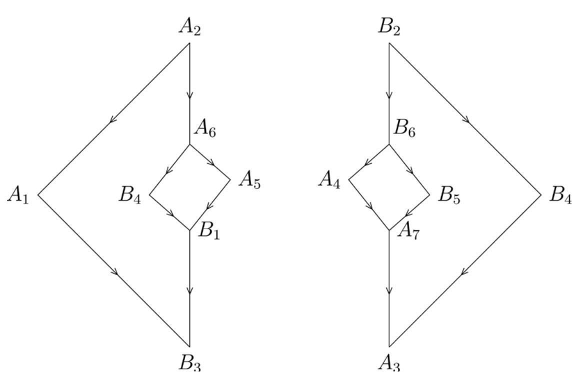

Diagram 1. The inclusion relationship of the 14 sets.

The papers [3]-[8] are closely related to the Kuratowski's theorem. The paper of Gardner and Jackson in [9] gave detailed description and information of the Kuratowski 14 set problem. The papers [3] [10]-[23] also studied and investigated the problems which are closely related to the closure and interior operators of a given set.

2. Algebraic Operator and the Algebraic Interior Operators

Throughout this section, we use f and i to denote the algebraic closure operator, respectively. According to Kuratowski [3], the algebraic closure operator has the following properties:

i) Monotonic property, *i.e.*, if $B \subseteq A$, then $f(B) \subseteq f(A)$,

ii) Expansive property, *i.e.*, $A \subseteq f(A)$,

iii) Idempotent property, *i.e.*, $f(f(A)) = f(A), f^2 = f$,

iv) Additive property, *i.e.*, $f(A \cup B) = f(A) \cup f(B)$.

On the other hand, the algebraic interior operator i has the following properties:

a) Monotonic property, *i.e.*, if $B \subseteq A$, then $i(B) \subseteq i(A)$,

b) Shrinking property, *i.e.*, $i(A) \subseteq A$,

c) Idempotent property, *i.e.*, $i(i(A)) = i(A)$, *i.e.*, $i^2 = i$,

d) Additive property, *i.e.*, $i(A \cup B) = i(A) \cup i(B)$.

Consider $A \subseteq B$. By using the above properties of algebraic closure operator and the algebraic interior operator, we obtain immediately the following crucial theorem.

Theorem 2.1. *Let* f, i *be the closure and interior operators. Then we derive the following identities,* $(fi)^2 = fi$ *and* $(if)^2 = if$ *.*

Proof. By using the shrinking property of i, we have $ifi \le fi$. By applying the monotonic property and the idempotent property of f, we can easily deduce the equations $(fi)^2 = fifi \le f(fi) = f^2 i = fi$. On the other hand, because $i \le fi$ we have $i = i^2 \le ifi$, consequently, we derive that $fi = (fi)^2$. Recall that $i = cfc$ and $f = cic$, by acting these two identities in the above equations, we obtain $if = (if)^2$. Thus, our theorem is proved. □

Corollary 2.2. *Because* $f(A) = cif(A)$ *and* $i(A) = cfc(A)$*. We immediately see in the above* **Table 1**, *we have* $A_4 = A_8$, $B_8 = B_4$*. Since the sets* A_1, A_2, A_3, A_4, A_5, A_6, A_7, *and* B_1, B_2, B_3, B_4, B_5, B_6, B_7 *are possibly distinct. We therefore conclude that at most* 14 *distinct sets can be generated by taking closure and complementation operator on the set* A, *repeatedly in any order.*

Let A, B be subsets of a given set M. Then, by applying the complementation operator c on these two sets. we have the following properties:

a) The inversion property, If $A \subseteq B$, then $c(A) \supseteq c(B)$, we simply denote this situation by $c \downarrow$.

b) The involution property, $c(c(A)) = A$, *i.e.*, $c^2 = e$, the identity.

c) The separation property. Since $c(A) = M - A$, $c(A) \cap A = \varnothing$, we simply denote this case by $c \wedge e = 0$.

Now, we consider the complementation operator c on a bounded lattice $(L,\wedge,\vee,0,1)$, where 0 is the minimum element of the lattice L and 1 is the greatest element of the lattice L. In general, the bounded lattice can be regarded as a power set of a give set L, if $\alpha(x)\le\beta(x)$, then we write $\alpha\le\beta$. The $0\in L$ can always be regarded as the zero function. We notice that if $c\uparrow$, then $c(g\wedge h)\le cg\wedge ch$, $c(g\vee h)\ge cg\vee ch$. Moreover, if $g\le h$ and $\alpha\uparrow$, then we always have $\alpha g\le\alpha h$.

For the complementation operator c, we have the following theorem.

Theorem 2.3. *Let* c *be a complementation operator and* e *the identity operator on a bounded lattice* $(L,\wedge,\vee,0,1)$. *Then* $c(1)=0$ *and* $c(0)=1$.

Proof. The proof is quite easy. For the sake of the interest of the readers, we provide here two proofs.

Proof i). Because $c^2=e$, c is clearly surjective and injective. Thus, for $0\in L$, by the inversion property of c, *i.e.*, $c\downarrow$, we have $0=c(x_L)\ge c(1)$ and hence $c(1)=0$. Now, by the involution property of c, we deduce that $1=c(c(1))=c(0)$.

Proof ii). Suppose that $c(x)\ne 0$ for any $x\in L$. Then we have

$c(c(x))\ne 0$. This contradicts to $x=0$, $c(c(0))=e(0)=0$. Hence, there exists $x_L\in L$ such that $c(x_L)=0$. Consequently, $0\le x_l\le 1$, $0=c(x_L)\ge c(1)$. Therefore, $c(1)=0$. The proof is now completed.

3. The Boundary of a Subset in a Topological Space

For any subset A in a topological space, the boundary of A is defined by

$b(A)=A^-\cap c(A)^-$. Hence, the boundary operator of A is denoted by

$b=f\wedge fc$. Similarly, we can define the rim operator, $r=e\wedge fc$ and the side operator by $s=f\wedge c$. Obviously, the boundary operator, the rim operator and the side operator are closely related on a distributive lattice. It is noted that we always have $b=r\vee s$, $e=i\vee r$. It is obvious that the boundary, the rim and the side of a subset A in a topological space are different concepts. The following example is an example of the above three notations.

Example 3.1. *Let* $A=[2,3)$ *be a closed-open interval in the real line* R. *Then according to the definitions of the boundary, the rim and the side of the given set* A *in a topological space* X, *we immediately see that* $b(A)=\{2,3\}$, $r(A)=\{2\}$ *and* $s(A)=\{3\}$. *It is clear that the above three concepts are distinct concepts, in fact, the rim and the side of a subset* A *in* X *are parts of the boundary of* $b(A)$.

It is noteworthy that the sequence $s,s^2,s^3,\cdots,s^n,\cdots$ of powers of s is sometimes infinite(for $i\ne j$, $s^i\ne s^j$).

In the following example, we construct an infinite sequence of side operator.

The construction of such an infinite sequence of side operator in a subset of positive integers is quite technical.

Example 3.2. *We first consider the set* $N=\{1,2,3,\cdots\}$, $F(N)=\{A\subseteq N\}$. *Now, we make* $F(N)$ *a bounded lattice* $(L,\wedge,\vee,0,1)$. *In this bounded lattice* L, "0" *is the empty set and* "1" *is the set* N *itself. The operation* "$\vee$" *can be* *regarded as the set union and the operation* "$\wedge$" *can be regarded as the set intersection. Consider the subsets in this bounded lattice* $(L,\wedge,\vee,0,1)$.

Let N be the set of all positive integers. For all $n\in N$, let $F_n=\{n,n+1,\cdots\}$, denoted by $[n,\infty)$ with $F_\infty=\varnothing$. Consider the set

$$F_\Sigma=\{F_\infty,F_1,F_2,\cdots,F_n,F_{n+1},\cdots\}.$$

Now for $A\subseteq N$. Define $f(A)=\bigcap\{F_i:F_i\in F_\Sigma \text{ and } F_i\supseteq A\}$. Then, we can easily verify that f is an algebraic closure operator of $\mathcal{P}(N)\to\mathcal{P}(N)$, where $\mathcal{P}(N)$ is the power set of the set N. We first take $N=\{1,3,5,\cdots\}$. Then we have $s(A)=\{2,4,6,\cdots\}$, and $s^2(A)=\{3,5,7,\cdots\}$; $s^3(A)=\{4,6,8,\cdots\}$, and so $s^{2k-1}(A)=\{2^k,2^k+2,2^k+4,\cdots\}$, $s^{2k}(A)=\{2^k+1,2^k+3,\cdots\}$.

