Scientific Research Publishing

Advances in Pure Mathematics, 2017, 7, 366-374
http://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

From Braided Infinitesimal Bialgebras to Braided Lie Bialgebras

Shengxiang Wang[1,2]

[1]Department of Mathematics, Nanjing University, Nanjing, China
[2]School of Mathematics and Finance, Chuzhou University, Chuzhou, China
Email: wangsx-math@163.com

How to cite this paper: Wang, S.X. (2017) From Braided Infinitesimal Bialgebras to Braided Lie Bialgebras. *Advances in Pure Mathematics*, 7, 366-374.
https://doi.org/10.4236/apm.2017.77023

Received: May 22, 2017
Accepted: July 14, 2017
Published: July 17, 2017

Abstract

The present paper is a continuation of [1], where we considered braided infinitesimal Hopf algebras (*i.e.*, infinitesimal Hopf algebras in the Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ for any Hopf algebra *H*), and constructed their Drinfeld double as a generalization of Aguiar's result. In this paper we mainly investigate the necessary and sufficient condition for a braided infinitesimal bialgebra to be a braided Lie bialgebra (*i.e.*, a Lie bialgebra in the category ${}^H_H\mathcal{YD}$).

Keywords

Braided Infinitesimal Bialgebra, Braided Lie Bialgebra, Yetter-Drinfeld Category, Balanceator

1. Introduction

An infinitesimal bialgebra is a triple (A, m, Δ), where (A, m) is an associative algebra (possibly without unit), (A, Δ) is a coassociative coalgebra (possibly without counit) such that

$$\Delta(xy) = xy_1 \otimes y_2 + x_1 \otimes x_2 y, x, y \in A.$$

Infinitesimal bialgebras were introduced by Joni and Rota in [2], called infinitesimal coalgebra there, in the context of the calculus of divided differences [3]. In combinatorics, they were further studied in [4] [5] [6]. Aguiar established the basic theory of infinitesimal bialgebras in [7] [8] by investigating several examples and the notions of antipode, Drinfeld double and the associative Yang-Baxter equation keeping close to ordinary Hopf algebras. In [9], Yau introduced the notion of Hom-infinitesimal bialgebras and extended Aguiar's main results in [7] [8] to Hom-infinitesimal bialgebras.

One of the motivations of studying infinitesimal bialgebras is that they are closely related to Drinfeld's Lie bialgebras (see [10]). The cobracket Δ in a Lie bialgebra is a 1-cocycle in Chevalley-Eilenberg cohomology, which is a 1-cocycle in Hochschild cohomology (*i.e.*, a derivation) in a infinitesimal bialgebra. So the compatible condition in a infinitesimal bialgebra can be seen as an associative analog of the cocycle condition in a Lie bialgebra.

Motivated by [1], in which we considered infinitesimal Hopf algebras in the Yetter-Drinfeld categories, called braided infinitesimal Hopf algebras, the natural idea is whether we can obtain braided Lie bialgebras (called generalized *H*-Lie bialgebras in [11] [12]) from braided infinitesimal Hopf algebras. This becomes our motivation of writing this paper.

To give a positive answer to the question above, we organize this paper as follows.

In Section 1, we recall some basic definitions about Yetter-Drinfeld modules and braided Lie bialgerbas. In Section 2, we introduce the notion of the balanceator of a braided infinitesimal bialgerba and show that a braided infinitesimal bialgerba gives rise to a braided Lie bialgerba if and only if the balanceator is symmetric (see Theorem 2.3).

2. Preliminaries

In this paper, k always denotes a fixed field, often omitted from the notation. We use Sweedler's ([13]) notation for the comultiplication: $\Delta(h)=h_1\otimes h_2$, for all $h\in H$. Let H be a Hopf algebra. We denote the category of left *H*-modules by ${}_H\mathcal{M}$. Similarly, we have the category ${}^H\mathcal{M}$ of left *H*-comodules. For a left *H*-comodules (M,ρ), we also use Sweedler's notation: $\rho(m)=m_{(-1)}\otimes m_0$, for all $m\in M$.