Thus, for $i\neq j$, we have $s^i(A)\neq s^j(A)$. This shows that $\{s,s^2,s^3,\cdots\}$ is an infinite sequence.

We now call an expansive operator g a quasi algebraic closure operator if $g^2\neq g$ but $g^2=g^3$. Obviously, the quasi algebraic closure operator is a generalized algebraic closure operator. Naturally, one would ask whether the Kuratowski's theorem still holds for the quasi algebraic closure operator?

In the following example, we show that Kuratowski's theorem does not hold for the quasi algebraic closure operator.

Example 3.3. *For any arbitrary integer* n, *consider the subset* $S=\{(n,n+1)\}\cup\{\mathcal{R}\setminus(n,n+1)\}$.

We first let $\mathcal{P}(\mathcal{R})$ be the power set of the real line $\mathcal{R}$.

Define the function $g_0:S\to\mathcal{P}(\mathcal{R})$ by

$$g_0(n,n+1)=(-\infty,n+1]\cup[n+1,\infty)=\mathcal{R}\setminus(n+1,n+2),$$

and

$$g_0(\mathcal{R}\setminus(n,n+1))=\mathcal{R}.$$

Now, we define a function $g:\mathcal{P}(\mathcal{R})\to\mathcal{P}(\mathcal{R})$ as follows:

$$g(x)=x\cup\{\cup g_0(y)\mid y\subseteq x, y\in S\},$$

where, $x\subseteq\mathcal{R}$. Then, after verification, we can easily see that if $x_1\subset x_2$, then $g(x_1)\subseteq g(x_2)$, also $x\subseteq g(x)$. Thus, g is an expansive operator.

By the definition of the operator g again, we can verify that $g^2\neq g$ but $g^2=g^3$. Hence, g is indeed a quasi algebraic closure operator.

Let $A=(0,1)$. Then by applying the operator g on the set $A=(0,1)$, we have $g((0,1))=(-\infty,1]\cup[2,\infty)$.

Applying c and g by repetition on the above equality, we obtain the following sequence of sets:

$$cg((0,1))=(1,2),$$

$$\vdots$$

$$(cg)^2((0,1))=((2,3)),$$

$$(cg)^k((0,1))=((k,k+1)),$$

$$\vdots$$

Thus, it is clear that $\left((cg)^k\right)_{k=1}^{\infty}$ is an infinite sequence of sets.

This example illustrates that the theorem of Kuratowski does not hold anymore for quasi algebraic closure operators.

For the boundary operator and the rim operator, we have the following theorem. (see [24])

Theorem 3.4. (*i*) $b^3 = b^2$ (*ii*) $r^2 = r$.

Proof of i):

$$b^2 = b(b) = f(b) \wedge fcb = f(f \wedge fc) \wedge fcb \leq f^2 \wedge fc \wedge fcb$$
$$\leq f \wedge fc \wedge * \leq f \wedge fc = b$$

Thus we have $b^3 = b^2(b) \leq b(b) = b^2$. $b^2 \wedge fcb = (fb \wedge fcb) \wedge fcb$. On the other hand, by $b^2 \leq b$, we deduce that $cb^2 \geq cb$, $fcb^2 \geq fcb$ and $b^3 = b(b^2) = fb^2 \wedge fcb^2 \geq fb^2 \wedge fcb \geq b^2 \wedge fcb = (fb \wedge fcb) \wedge fcb = fb \wedge fcb = b(b)$ $= b^2$. Hence, we have proved that $b^2 = b^3$.

Proof of ii): Clearly, $r^2 = r(r) = r \wedge fcr \leq r$. Also by $r = e \wedge fc$, we have $r \leq fc$ and thereby we obtain $r^2 = r(r) = r \wedge fcr \geq r \wedge fc = e \wedge fc \wedge fc = e \wedge fc = r$. Thus, we deduce that $r^2 \leq r \leq r^2$, and consequently $r^2 = r$.

Remark. The amalgamation of the boundary operator with the other algebraic operators also produces a number of useful equalities. (see [25])

Proposition 3.5. (*i*) $b^2 f = bf$.

Proof. By $bf = f^2 \wedge fcf = f \wedge fcf$, $f \wedge fcf \leq fc$, $f \wedge fcf \leq fcf$. Hence we have $c(fcfcf) \geq cf$. Consequently,

$$c(f \wedge fccf) \geq f(cfcfcf) \geq fcffcfcf \geq fcf.$$

This implies that $fcbf \geq fcf$. Thus,
$b^2 f = b(bf) = fbf \wedge fcbf = bf \wedge fcbf = f \wedge fcfcbf = f \wedge fcf = b$.

Remark 3.6. *Using the equalities* $i = cfc$, $f = cic$, $r = f \wedge fc$, $c^2 = e$ *we can easily deduce that* $bc = b$, $bf = bic$, $b^2 i = bi$, $ibi = 0$ (*that is, the zero operator*), $ib^2 = 0$, $ir = ri = 0$, $bf = fr$, $b^2 = rb$, $b^2 r = rbr = rfr$, $frf = rf$, $re = s$, $sc = r$, $\cdots$, etc.

Finally, we consider some questions which are closely related to the theorem of Kuratowski.

Let A, B be subsets of a space *M*. Then we call an operator c' an algebraic quasi-complmentation operator if c' satisfies the separation property $c'(A) = M - A$, $c'(A) \cap A = \varnothing$, simply denoted it by $c' \wedge e = 0$; also c' satisfies the inversion property, *i.e.*, if $A \subseteq B$, then $c(A) \supseteq c(B)$, we simply denote this situation by $c' \downarrow$, but c' does not satisfies the involution property, $c(c(A)) \neq A$, *i.e.*, $c^2 \neq e$, the identity

We now pose the following question.

Question 3.7. *a*) *Let* A *be a given subset in a topological space* X. *How many distinct sets can be generated by applying the algebraic closure operator and the algebraic quasi-complementation operator successfully on* A *in any order*?

b) Likewisely, we call a shrinking operator j an algebraic quasi-interior operator if $j^2 = j^3$ but not $j^2 = j$ .Can we construct an algebraic quasi-interior open set for the set A, that $j(A) = A$?

Now, we ask when will an algebraic quasi-interior operator j and the complementation operator c generate a finite semigroup under the usual operator composition?

In order to help answer the above questions, we construct an algebraic quasi-complementation operator c'.

Let R be a real line. For any element $x \in R$, define $c'(x) = (-\infty, -x]$, $c'(-x) = [x, \infty)$ and $c'(0) = \{0\}$. Obviously, $c' \wedge e = 0$; and $c' \downarrow$ because if $x \le y$,then $c'(x) \ge c'(y)$, but $c'^2 \ne e$. Thus, c' is indeed an algebraic quasi-complementation operator.

By using the idea of algebraic quasi closure operator described in Example 2.6, one can therefore construct an algebraic quasi interior operator. Hence, by the idea of the construction of an algebraic quasi interior operator, the reader should be able to answer the above questions.

4. Further Relationship of Algebraic Closure and the Algebraic Interior Operations

We notice that an equivalent formulation of the Kuratowski's theorem is that at most 7 distinct sets can be obtained by the composition of the algebraic closure operator f and the algebraic interior operator i. In order to show clearly the mutually inclusion relationship of these 7 distinct sets composed by f and i on a given set A. we give again the following Hasse diagram (Diagram 2).

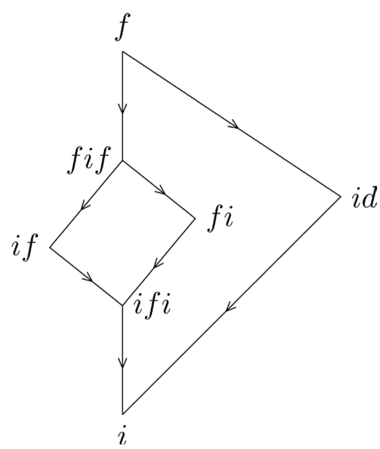

Diagram 2. The inclusion relationship of 7 distinct sets.

For a given subset A in a topological space X, we call the sets generated from A by successive applications of the algebraic closure and complementation operations in any order on A the relatives of A. Thus, by Kuratowski's theorem, a K-set A always have 14 relative sets.

It is natural to ask whether there exists a subset in a topological space with less than 14 relative sets? The following theorem gives an answer to the above question.

Theorem 4.1. *If the rim of a set* A *is nowhere dense, then by successive applications of closure operator and complementation operator, in any order, on* A, A *has at most* 10 *relative sets.*

Proof. We first denote the interior of the set A by $i(A)$. Then $A = i(A) \cup r(A)$. Consequently, by using the algebraic closure operator, we have $f(A) = fi(A) \cup fr(A)$. Now, $if(A) = i(fi(A) \cup fr(A)) \subseteq fi(A) \cup ifr(A)$. Because the rim of the set A, say $r(A)$ is nowhere dense, we have $fr(A) = \varnothing$ and so, $f(A) \subseteq f(A)$. Thus, we see immediately that $i(if(A)) \subseteq ifi(A)$, that is, $f(A) \subseteq ifi(A)$. Conversely, $A \supseteq i(A)$ implies that $f(A) \supseteq ifi(A)$. Hence, we deduce that

$$f(A) = ifi(A). \tag{1}$$

Now we put $A_1 = A, A_2 = f(A)$ and so on. For $n \geq 2$, we put $A_{2n} = (fc)^{n-1} f(A)$, for $n > 1$, put $A_{2n+1} = (cf)^n (A)$. Also, similarly, we put $B_1 = c(A)$. For $n \geq 1$, $B_{2n} = (fc)^n (A)$, $B_{2n+1} = c(fc)^n (A)$. By using Kuratowski's theorem again, we always have $B_8 = B_4$, $A_8 = A_4$. By the equation obtained above, we see that $A_5 = B_8 = B_4$. Thus *A* has at most 10 distinct relative sets.

In the literature, P. Halmos [12] first called a set *A* a regular open set if $f(A) = A$. Dually, he called a set *B* regular closed if $fi(B) = B$. If *X* is a topological space containing a K-set, then we always have $(if)^2 = if$, $(fi)^2 = fi$. This means that the space *X* simultaneously contains a regular closed set and a regular open set.

The following theorem is also an interesting theorem.

Theorem 4.2. *If the rim of a set* A *is dense in a regular closed subset of a topological space* X, *then* A *has at most* 12 *distinct relative sets by taking the algebraic closure operator* f *and the the complementation operators* c *repeatedly in any order on* A.

Proof. Let T be a regular closed set. Then by definition, we have $fi(T) = T$. Since $A = i(A) \cup r(A)$, by the additive property of the algebraic closure, we have $f(A) = fi(A) \cup fr(A) = fi(A) \cup T$. This is because we assume that $r(A)$ is dense in T. Now, we have $f(A) \supseteq i(fi(A)) \cup i(T)$. This leads to $fif(A) \supseteq fif(A) \cup T = fi(A) \cup T$, and thereby $fif(A) \supseteq f(A)$. Trivially, $fif(A) \subseteq f(A)$. Hence, we have proved that $fif(A) = f(A)$, that is, $A_6 = A_2$. By the result of Kuratowski's theorem, we always have $B_8 = B_4$, $A_8 = A_4$. Thus, we can easily see that A has at most 12 distinct relative.

It was stated by D. Sherman [22] that 13 distinct operators can be produced by the combination of the algebraic closure operation f, the algebraic interior operation i and the intersection operation "$\wedge$" on a given set A. This result is the answer to the well known problem $p(f, i, \wedge)$. If the union operation "$\vee$" is also allowed to use, then D.Sherman in [22] has shown that at most 35 different operators can be derived.This problem is called the $p(f, i, \wedge, \vee)$ problem. The solution of the $p(f, i, \wedge)$ problem and its dual problem $p(f, i, \vee)$ are shown in the following Hasse diagrams [23] (**Diagram 3**).

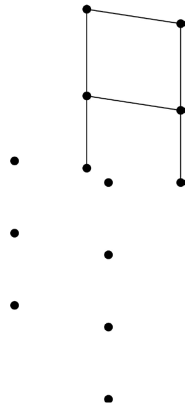

Diagram 3. The solution of the $p(f,i,\wedge)$ problem and its dual problem $p(f,i,\vee)$.

Proof. Let T be a regular closed set. Then $fi(T)=T$. Since $A=i(A)\cup r(A)$, by the additive property of the relation "$\wedge$" on a given set A. This result is the answer to the well known problem $p(f,i,\wedge)$. If the union operation "$\vee$" is also allowed to use, then at most 35 different operators can be formed. This problem is now called the $p(f,i,\wedge,\vee)$ problem. The solution of $p(f,i,\wedge)$ and $p(f,i,\vee)$ are shown in the following Hasse diagrams [23] (**Diagram 4** and **Diagram 5**).

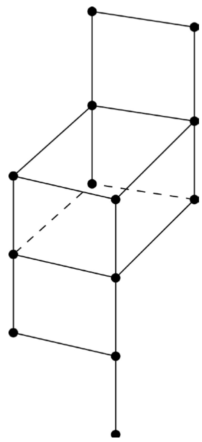

Diagram 4. $p(f,i,\wedge)$.

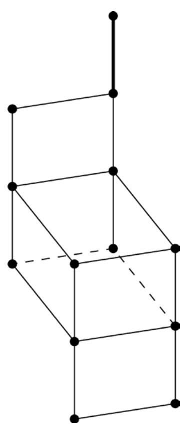

Diagram 5. $p(f,i,\vee)$.

It is clear that the $p(f,i,\vee)$ problem is the dual of the $p(f,i,\wedge)$ problem.

One can verify the following subset T of the real line R

$$T=\left[\{1/n, n\in N\}\right]\cup[2,4]-\{3+1/n\}\cup\left[[5,7]\cap\left(Q\cup\left(6+1/2n\pi, 6+1/(2n-1)\pi\right)\right)\right]$$

is a Shermann set, that is, it can generate 13 distinct sets by successively applications of the algebraic closure operator,the algebraic interior operator and the set intersection operations on T in any order (see also [22]).

5. An Example of Kuratowski 14 Sets in the Set of Positive Integers

Let M be the set of all positive integers and $A\subseteq M$. Then Hammer in 1960 defined $h(A)=\bigcup_{n=1}^{\infty}A^n$, where A^n is the set product of n copies of A. Clearly, $h(A)\supseteq A$ and $h(h(A))=h(A)$. Thus, h is an expansive and idempotent operator, we now call h the Hammer closure. Now, we define $i=chc$ and call it the Hammer interior operator. It is clear that i is also a shrinking and idempotent operator.

The following interesting question was asked by Hammer in 1960 [12]. Do h and c generated exactly 14 distinct subsets by composition of h and c in any order, in the set of positive integers? By using the expansive property of h and the shrinking property of $i=chc$, we can easily prove that $(ih)^2=ih$, $(hi)^2=hi$. Write $h=cic$ and $i=chc$. Then we obtain the following 14 distinct sets listed in **Table 1**.

We now give the following example to answer the open problem of Hammer in 1960. This example was given by Shum in 1996. (see [26]).

Example 5.1. *The following subset of the set of positive integers*

$$A=\left\{5,2^2\times3^2,7,2\times5^2,2^2\times5^3,3\times7^2,3^2\times7^3\right\}$$

is a K-set.

We list bellow the family of all the 14 relatives of the set A. This would help the readers to check the validity of the example.