A left-left Yetter-Drinfeld module M is both a left *H*-module and a left *H*-comodule satisfying the compatibility condition

$$h_1m_{(-1)}\otimes h_2\cdot m_0=(h_1\cdot m)_{(-1)}h_2\otimes(h_1\cdot m)_0 \tag{2.1}$$

for all $h\in H$ and $m\in M$. The equation (1.1) is equivalent to

$$\rho(h\cdot m)=h_1m_{(-1)}S(h_3)\otimes(h_2\cdot m_0). \tag{2.2}$$

By [14] [15], the left-left Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ is a braided monoidal category whose objects are Yetter-Drinfeld modules, morphisms are both left *H*-linear and *H*-colinear maps, and its braiding $C_{-,-}$ is given by

$$C_{M,N}(m\otimes n)=m_{(-1)}\cdot n\otimes m_{(0)},$$

for all $m\in M\in{}^H_H\mathcal{YD}$ and $n\in N\in{}^H_H\mathcal{YD}$.

Let A be an object in ${}^H_H\mathcal{YD}$, the braiding τ is called symmetric on A if the following condition holds:

$$\left(\left(a_{(-1)}\cdot b\right)_{(-1)}\cdot a_0\right)\otimes\left(a_{(-1)}\cdot b\right)_0=a\otimes b, \tag{2.3}$$

which is equivalent to the following condition:

$$a_{(-1)}\cdot b\otimes a_0=b_0\otimes S^{-1}\left(b_{(-1)}\right)\cdot a, \tag{2.4}$$

for any $a,b\in A$.

In the category ${}_H^H\mathcal{YD}$, we call an (co)algebra simply if it is both a left H-module (co)algebra and a left H-comodule (co)algebra. For more details about (co)module-(co)algebras, the reader can refer to [16] [17].

A braided Lie algebra ([11]) in ${}_H^H\mathcal{YD}$, called generalized H-Lie algebra there, is an object L in ${}_H^H\mathcal{YD}$ together with a bracket operation $[,]:L\otimes L\to L$, which is a morphism in ${}_H^H\mathcal{YD}$ satisfying

(1) H-anti-commutativity: $[l,l']=-[l_{(-1)}\cdot l',l_0],l,l'\in L$,

(2) H-Jacobi identity:

$$\{l\otimes l'\otimes l''\}+\{(\tau\otimes 1)(1\otimes\tau)(l\otimes l'\otimes l'')\}+\{(1\otimes\tau)(\tau\otimes 1)(l\otimes l'\otimes l'')\}=0,$$

for all $l,l',l''\in L$, where $\{l\otimes l'\otimes l''\}$ denotes $[l,[l',l'']]$ and τ the braiding for L.

Let A be an associative algebra in ${}_H^H\mathcal{YD}$. Assume that the braiding is symmetric on A. Define

$$[a,b]=ab-(a_{(-1)}\cdot b)a_0, a,b\in A.$$

Then $(A,[,])$ is a braided Lie algebra (see [11]).

A braided Lie coalgebra ([12]) Γ is an object in ${}_H^H\mathcal{YD}$ together with a linear map $\delta:\Gamma\to\Gamma\otimes\Gamma$ (called the cobracket), which is also a morphism in ${}_H^H\mathcal{YD}$ subject to the following conditions:

(1) H-anti-cocommutativity: $\delta=-\tau\delta$,

(2) H-coJacobi identity:

$$(id+(id\otimes\tau)(\tau\otimes id)+(\tau\otimes id)(id\otimes\tau))(id\otimes\delta)\delta=0,$$

where τ denotes the braiding for L.

Dually, let (C,Δ) be a coassociative coalgebra in ${}_H^H\mathcal{YD}$. Assume that the braiding on C is symmetric. Define $\delta:C\to C\otimes C$, by

$$c\mapsto c_1\otimes c_2-c_{1(-1)}\cdot c_2\otimes c_{10}, c\in C.$$

Then (C,δ) is a braided Lie coalgebras in ${}_H^H\mathcal{YD}$ (see [12]).

A braided Lie bialgebra ([18]) is $(L,[,],\delta)$ in ${}_H^H\mathcal{YD}$, where $(L,[,])$ is a braided Lie algebra, and (L,δ) is a braided Lie coalgebra, such that the compatibility condition holds:

$$\delta[x,y]=(([,]\otimes id)(id\otimes\delta)+(id\otimes[,])(\tau\otimes id)(id\otimes\delta))(id\otimes id-\tau)(x\otimes y), x,y\in L,$$

where τ denotes the braiding for L.

3. Main Results

In this section, we will study the relation between braided infinitesimal bialgebras and braided Lie bialgebras as a generalization of Aguiar's result in [8].

Let (A,m,Δ) be a braided ε-bialgebra in ${}_H^H\mathcal{YD}$. For any $x,y,z\in A$, define an action of A on $A\otimes A$ by

$$x\to(y\otimes z)=xy\otimes z-x_{(-1)}\cdot y\otimes(x_{0(-1)}\cdot z)x_{00}.$$

Then the action $\rightarrow$ is a morphism in ${}_{H}^{H}\mathcal{YD}$. In fact, for any $x,y,z\in A$ and $h\in H$, we have

$$\begin{aligned}
&h_1\cdot x\rightarrow h_2\cdot(y\otimes z)=h_1\cdot x\rightarrow(h_2\cdot y\otimes h_3\cdot z)\\
&=(h_1\cdot x)(h_2\cdot y)\otimes(h_3\cdot z)-(h_1\cdot x)_{(-1)}\cdot h_2\cdot y\otimes\left((h_1\cdot x)_{0(-1)}\cdot h_3\cdot z\right)(h_1\cdot x)_{00}\\
&=(h_1\cdot x)(h_2\cdot y)\otimes(h_3\cdot z)-h_{11}x_{(-1)}S(h_{13})\cdot h_2\cdot y\otimes\left((h_{12}\cdot x_0)_{(-1)}\cdot h_3\cdot z\right)(h_{12}\cdot x_0)_0\\
&=(h_1\cdot x)(h_2\cdot y)\otimes(h_3\cdot z)-h_1x_{(-1)}\cdot y\otimes\left((h_2\cdot x_0)_{(-1)}\cdot h_3\cdot z\right)(h_2\cdot x_0)_0\\
&=(h_1\cdot x)(h_2\cdot y)\otimes(h_3\cdot z)-h_1x_{(-1)}\cdot y\otimes\left(h_{21}x_{0(-1)}S(h_{23})\cdot h_3\cdot z\right)(h_{22}\cdot x_{00})\\
&=(h_1\cdot x)(h_2\cdot y)\otimes(h_3\cdot z)-h_1x_{(-1)}\cdot y\otimes\left(h_2x_{0(-1)}\cdot z\right)(h_3\cdot x_{00})\\
&=h_1\cdot(xy)\otimes(h_2\cdot z)-h_1x_{(-1)}\cdot y\otimes h_2\cdot\left(\left(x_{0(-1)}\cdot z\right)x_{00}\right)\\
&=h\cdot\left(xy\otimes z-x_{(-1)}\cdot y\otimes\left(x_{0(-1)}\cdot z\right)x_{00}\right).
\end{aligned}$$

So $\rightarrow$ is left H-linear. To show the left H-colinearity of the action $\rightarrow$, we compute

$$\begin{aligned}
&\rho(x\rightarrow(y\otimes z))=\rho\left(xy\otimes z-x_{(-1)}\cdot y\otimes\left(x_{0(-1)}\cdot z\right)x_{00}\right)\\
&=x_{(-1)}y_{(-1)}z_{(-1)}\otimes x_0y_0\otimes z_0-\left(x_{(-1)}\cdot y\right)_{(-1)}\left(x_{0(-1)}\cdot z\right)_{(-1)}x_{00(-1)}\otimes\left(x_{(-1)}\cdot y\right)_0\otimes\left(x_{0(-1)}\cdot z\right)_0x_{000}\\
&=x_{(-1)}y_{(-1)}z_{(-1)}\otimes x_0y_0\otimes z_0-x_{(-11)}y_{(-1)}S\left(x_{(-13)}\right)x_{(-1)4}z_{(-1)}S\left(x_{(-1)6}\right)x_{(-1)7}\otimes x_{(-1)2}\cdot y_0\otimes\left(x_{(-1)5}\cdot z_0\right)x_0\\
&=x_{(-1)}y_{(-1)}z_{(-1)}\otimes x_0y_0\otimes z_0-x_{(-11)}y_{(-1)}z_{(-1)}\otimes x_{(-1)2}\cdot y_0\otimes\left(x_{(-1)3}\cdot z_0\right)x_0,
\end{aligned}$$