$A_1=A=e(A)=\left\{5,2^2\times3^2,7,2\times5^2,2^2\times5^3,3\times7^2,3^2\times7^3\right\};$

$A_2=h(A)=A^-=A_1\cup\left\{5\times7,2^2\times5^4,2^2\times3^2\times5\times7,2^2\times3^2\times5^3\times7^3,\cdots\right\}\cup\cdots;$

$A_3=ch(A)=B^\circ=\left\{1,2,3,2\times3,2^2\times7^3,3^2\times5^3,2^2\times7,3^2\times5,\cdots\right\}\cup\cdots;$

$A_4=h(h(A))=B^{\circ-}=A_3\cup\left\{2^2\times3^2,2^2\times3^2\times5^3\times7^3,2^2\times3^2\times5\times7,\cdots\right\}\cup\cdots;$

$A_5=chch(A)=A^{-\circ}=\left\{5,7,2^2\times5^3,3^2\times3,2\times5^2,3\times7^2,\cdots\right\}\cup\cdots;$

$A_6=hchch(A)=A^{-\circ-}=A_5\cup\left\{5\times7,2^2\times3^2,2^2\times3^2\times5^3\times7^3,2\times3\times5^2\times7^2,\cdots\right\}\cup\cdots;$

$A_7=chch(A)=A^{\circ-\circ}=\left\{1,2,3,2^2\times3^2\times5\times7,\cdots\right\};$

$B_1=c(A)=\left\{1,2,3,7^2,2\times3,2\times5,2^2\times5^4,2^2\times5^3,3\times7^3,\cdots\right\};$

$B_2=hc(A)=B^-=B_1\cup\left\{2^2\times3^2,3\times7^2,3\times7^3,3^2\times7^3,\cdots\right\}\cdots;$

$B_3 = chc(A) = A^{\circ} = \{5, 7, 2\times 5, 3\times 7, 2\times 5^3, 3^2\times 5^3, 2\times 7^3\}\cdots;$

$B_4 = hchc(A) = A^{\circ -} = B_3 \cup \{5\times 7, 2^2\times 3^2\times 5^3\times 7^3, 2^2\times 5^4, 3\times 7^3, \cdots\}\cdots;$

$B_5 = chchc(A) = B^{-\circ} = \{1, 2, 3, 2\times 3, 3^2\times 7, 2^2\times 5^3, 3^2\times 7^3\times 2\times 3\times 5\times 7, \cdots\}\cdots;$

$B_6 = hchchc(A) = B^{-\circ -} = B_5 \cup \{2^2\times 3^2\times 5^3\times 7^3, 2^2\times 3^2\times 5\times 7, \cdots\}\cdots;$

$B_7 = chchchc(A) = B^{\circ -\circ} = \{5, 7, 5\times 7, 3\times 7^3, \cdots\}.$

The relationship of the above 14 sets can be seen in Diagram 6.

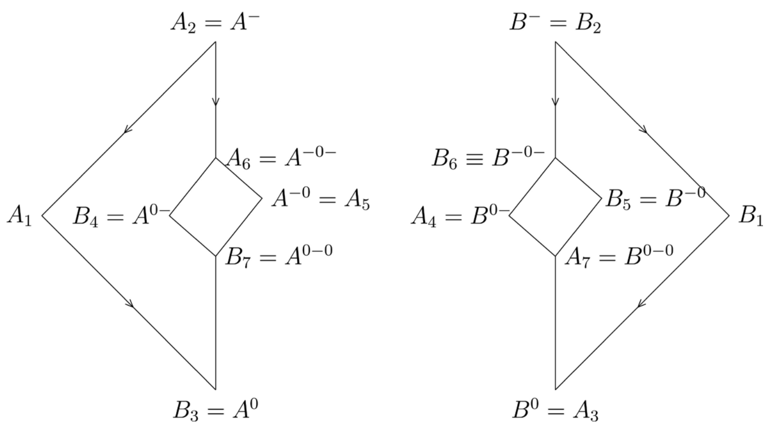

Diagram 6. The relationship of the 14 sets.

Recall that for a subset A in the set of positive integers Z_+, the Hammer closure of A is defined by $h(A) = \bigcup_{n=1}^{\infty}$, where A^n is the set product of n copies of A. It is natural to ask how to construct a K-set in the set Z_+? The following lemma gives some ideas and hints.

Lemma 5.2. *Let* p, q, r, s *be prime numbers and* $i(A) = chc(A)$. *If* $p^n \in i(A)$, *then* $p \in i(A)$, *where* $n > 1$.

Proof. If $p \notin i(A) = chc(A)$, then $p \in hc(A)$. This implies that $p^n \in hc(A)$ and so we have $p^n \notin chc(A) = i(A)$, a contradiction.

Lemma 5.3. *Assume that* $i(A) = \{p, p^n\}$. *Then* $n \le 3$.

Proof. If $n \ge 4$, then by our assumption, we have $p^{n-2} \notin i(A) = chc(A)$, and so $p^2 \notin chc(A)$. This leads to $p^{n-2} \in hc(A)$, and hence $p^n = p^{n-2}p^2 \in hc(A)$. Consequently, we deduce that $p^n \notin chc(A) = i(A)$, a contradiction.

By using the arguments $a \notin i(A)$, $b \in i(A)$, $ab \notin i(A)$ repeatedly, we deduce the following lemma.

Lemma 5.4. 1) Let m be a composite number and p is a prime number. If $i(A) = \{p, m\}$, then $p \mid m$;

2) If $i(A) = \{p, p^n q\}$ or $i(A) = \{p, pq^n\}$, then $n = 1$;

3) If $i(A) = \{p, p^2, n\}$, n is a composite number, then $n = p^3, p^4, p^5, ps$ or $p^2 s$, where s is a prime number but is not equal to p;

4) If $p^2 q \in i(A)$, $p^{n-1} \notin i(A)$, then $p \in i(A)$;

5) $i(A) = \{p, pqr\}$ is impossible;

6) Let $d(m)$ be the prime decomposition length of the composite number

m. If $i(A)=\{p,m\}$, then $d(m)\leq 2$;

7) $i(A)=\{m,n\}$ is impossible if m,n are both composite numbers.

Proof. The proofs of this lemma are routine but quite tedious, we hence omit the details.

By summing up the above lemmas, we state the following theorem.

Theorem 5.5. *If* $|i(A)|\leq 3$, *then* A *is not a K-set.*

Proof. The proof is to check $|i(A)|=1,2,3$ case by case. For example, we consider $i(A)=\{p\}$, $i(A)=\{p,q\}$, $i(A)=\{p,q,r\}$, $i(A)=\{p,m,n\}$ or $i(A)=\{m,n,u\}$ until all possible cases are exhausted and are impossible. Therefore we conclude that A is not a K-set if $|i(A)|\leq 3$.

Theorem 5.6. *Let* A *be a subset of the set of all positive integers and* p,q,r,s *are prime numbers. Then we have the following statements*:

1) If A is a K-set, then A contains at least five elements and the elements of A can be composed of not more than three distinct prime numbers.

2) If A is a K-set with five elements, then $i(A)$ must be one of the following forms:

a) $i(A)=\{p,r,pr,pqr\}$;

b) $i(A)=\{p,pq,r,rs\}$;

c) $i(A)=\{p,pq,pr,pqr\}$;

d) $i(A)=\left\{p,pq,p^{\alpha+1},p^{\alpha+1}q \,\middle|\, \alpha\geq 1\right\}$.

For more information concerning the construction of a K-set, the reader is referred to K. P. Shum [26].

In view of **Diagram 5**, there are four distinct sets, namely the sets $B_4=A^{\circ-}$, $A_4=A^{-\circ-}$, $B_7=A^{\circ-\circ}$ between the sets $A_2=A^{-}$ and $B_3=A^{\circ}$. Likewisely, there are also four distinct sets, namely, the sets $B_6=B^{-\circ-}$, $A_4=B^{\circ-}$, $B_5=B^{-\circ}$ and $A_7=B^{\circ-\circ}$ between the sets $B_2=B^{-}$ and $A_3=B^{\circ}$.

Now, we perform the usual set operations of union intersection and subtraction on the above sets, we obtain the following equalities:

1) $A^{\circ}=A^{\circ}$;

2) $A^{\circ-\circ}=A^{\circ}\cup\left(A^{\circ-\circ}\setminus A^{\circ}\right)$;

3) $A^{\circ-}=A^{\circ}\cup\left(A^{\circ-\circ}\setminus A\right)\cup\left(A^{\circ-}\cap A^{-\circ}\setminus A^{\circ-\circ}\right)\cup\left(A^{\circ-}\setminus A^{-\circ}\right)$;

4) $A^{-\circ}=A^{\circ}\cup\left(A^{\circ-\circ}\setminus A^{\circ}\right)\cup\left(A^{\circ-}\cap A^{-\circ}\setminus A^{\circ-\circ}\right)\cup\left(A^{-\circ}\setminus A^{\circ-}\right)$;

5) $$A^{-\circ}=A^{\circ}\cup\left(A^{\circ-\circ}\setminus A^{\circ}\right)\cup\left(A^{\circ-}\cap A^{-\circ}\setminus A^{\circ-\circ}\right)\cup\left(A^{-\circ}\setminus A^{\circ-}\right)\cup\left(A^{\circ-}\setminus A^{-\circ}\right)$$
$$\cup\left(A^{-\circ-}\setminus A^{\circ-}\cap A^{-\circ}\right);$$

6) $A=A^{\circ}\cup\left(A\setminus A^{\circ}\right)$;

7) $A^{-}=A^{-\circ-}\cup\left(A^{-}\setminus A^{-\circ-}\right)$.

Because the boundary of a set A in the topological space X is defined by $b(A)=A^{-}\cap c(A)^{-}=f(A)\cap fc(A)$.

Since the rim of a set A is defined by $rb(A)=A\cup fc(A)$ and so $rb(A)\subseteq A^{-}\cap c(A)^{-}=b(A)$. This means that the rim of A, $rb(A)$ is a part of the boundary $b(A)$.

By using the equation $i=cfc$ and $f=cic$, we immediately see that the gain

boundary of A
$Gb(A) = A^{\circ-\circ} \setminus A^{\circ} = ifi(A) \cap ci(A) = ifi(A) \cap c(cfc(A)) \subseteq f(A) \cap fc(A) = b(A)$.
Likewisely, we can also see that the inner boundary of A,
$I(A^{\circ-} \cap A^{-\circ} \setminus A^{\circ-\circ}) \subseteq f(A) \cap fc(A) = b(A)$; also, the public boundary of A,
$Pb(A) = A^{\circ-} \setminus A^{-\circ} = fi(A) \cap ifif(A) \subseteq fi(A) \cap cf(A) \subseteq fi(A) \cap cf(A) = b(A)$;
the neutral boundary $Nb(A) = A^{-\circ} \setminus A^{\circ-} \subseteq f(A) \cap fc(A) = b(A)$; the outer boundary of A, $Ob(A) = A^{-\circ-} \setminus A^{\circ-} \cap A^{-\circ} \subseteq f(A) \cap fc(A) = b(A)$; the loss boundary of A
$Lb(A) = A^{-} \setminus A^{-\circ-} = f(A) \cap c(fif(A)) \subseteq f(A) \cap ifc(A) \subseteq f(A) \cap fc(A) = b(A)$.

Now, by using the above concepts of the components of the boundary of a set A, we obtain the following equalities related to the above components of the boundaries and the 7 Kuratowski sets. (see [26])

1) $A = A^{\circ} \cap Rb(A)$;
2) A°;
3) $A^{\circ-\circ} = A^{\circ} \cup Gb(A)$;
4) $A^{\circ-} = A^{\circ} \cup Gb(A) \cup Ib(A) \cup Pb(A)$;
5) $A^{-\circ} = A^{\circ} \cup Gb(A) \cup Ib(A) \cup Nb(A)$;
6) $A^{-\circ-} = A^{\circ} \cup Gb(A) \cup Ib(A) \cup Nb(A) \cup Pb(A) \cup Ob(A)$;
7) $A^{-} = A^{\circ} \cup Gb(A) \cup Ib(A) \cup Nb(A) \cup Lb(A)$.

We now consider the Structure of the boundary of a set A in a topological space.

The boundary of a set A and all its components in a topological space can be interpreted in the following "envelope diagram" (see [27]) (**Diagram 7**).

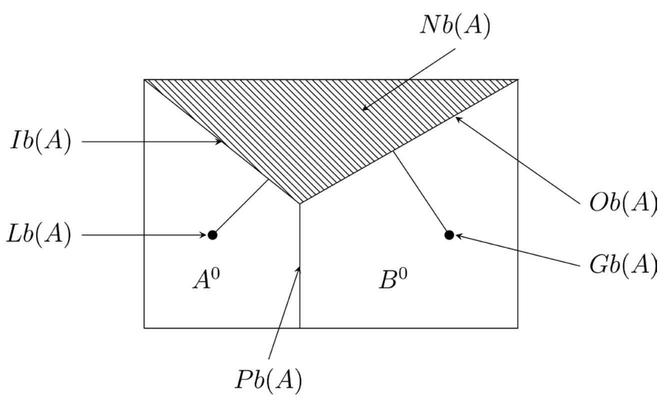

Diagram 7. The boundary of a set A and all its components in a topological space.

In **Diagram 6**, A° can be regarded as a troop of soldiers and B° is the troop of the enemies of A°.

In the diagram, $Pb(A)$ is the public boundary at which the troops A° and B° are in combatomg.

$Nb(A)$ can be regarded as the ceased fire zone or the negotiation area between the troops A° and B°, and therefore, we call this part of boundary the neutral boundary of A.

$Ib(A)$, the inner boundary is the final defence frontier of A°, that is, it can be regarded as the Mignot defence line of the troop A°.

$Gb(A)$, the gain boundary of A°, it can be interpreted as the picket posts of the troop B° and these posts will be seized and occupied by the troop A° when the war begins.

$Ob(A)$, the outer boundary of A° which is the inner boundary of the troop B°. Roughly speaking, the inner boundary of A° can be regarded as the skin of A° and the outer boundary can be regarded as the clothes of A°. Then one can easily distinct the inner boundary and the outer boundary of a set A in a given space X.

$Lb(A)$, the loss boundary of A° which can be described as the gain boundary of the troop B°.

$Rb(A)$, the rim boundary of A. Of course,the rim of a set A is a part of the boundary of the set A but it is not necessarily belongs to the given set A. In considering the boundary of a set A, many people always overlook that there are rims of the set A.

Thus,in terms of the closure operator f, the interior operator i and the complementation operator c, we can express the above components of the boundary of the set A in the topological space X as follows:

We always use the equalities $i = cfc$ and $f = cic$ to describe the different components of the boundary of the set A. These different components of the boundary of the set A are the followings:

$$Pb(A) = fi(A) \cap fcf(A) = fi(A) \cap fic(A) = B_4 \cap A_4;$$

$$Nb(A) = cfi(A) \cap fic(A) = cfcfc(A) \cap cfcfc(A) = B_5 \cap B_5;$$

$$Ib(A) = fi(A) \cap cfcf(A) \cap fcfcfc(A) = B_4 \cap A_5 \cap B_6;$$

$$Ob(A) = fic(A) \cap cfcfc(A) \cap fcfcf(A) = A_4 \cap B_5 \cap A_6;$$

$$Gb(A) = ifi(A) \cap fc(A) = B_7 \cap B_2;$$

$$Lb(A) = f(A) \cap cfif(A) = A_7 \cap A_2;$$

$$Rb(A) = A \cap fc(A) = A_1 \cap B_2.$$

After we have displayed all the possible components of the boundary of a set A, we can easily find out the conditions for a set A which leads to the set A to be a K-set. In fact, we have the following three different types of sets.

Type I. A set A is called a standard set if $Pb(A), Nb(A), Ib(A), Ob(A)$, $Gb(A), Lb(A)$ and $Rb(A)$ are all non-empty sets.

One can easily observe that a set A is a K-set if and only if A is a standard set.

It may happen that $Ib(A) = Ob(A) = Nb(A) \neq \varnothing$.

Type II. A set A is called an abnormal set if $Pb(A) = \varnothing$. It can be observed that an abnormal set A is a K-set if and only if $Nb(A), Ib(A), Ob(A), Gb(A), Lb(A)$ and $Rb(A)$ are all non-empty sets.

Type III. A set A is called a normal set if $Pb(A) \neq \varnothing$. It is clear that a set A is a K-set if and only if $Nb(A), Ib(A), Ob(A), Gb(A), Lb(A)$ and $Rb(A)$

are all non-empty, moreover, $Ib(A)$ and $Ob(A)$ may be possibly empty.

Remark 5.7. *Because* $Ib(A) \subseteq Nb(A)$ *and* $Ob(A) \subseteq Nb(A) = A_3 \cap B_5$. *In this case, the set* A *is also a standard K-set.*

Below are some examples of normal set and abnormal set.