and

$$\begin{aligned}
&(id\otimes\rightarrow)\rho(x\otimes y\otimes z)=(id\otimes\rightarrow)\left(x_{(-1)}y_{(-1)}z_{(-1)}\otimes x_0\otimes y_0\otimes z_0\right)\\
&=x_{(-1)}y_{(-1)}z_{(-1)}\otimes x_0y_0\otimes z_0-x_{(-1)}y_{(-1)}z_{(-1)}\otimes x_{0(-1)}\cdot y_0\otimes\left(x_{00(-1)}\cdot z_0\right)x_{000}\\
&=x_{(-1)}y_{(-1)}z_{(-1)}\otimes x_0y_0\otimes z_0-x_{(-1)1}y_{(-1)}z_{(-1)}\otimes x_{(-1)2}\cdot y_0\otimes\left(x_{(-1)3}\cdot z_0\right)x_0,
\end{aligned}$$

as desired.

Definition 2.1. Let (A,m,Δ) be a braided infinitesimal bialgebra and τ the braiding of A. The map $B:A\otimes A\rightarrow A\otimes A$ defined by

$$B(x,y)=x\rightarrow\tau\Delta(y)+\tau\left(y\rightarrow\tau\Delta(x)\right),\ x,y\in A,\tag{3.1}$$

is called the balanceator of A. The balanceator B is called symmetric if $B=B\circ\tau$. The braided infinitesimal bialgebra A is called balanced if $B\equiv0$ on A.

Condition (2.1) can be written as follows:

$$\begin{aligned}
B(x,y)=&\,x\left(y_{1(-1)}\cdot y_2\right)\otimes y_{10}-x_{(-1)}y_{1(-1)}\cdot y_2\otimes\left(x_{0(-1)}\cdot y_{10}\right)x_{00}\\
&+\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_{01}\otimes\left(x_{(-1)}\cdot y\right)_0x_{02}\ -\left(\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_{01}\right)\left(x_{(-1)}\cdot y\right)_0\otimes x_{02}
\end{aligned}$$

Obviously,

$$\begin{aligned}
B\left(x_{(-1)}\cdot y,x_0\right)=&\left(x_{(-1)}\cdot y\right)\left(x_{0(-1)}\cdot x_{02}\right)\otimes x_{010}-\left(x_{(-1)}\cdot y_1\right)x_0\otimes y_2+x_{(-1)}\cdot y_1\otimes x_0y_2\\
&-\left(x_{(-1)}\cdot y\right)_{(-1)}x_{01(-1)}\cdot x_{02}\otimes\left(\left(x_{(-1)}\cdot y\right)_{0(-1)}\cdot x_{010}\right)\left(x_{(-1)}\cdot y\right)_{00}.
\end{aligned}$$

Lemma 2.2. Let (A,m,Δ) be a braided infinitesimal bialgebra and $x,y\in A$. Assume that the braiding τ on A is symmetric. Then the following equations hold:

(1) $\left(\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_{01}\right)\otimes\left(\left(x_{(-1)}\cdot y\right)_{0(-1)}\cdot x_{02}\right)\left(x_{(-1)}\cdot y\right)_{00}=x_1\otimes x_2y,$

(2) $\left(\left(x_{(-1)}\cdot y\right)_{(-1)}x_{01(-1)}\cdot x_{02}\right)\left(x_{(-1)}\cdot y\right)_0\otimes x_{010}=x_{1(-1)}\cdot\left(x_2y\right)\otimes x_{10},$

(3) $\left(x_{(-1)1}\cdot y_1\right)_{(-1)}\cdot\left(\left(x_{(-1)2}\cdot y_2\right)x_0\right)\otimes\left(x_{(-1)1}\cdot y_1\right)_0=\left(x_{(-1)}y_{1(-1)}\cdot y_2\right)x_0\otimes y_{10}.$

Proof. (1) Since the braiding τ on A is symmetric, for all $x,y\in A$, we have $\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_0\otimes\left(x_{(-1)}\cdot y\right)_0=x\otimes y$, then

$$(id\otimes m)(\Delta\otimes id)\left(\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_0\otimes\left(x_{(-1)}\cdot y\right)_0\right)=(id\otimes m)(\Delta\otimes id)(x\otimes y)$$

that is,

$$\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_{01}\otimes\left(\left(x_{(-1)}\cdot y\right)_{0(-1)}\cdot x_{02}\right)\left(x_{(-1)}\cdot y\right)_{00}=x_1\otimes x_2y.$$

So (1) holds.