Example 5.8. $A = \{2, 5, 2\times 5, 2\times 3\times 5, 2^2\times 3^2\}$ *is a normal set. In this set set, we can check that* $Pb(A) = \{2^2\times 3^2\times 5^2, \cdots\}$, $Nb(A) = \{2^4\times 3, \cdots\}$, $Ob(A) = \varnothing$, $Ib(A) = \varnothing$, $Gb(A) = \{2^2\times 3\times 5, \cdots\}$, $Lb(A) = \{2^3\times 3^2, \cdots\}$, $Rb(A) = \{2^2\times 3^2\}$.

Therefore A is a normal K-set.

Example 5.9. *The* $A = \{2, 2\times 3, 5, 2\times 7, 3\times 7\}$ *is a abnormal set.*

It is clear to see that $i(A) = \{2, 2\times 3, 5, 2\times 7\}$. Since $3 \notin i(A)$, $7 \notin i(A)$, we know that $3\times 7 \notin A \cap fc(A) = Rb(A)$. Because we can easily see that $3, 7, 5\times 7, 3\times 5$ and $2\times 3\times 7, \cdots$ are all in the interior of $c(A)$. Thus $Pb(A) = fi(A) \cap fi(c(A)) = \varnothing$. This shows that A is an abnormal set.

The following example is also a standard *K*-set.

Example 5.10. $A = \{2, 2\times 3, 3^3, 2^3\times 3, 3\times 5^2\}$ *is a standard set. In this example,* $3 \notin i(A)$, $5^2 \notin i(A)$ *and so* $3\times 5^2 \notin i(A)$. *Thus* $i(A) = \{2, 2\times 3, 3^3, 2^3\times 3\}$. *Because* $ic(A) = cf(A)\{3, 5, 7, 2\times 5, 3\times 5, 2\times 5^2, 2\times 3\times 5, 3^2\times 5, 2^2\times 7, \cdots\}$. *By using the definitions of the components of the boundary, we can find that* $Pb(A) = \{2^3\times 3^3, \cdots\}$, $Nb(A) = \{2\times 3^2, \cdots\}$, $Ob(A) = \{2^2\times 3^2, \cdots\}$, $Gb(A) = \{2^4\times 3, \cdots\}$, $Lb(A) = \{2\times 3\times 5^2, \cdots\}$ *and* $Ib(A) = \{3\times 2^2, \cdots\}$, $Rb(A) = \{3\times 5^2\}$. *Thus, the boundary of the set* A *has components and so* A *is standard set. We have therefore verified that* A *is indeed a K-set.*

6. The K-Sets in a Topological Space

In the above section, we have already considered the construction of a K-set in the set of positive integers. We now turn to the topological space X containing a K-set K. We concentrate on the so called 7 point space theorem, that is, if A is a subset of a n-point space then A has at most 2n relations by taking closure and complementation on A successively, in any order. This is a significant result in general topology. (see [3])

Theorem 6.1. *Let* X *be a topological space and* A *a K-set in* X. *If exactly* $2n$ $n \le 7$ *distinct sets can be generated by* A *by successive applications, in any order of closure and complementation on* A, *then then the number of the elements of the space* X *can not be less that* 7. (*see* [3]).

Before we prove this theorem, we recall the process of solving the Kuratowski problem. We first take $A = A_1$. Then $A_2 = f(A_1)$, $A_3 = c(A_2), \cdots$ and $B_1 = c(A_1)$, $B_2 = f(c(A_1))$, $B_3 = cfc(A_1), \cdots$ and so on.

We first let l be the maximum number of distinct sets generated by A in the above process of Kuratowski's theorem.

Similarly, we let m be the maximum number of distinct sets generated by B in the above process. In order to prove Theorem 5.1, we need the following lemmas.

Lemma 6.2. *If* $l = 7$, *then* $|A_4| \ge 3$.

Proof. Let $A = A_1$. Then $A_2 = f(A_1)$, $A_3 = cf(A_1) = c(A_2)$, $A_4 = fcf(A_1), \cdots$

and $A_7 = cfcfcf(A_1)$. It is clear to see that A_3 is an open set and is contained in A_4. By $i = cfc$, we see that A_7 is in $i(A_4)$. Hence, we have $A_4 \supseteq A_7 \supseteq A_3$. Thus, $|A_4| \geq 3$.

By using the identities $i = cfc$ and $f = cic$ repeatedly, we can prove the following lemmas.

Lemma 6.3. *If* $l \geq 5, m \geq 5$, *then* $|A_3 \cap B_5| \geq 2$, $|A_5| \geq 3$ *and* $|B_5| \geq 3$.

Lemma 6.4. *If* $l \geq 5, m \geq 3$, *then* $|A_5| \geq 2$.

In the above section, we have already discussed the boundary structure of a K-set in a topological space X, in particular, we have $A_5 = A^{\circ -}$ and $B_5 = A^{-\circ}$. By notice that the neutral boundary $Nb(A)$ of a K-set is non-empty, we can easily verify the following lemma.

Lemma 6.5. *If the set* A *is a K-set, that is, the maximal number of relatives is attained, then* $|A_3| \leq 1$ *and* $|B_3| < 1$.

Now, by using the above lemmas, we can prove Theorem 5.1. In fact, we recall that the standard set, the normal set and the abnormal set can all possibly be the K-set of the required components of the boundary of the set A in the topological space all exist, then we see immediately that This result was given by Anusiak and Shum in 1971. (see [3]). We now state the following theorem for the 7 points space.

Theorem 6.6. *Let* X *be an n-point set with* $n \leq 7$. *If* A *is a subset of* X *is a K-set under some "good" topology, then the number of non-homomorphic topologies* $T(n)$ *for each* n *can be determined and is shown by the following table* (**Table 2**).

Table 2. The number of non-homomorphic topologies $T(n)$.

n	1	2	3	4	5	6	7
$T(n)$	1	2	3	5	7	4	6

In view of the structure of the boundary of a given set and its seven components in a topological space. We can state the following theorem for a K-set in a topological space.

Theorem 6.7. *Let* A *be a discrete set in a topological space* X. *If the set* A *is a* K *set, then the cardinality of the set* A *is* ≥ 7, *conversely, if the cardinality of the set* A *is* ≥ 7 *then it is possible for the set* A *to be a* K *set. However, we still do not know whether* A *will be a* K *set or not if the cardinality of the set* A *is* 6 *or less.*

7. Topics Related to the Closure and Complementation Operators

We now discuss some topics related to the closure and the complementation operators.

(A) Abstract algebras.

By an abstract algebra $(X; F)$, we mean a set X and a family of fundamental operations consisting of X-valued functions of several variables running over x of $X\{a,b,\cdots\}$ and $F=\{\varphi,\psi,\cdots\}$. We sometimes write $(X;\varphi,\psi,\cdots)$ to represent the abstract algebra $(X;F)$. If A is a non-empty subset of the set X, then the smallest subalgebra containing A which is closed to an algebraic operator Φ is called the algebraic closure of Φ denoted by $\hat{A}$ of $\hat{\Phi}$. In application, we call the n-ary operations $e_k^n(x_1,\cdots,x_n)=x_k$, where $(k=1,2,\cdots,n;n=1,2,\cdots)$. The smallest class which contains trivial operations and is closed under composition if these fundamental operations is called the class of algebraic operators. The values of constant algebraic operations are called the algebraic constants. We have the following version of Kuratowski's theorem in the abstract algebra $(X;F)$.

Theorem 7.1. *can be constructed form* A *by taking the algebraic closure operation and complementation operator successively in any order.*

Remark. According to Kuratowski's theorem,the maximal number of relatives of a subset A in an n-element topological space X is $2n(n\le 7)$. However, the same conclusion does not hold in abstract algebra.It should be noted that the classification of topological spaces by using closure and interior operators was also given by C.E. Aull in 1967 (see [4]).

Example 7.2 *Consider the abstract algebra* $(X;F)=\{a,b,c;\varphi,\psi\}$, *where* $\varphi(x)=b$, $\varphi(a)=b=\psi(b)$, $\psi(c)=a$. *Starting with* $A=\{a\}\subset X$, *we have* $\{\hat{a}\}=\{a,b\}$, $\{\hat{c}\}=X$, $\Phi=\{b\}$. *Thus the maximal number generated by* $\{c\}\subset X$ *is* $v(3)=8$.

By routine verification, we have the following theorem. (see [3]).

Theorem 7.3. *Let* X *be an n-element set. Then there exists an abstract algebra* $(X;F)$ *and a subset* A_1 *of* X *such that, by taking algebraic closure and complementation operations to* A_1, *in any order. We obtain we obtain the following results*:

1) 2^n relatives of A if $n\le 3$;

2) $2(n+1)$ relatives of A if $3\le n\le 6$;

3) 14 relatives of A if $n\ge 6$.

The number of relatives of A cannot be enlarged.

(B) An application of Kuratowski's theorem in social science.

Suppose that the immigration rule for the Hong Kong citizens to apply to immigrate to the country U is the following:

1) The father can apply himself, his spouses and his dependent children to immigrate with him.

2) The mother can apply herself and her dependent children to immigrate with her.

3) The married son can apply himself, his spouses and his unmarried dependent brothers and sisters in the family to go with him.

4) The married daughter can only apply herself and the unmarried dependent bothers and sisters in the family to go with her.

5) The son in law in the family can only applying himself and his spouses to go with him but not the others.

6) The daughter in law in the family and also the independent children in the family do not have the privilege of application.

Now, Suppose a family have father g, mother m, son a, daughter in law b, daughter d, son in law e, unmarried dependents y.

If the following family members $\{a,m,d\}$ first apply to immigrate to UK, then the rest of the family apply next.

The question is how many possible cases will be there for this family in total?

The answer is 14 possible cases because this is a typical application of Kuratowski's theorem and the algebraic closure operator is the immigration rule f.

Let $A_1=\{a,m,d\}$. Then under the above immigration rule, there will be the following 7 cases:

$$A_2=f(A_1)=\{a,m,d,b,e,y\};$$

$$A_3=c(A_2)=\{g\};$$

$$A_4=f(A_3)=\{g,m,y\};$$

$$A_5=c(A_4)=\{a,b,e,d\};$$

$$A_6=f(A_5)=\{a,b,e,d,y\};$$

$$A_7=c(A_6)=\{m,g\};$$

$$A_8=f(A_7)=\{m,g,y\}.$$

On the other hand, after the father has first applied to immigrate to country U. We have the following situations:

$B_1=c(A_1)=\{b,e,g,y\}$. and then

$$B_2=f(B_1)=\{b,d,e,m,g,y\};$$

$$B_3=c(B_2)=\{a\};$$

$$B_4=f(B_3)=\{a,b,y\};$$

$$B_5=c(B_4)=\{d,e,g,m\};$$

$$B_6=f(B_5)=\{d,e,g,m,y\};$$

$$B_7=c(B_6)=\{a,b\};$$

$$B_8=f(B_7)=\{a,b,y\}=B_4.$$

Thus, in this immigration application for the family $\{a,m,d,b,e,y\}$, there are at most 14 distinct combinations (see [18]).

(C) The Closure and interior operators in general algebras.

Let f be the closure operator and $i=cfc$, the interior operator acting on a subset A in a topological space X.

In the literature, a set A in a topological space X is called a regular open set by Halmos if $f(A)=A$. Dually, a set B in X is called regular closed if $fi(B)=B$. Because it has been known in Kuratowski problem that $(if)^2=fi$.

Thus, if X is a K-space, then X contains simultaneously a regular closed set and a regular open set. It was also noticed by Shum [28] that the Boolean algebra formed by the set of regular open sets is isomorphic to the Boolean algebra formed by the set of regular closed sets.

We call a mapping $\varphi: L \to L$ an interior operator on the lattice $L(\wedge, \vee, 0, 1)$ if φ is a shrinking mapping, *i.e.*, $\varphi(A) \le A$ and is an idempotent mapping, *i.e.*, $\varphi(\varphi(A)) = \varphi(A)$. The mapping φ is also called meet preserving if $\varphi(\varphi(X)) \wedge \varphi(Y) = \varphi(X \wedge Y)$.

For the closure operator of a lattice, the reader is referred to [29].

We give below the following definitions.

Definition 7.4. *An interior operator* ϕ *is said to be a strong interior operator if* ϕ *is shrinking, idempotent and meet preserving.*

Definition 7.5. *A sub semilattice* S *of a complete lattice* L *is called a closed join subsemilattice, if for all* $x_i \in S \subseteq L$ *and* $V_L\{x_i \in S\}$ *exists then* $Sup_L\{x_i \in S\} \in S$. *In general, a closed join sub-semilattice is not necessarily closed. This fact can be seen in the following example* (**Diagram 8**).

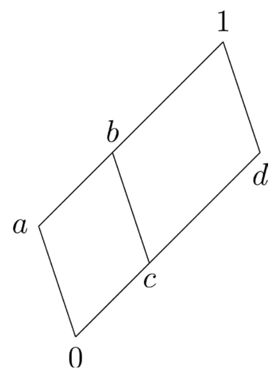

Diagram 8. A closed join sub-semilattice.

In the above lattice, $S = \{0, a, c, 1\}$ is a join semilattice of L. It is clear that $aV_L c = b \notin S$. Thus $S = \{0, a, c, 1\}$ is not a closed join semilattice.

By considering the closed join subsemilattice of a lattice L, we obtain the following result related with the Tarski fixed plank. (see [30] [31] and [32]).

For the strong interior operator on a closed join subsemilattice S, we have the following theorem.

Theorem 7.6. *The mapping* $\phi: L \to S$ *defined by* $\phi(x) = V_L\{y \in S \mid y \le x\}$ *is a strong interior operator on* S *and moreover,* $S = Range(\phi) = F_x(\phi)$, *the fixed plank of* ϕ.

Proof. We first let $x \in L$. Because S is a closed join semilattice in L. Then $\phi(x) = V_l\{y \in s \mid y \le x\}$ from $L \to S$ is clearly a strong interior operator on S and is also surjective.

Trivially, ϕ is a shrinking mapping and $\phi(x) \le x$ for all $x \in S$. Suppose that $u \le v$ in L. Then we have $\phi(u) = V_L\{y \in S \mid y \le u\} = V_L\Sigma_1$ and $\phi(v) = V_L\{y \in S \mid y \le v\} = V_L\Sigma_2$. It can be easily verified that $u \le v$ leads to $V_L\Sigma_1 \le V_L\Sigma_2$. Thus ϕ is an expansive mapping. On the other hand we can check that for all $x \in S$, we have $x \in \Sigma_x = \{y \in S | y \le x\}$ and thereby $V_L \Sigma_x \ge x$, that is $\phi(x) \ge x$. By the shrinking property of ϕ, we have $\phi(x) = x$ immediately. Thus x is a fixed point under the mapping ϕ. This means that $S \le Fix(\phi)$. Because $\phi(\phi(x)) = V_L \{y \in S | y \le \phi(x) = V_L \Sigma_{\phi(x)}\}$, where $\Sigma_{\phi(x)} = \{y \in S | y \le \phi(x)\}$. Hence, if we take $x \in \Sigma_x$, then by definition, we have $y \in S$ and $y \le x$. Using the expansive property of ϕ, we get $y = \phi(y) \le \phi(x)$. Consequently, $y \in \Sigma_{\phi(x)}$, that is, $\Sigma_x \subseteq \Sigma_{\phi(x)}$. Thus, we have proved that $\phi(x) \le \phi(\phi(x))$. Again by the shrinking property of $\phi(x)$, we get $\phi(\phi(x)) \le \phi(x)$. This proves that $\phi(x) = \phi(\phi(x))$ and hence ϕ is an idempotent mapping.

It remains to prove that the mapping ϕ is a meet preserving mapping. Since $x \wedge_L y \le x$ and $x \wedge_L y \le y$, by the shrinking property of ϕ again, we have $\phi(x \wedge_L y) \le \phi(x) \wedge_L \phi(y)$, also by $\phi(x) \le x$ and $\phi(y) \le y$, we have $\phi(x) \wedge_L \phi(y) \le x \wedge_L y$. Now, we use again the properties of ϕ, we deduce that $\phi(x \wedge_L y) = \phi(\phi(x \wedge_L y)) \le \phi(\phi(x) \wedge_L (y)) \le \phi(x \wedge_L y)$. This proves that ϕ is a strong interior operator of L. Because $Fix(\phi) \subseteq Range(\phi) \subset S$. Therefore, we have shown that $S = Range(\phi) = Fix(\phi)$, the fixed plank of ϕ.