(2) To show the Equation (2.2), we need the following computation:

$$\begin{aligned}
&\left(\left(x_{(-1)}\cdot y\right)_{(-1)}x_{01(-1)}\cdot x_{02}\right)\left(x_{(-1)}\cdot y\right)_0\otimes x_{010}\\
&=\left(\left(x_{1(-1)}x_{2(-1)}\cdot y\right)_{(-1)}x_{10(-1)}\cdot x_{20}\right)\left(x_{1(-1)}x_{2(-1)}\cdot y\right)_0\otimes x_{100}\\
&=\left(x_{1(-1)1}x_{2(-1)1}y_{(-1)}S\left(x_{2(-1)3}\right)S\left(x_{12(-1)3}\right)x_{10(-1)}\cdot x_{20}\right)\left(x_{1(-1)2}x_{2(-1)2}\cdot y_0\right)\otimes x_{100}\\
&=\left(x_{1(-1)1}x_{2(-1)1}y_{(-1)}S\left(x_{2(-1)3}\right)\cdot x_{20}\right)\left(x_{1(-1)2}x_{2(-1)2}\cdot y_0\right)\otimes x_{10}\\
&=\left(x_{1(-1)1}\left(x_{2(-1)}\cdot y\right)_{(-1)}\cdot x_{20}\right)\left(x_{1(-1)2}\left(x_{2(-1)}\cdot y\right)_0\right)\otimes x_{10}\\
&=x_{1(-1)}\cdot\left(\left(\left(x_{2(-1)}\cdot y\right)_{(-1)}\cdot x_{20}\right)\left(x_{2(-1)}\cdot y\right)_0\right)\otimes x_{10}=x_{1(-1)}\cdot\left(x_2y\right)\otimes x_{10}.
\end{aligned}$$

The last equality holds since τ is symmetric on A. Hence (2) holds.

(3) Finally, we check the Equation (2.3) as follows:

$$\begin{aligned}
&\left(x_{(-1)1}\cdot y_1\right)_{(-1)}\left(\left(x_{(-1)2}\cdot y_2\right)x_0\right)\otimes\left(x_{(-1)1}\cdot y_1\right)_0\\
&=\left(x_{(-1)11}y_{1(-1)}S\left(x_{(-1)13}\right)\right)\cdot\left(\left(x_{(-1)2}\cdot y_2\right)x_0\right)\otimes x_{(-1)12}\cdot y_{10}\\
&=\left(x_{(-1)1}y_{1(-1)}S\left(x_{(-1)3}\right)\right)\cdot\left(\left(x_{(-1)4}\cdot y_2\right)x_0\right)\otimes x_{(-1)2}\cdot y_{10}\\
&=\left(x_{(-1)11}y_{1(-1)1}S\left(x_{(-1)32}\right)x_{(-1)4}\cdot y_2\right)\left(x_{(-1)12}y_{1(-1)2}S\left(x_{(-1)31}\right)\cdot x_0\right)\otimes x_{(-1)2}\cdot y_{10}\\
&=\left(x_{(-1)11}y_{1(-1)1}\cdot y_2\right)\left(x_{(-1)12}y_{1(-1)2}S\left(x_{(-1)3}\right)\cdot x_0\right)\otimes x_{(-1)2}\cdot y_{10}\\
&=\left(x_{(-1)1}y_{1(-1)}\cdot y_2\right)\left(\left(x_{(-1)2}\cdot y_{10}\right)_{(-1)}\cdot x_0\right)\otimes\left(x_{(-1)2}\cdot y_{10}\right)_0\\
&=\left(x_{(-1)}y_{1(-1)}\cdot y_2\right)\left(\left(x_{0(-1)}\cdot y_{10}\right)_{(-1)}\cdot x_{00}\right)\otimes\left(x_{0(-1)}\cdot y_{10}\right)_0=\left(x_{(-1)}y_{1(-1)}\cdot y_2\right)x_0\otimes y_{10}.
\end{aligned}$$

The last equality holds since τ is symmetric on A. Hence (3) holds as required. □

Therem 2.3. Let (A,m,Δ) be a braided infinitesimal bialgebra. Assume that the braiding τ on A is symmetric. Then $(A,[,]=m-m\tau,\delta=\Delta-\tau\Delta)$ is a braided Lie bialgebra if and only if $B=B\circ\tau$.

Proof. Since (A,m) is an associative algebra and (A,Δ) is a coassociative coalgebra in ${}_H^H\mathcal{YD}$, $(A,[,]=m-m\tau)$ is a braided Lie algebra and $(A,\delta=\Delta-\tau\Delta)$ is a braided Lie coalgebra. Therefore it remains to check the compatible condition:

$$\delta[x,y]=\big(([,]\otimes id)(id\otimes\delta)+(id\otimes[,])(\tau\otimes id)(id\otimes\delta)\big)(id\otimes id-\tau)(x\otimes y),$$

for all $x,y\in A$. In fact, on the one hand, we have

$$\begin{aligned}
\delta[x,y]&=\delta\left(xy-\left(x_{(-1)}\cdot y\right)x_0\right)\\
&=(1-\tau)\Delta(xy)-(1-\tau)\Delta\left(\left(x_{(-1)}\cdot y\right)x_0\right)\\
&=(1-\tau)\left(x_1\otimes x_2y+xy_1\otimes y_2\right)\\
&\quad-(1-\tau)\left(\left(x_{(-1)1}\cdot y_1\right)\otimes\left(x_{(-1)2}\cdot y_2\right)x_0+\left(x_{(-1)}\cdot y\right)x_{01}\otimes x_{02}\right)\\
&=x_1\otimes x_2y+xy_1\otimes y_2-x_{1(-1)}\cdot\left(x_2y\right)\otimes x_{10}-\left(xy_1\right)_{(-1)}\cdot y_2\otimes\left(xy_1\right)_0\\
&\quad-\left(x_{(-1)}\cdot y\right)x_{01}\otimes x_{02}-\left(x_{(-1)1}\cdot y_1\right)\otimes\left(x_{(-1)2}\cdot y_2\right)x_0\\
&\quad+\left(x_{(-1)1}\cdot y_1\right)_{(-1)}\cdot\left(\left(x_{(-1)2}\cdot y_2\right)x_0\right)\otimes\left(x_{(-1)1}\cdot y_1\right)_0\\
&\quad+\left(\left(x_{(-1)}\cdot y\right)x_{01}\right)_{(-1)}\cdot x_{02}\otimes\left(\left(x_{(-1)}\cdot y\right)x_{01}\right)_0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\big(([,]\otimes id)(id\otimes\delta)+(id\otimes[,])(\tau\otimes id)(id\otimes\delta)\big)(id\otimes id-\tau)(x\otimes y)\\
&=\big(([,]\otimes id)(id\otimes\delta)+(id\otimes[,])(\tau\otimes id)(id\otimes\delta)\big)\left(xy-\left(x_{(-1)}\cdot y\right)x_0\right)\\
&=xy_1\otimes y_2-\left(x_{(-1)}\cdot y_1\right)x_0\otimes y_2-x\left(y_{1(-1)}\cdot y_2\right)\otimes y_{10}\\
&\quad+\left(x_{(-1)}y_{1(-1)}\cdot y_2\right)x_0\otimes y_{10}-\left(x_{(-1)}\cdot y\right)x_{01}\otimes x_{02}\\
&\quad+\left(\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_{01}\right)\left(x_{(-1)}\cdot y\right)_0\otimes x_{02}+\left(x_{(-1)}\cdot y\right)\left(x_{01(-1)}\cdot x_{02}\right)\otimes x_{010}\\
&\quad+x_{(-1)}\cdot y_1\otimes x_0y_2-\left(\left(x_{(-1)}\cdot y\right)_{(-1)}x_{01(-1)}\cdot x_{02}\right)\left(x_{(-1)}\cdot y\right)_0\otimes x_{010}\\
&\quad-x_{(-1)}\cdot y_1\otimes\left(x_{0(-1)}\cdot y_2\right)x_{00}-x_{(-1)}y_{1(-1)}\cdot y_2\otimes x_0y_{10}\\
&\quad+x_{(-1)}y_{1(-1)}\cdot y_2\otimes\left(x_{0(-1)}\cdot y_{10}\right)x_{00}-\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_{01}\otimes\left(x_{(-1)}\cdot y\right)_0x_{02}\\
&\quad+\left(x_{(-1)}\cdot y\right)_{(-1)}\cdot x_{01}\otimes\left(\left(x_{(-1)}\cdot y\right)_{0(-1)}\cdot x_{02}\right)\left(x_{(-1)}\cdot y\right)_{00}\\
&\quad+\left(x_{(-1)}\cdot y\right)_{(-1)}x_{01(-1)}\cdot x_{02}\otimes\left(x_{(-1)}\cdot y\right)_0x_{010}\\
&\quad-\left(x_{(-1)}\cdot y\right)_{(-1)}x_{01(-1)}\cdot x_{02}\otimes\left(\left(x_{(-1)}\cdot y\right)_{0(-1)}\cdot x_{010}\right)\left(x_{(-1)}\cdot y\right)_{00}.
\end{aligned}$$