Remark 7.7. *Theorem* 6.5 *can also be extended to topological lattices and further modification of the strong interior operator to the so called topological interior operator. In modifying the strong interior operator to the topological interior operator,the additional requirement* $\phi(x) = 1$ *is crucial. Thus, the Tarski-like results of subalgebras can also be established in a complete Boolean algebra* $(B, \le, \vee, \wedge, *, 0, 1)$ *by using a topological interior operator. It was stated in* [34] *that if* S *is a subalgebra of a complete Boolean algebra* $B = (B, \le, \vee, \wedge, *, 0, 1)$, *then we can prove that* S *is a subalgebra of* B *with girth* 2,*that is,* $\bigcap x_i \in S$, $|J| \ge 2$, where the elements $x_i \in J$ *if and only if there exists a topological interior operator* ϕ *and satisfies the equality* $Fix(\phi) = Range(\phi) = S$. *We omit the details,The reader is referred to K.P.Shum and A. Yang in* [31] *and* [33].

(D) Galois connection

Let $P(A)$ be the power set of a set A, $P(B)$ be the power set of a set B. If $R \subseteq A \times B$ is a relation between the sets A and B, then, we define the operator $\sigma : P(A) \to P(B)$ and the operator $\tau : P(B) \to P(A)$ as follows:

$$\sigma(x) = \{y \in B | \forall x \in X, (x, y) \in R\}, X \subseteq A;$$

$$\tau(y) = \{x \in A | \forall y \in Y, (x, y) \in R\}, Y \subseteq B.$$

Then, we can easily see that

$$X \subseteq X' \Rightarrow \sigma(X) \ge \sigma(Y') : Y \subseteq Y' \Rightarrow \tau(Y) \ge \tau(Y') \subseteq \tau\sigma(X) : Y \le \sigma\tau(Y),$$

where X, X' are the subsets of A, Y, Y' are subsets of B.

One can prove immediately that if A and B are partially ordered sets, then the mappings $\sigma\tau$ and $\tau\sigma$ are the pair of closure operators of A and B respectively. We now call the pair $(\sigma\tau, \tau\sigma)$ a Galois connection of the partially ordered sets A and B. In fact, formed by all the transformations of a three dimensional space then we should be able to use all the invariant classes of

G to define some kind of geometry and the new geometry will be generated by using some kind of Galois connection pairs. Hence, the affine geometry is closely linked with the invariant classes of geometry under the affine transformations. The relationship of the transformation groups and the invariant classes of geometry under transformations can be regarded as a pair of Galois connection. Thus, we can always make use of group theory to describe the geometry and conversely,we can also describe the extension of fields by using the structures of permutation groups. The set of identities can be used to describe the varieties of algebras concerning invariant properties and vice versa. To link the operation pair (f,i) with some other kinds of Galois connection pairs would be a useful tool in the future to study the recent developing topics in fuzzy algebraic structures, may be in the soft algebraic structures and in theoretical computer Science. There is an increasing need to use some kind of Galois connections to cope with many uncertainty problems of the real world to the existing mathematical systems. The readers are referred to the article on Galois connection by Denecke and Wismath in [34].

8. Closure in Topological Semigroups

After the concept of Closure introduced by Kuratokski, a topological space was established by using the Kuratowski closure.

Recall that a topological semigroup S is a Hausdorff space endowed with a jointly continuous multiplication which is associative on the space. Therefore, the structure of the topological semigroup is closely related to the topological closure of the topological semigroup S.

Now, let any $b \in S$. Form the semigroup generated by b, that is, $\langle b \rangle = \{b, b^2, \cdots\}$. Denote this semigroup by $\Gamma(b)$. Take the topological closure closure on $\Gamma(b)$. Then $\overline{\Gamma(b)}$ is clearly a compact semigroup of S. Consider the set of accumulation points of $\Gamma(b)$, that is, $K(b) = \bigcap_{n=1} \overline{\{b^i \mid i \geq n\}}$. It was proved by A. D. Wallace in 1955 that $K(b)$ is a minimal ideal of S and is also a group. This is the well known theorem of A.D. Wallace [28] in topological semigroup. Another interesting theorem in topological semigroup is the prime ideal theorem of K. Numakura [20], it states that if $e^2 = e$ is an idempotent element of S, then the maximal ideal contained in the open set $(S - e)$, namely $J_{\circ}(S \setminus e)$ is an open prime ideal of S.

We mention below a theorem concerning the expression of a prime ideal in a compact semigroup.

Theorem 8.1. *Let $P = J_{\circ}(S \setminus e)$ be an open prime ideal in a compact semigroup S with $e^2 = e \in S$. Consider the group $H(e)$, the maximum group generated by the idempotent $e \in S$. Then we have $J_{\circ}(S \setminus e) = J_{\circ}(S \setminus g)$ for any group element $g \in H(e)$. In other words, the idempotent element e in the open prime ideal $J_{\circ}(S \setminus e)$ can be replaced by any group element $g \in H(e)$.*

Proof. We first prove that $J_{\circ}(S \setminus e) \cap \overline{H(e)} = \varnothing$. If $J_{\circ}(S \setminus e) \cap \overline{H(e)} \neq \varnothing$, then there exists $t \in J_{\circ}(S \setminus e) \cap \overline{H(e)}$. Because $J_{\circ}(S \setminus e)$ is the maximal ideal contained in the open set $(S - e)$ and hence, the open neighborhood of t, namely $V(t) \subset J_{\circ}(S \setminus e)$. Since $t \in \overline{H(e)}$ as well so that the open neighborhood $V(t) \cap H(e) \neq \varnothing$. Now, we let $h \in V(t) \cap H(e)$, then we can find an element h^{-1} in the group $H(e)$ so that $e = hh^{-1} \in J_{\circ}(S \setminus e)$ as $h \in V(t) \subset J_{\circ}(S \setminus e)$, a contradiction. Thus we have proved that $J_{\circ}(S \setminus e) \subset S \setminus H(e)$. Because by definition $J_{\circ}(S \setminus e)$ is the maximal ideal contained in the open set $(S - e)$, consequently, we have $J_{\circ}(S \setminus H(e)) \subset J_{\circ}(S \setminus e)$. This means that we can always replace the idempotent $e^2 = e \in J_{\circ}(S \setminus e)$ by any group element $g \in H(e)$. Thus, the open prime ideal $J_{\circ}(S \setminus e)$ in the topological semigroup S will have more different expressions. The theorem is proved.

For more information concerning the structure of topological semigroups and the properties of ideals and radicals in a topological semigroup, the reader is refereed to the monograph of Wallace [35] and the paper by Shum and Hoo [36].

In closing this paper, we conclude that the closure and complementation theorem is a very interesting theorem to study, especially the concept of the closure operation has penetrated into many branches in mathematics. My contribution to the Kuratowski Theorem is that I discovered the cardinality of a K-set is seven, that is, a K-set in a topological space contains at least seven points. (see [3]). I also observe the boundary of a set is composed by Neutral boundary, Public boundary,Gain boundary, Loss boundary,Inner boundary and Outer boundary. This part of work can be seen in **Diagram 7** in this paper and in [27]. The example 5.1 given by me is also an interesting finding in the literature.

We now pose the following open questions for solution.

Open problem Let A be a subset of the set of integers M. Define the Hammer closure of A to be $h(A) = \bigcup_{n=1}^{\infty} \{A^n\}$, where n is a positive integer and A^n is the cartesian set product of n copies, under usual multiplication. Let $c(A)$ be the set complementation of A in M.

1) Find a subset A in M so that A will generate exactly 12 distinct sets by taking h and c successively on A in in any order.

2) Find a subset B in M so that B will generate exactly 10 distinct sets by acting h and c successively on B in any order.

We do not have an answer in hand at this moment for this problem, but Theorem 3.3, Theorem 3.4 and also the description of the boundary of the set A in a topological space have already given some hints to tackle this problem.

As an exercise, the readers are also encouraged to construct a subset A in the real line which will generate exactly 12 distinct subsets and 10 distinct subsets by taking the topological closure operation and the complementation operation, repeatedly on the set A in any order.

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