According to Lemma 2.2, we have

$$xy_1 \otimes y_2 + \left(x_{(-1)}y_{1(-1)} \cdot y_2\right)x_0 \otimes y_{10} - \left(x_{(-1)} \cdot y\right)x_{01} \otimes x_{02}$$
$$-\left(\left(x_{(-1)} \cdot y\right)_{(-1)} x_{01(-1)} \cdot x_{02}\right)\left(x_{(-1)} \cdot y\right)_0 \otimes x_{010}$$
$$-x_{(-1)} \cdot y_1 \otimes \left(x_{0(-1)} \cdot y_2\right)x_{00} - x_{(-1)}y_{1(-1)} \cdot y_2 \otimes x_0 y_{10}$$
$$+\left(x_{(-1)} \cdot y\right)_{(-1)} \cdot x_{01} \otimes \left(\left(x_{(-1)} \cdot y\right)_{0(-1)} \cdot x_{02}\right)\left(x_{(-1)} \cdot y\right)_{00}$$
$$+\left(x_{(-1)} \cdot y\right)_{(-1)} x_{01(-1)} \cdot x_{02} \otimes \left(x_{(-1)} \cdot y\right)_0 x_{010}$$
$$= xy_1 \otimes y_2 + \left(x_{(-1)1} \cdot y_1\right)_{(-1)} \cdot \left(\left(x_{(-1)2} \cdot y_2\right)x_0\right) \otimes \left(x_{(-1)1} \cdot y_1\right)_0$$
$$-\left(x_{(-1)} \cdot y\right)x_{01} \otimes x_{02} - x_{1(-1)} \cdot \left(x_2 y\right) \otimes x_{10}$$
$$-x_{(-1)1} \cdot y_1 \otimes \left(x_{(-1)2} \cdot y_2\right)x_0 - \left(xy_1\right)_{(-1)} \cdot y_2 \otimes \left(xy_1\right)_0$$
$$+x_1 \otimes x_2 y + \left(\left(x_{(-1)} \cdot y\right)x_{01}\right)_{(-1)} \cdot x_{02} \otimes \left(\left(x_{(-1)} \cdot y\right)x_{01}\right)_0$$
$$= \delta[x,y].$$

Therefore,

$$\left(\left([,]\otimes id\right)\left(id \otimes \delta\right) + \left(id \otimes [,]\right)\left(\tau \otimes id\right)\left(id \otimes \delta\right)\right)\left(id \otimes id - \tau\right)\left(x \otimes y\right)$$
$$= \delta[x,y] - x\left(y_{1(-1)} \cdot y_2\right) \otimes y_{10} + \left(\left(x_{(-1)} \cdot y\right)_{(-1)} \cdot x_{01}\right)\left(x_{(-1)} \cdot y\right)_0 \otimes x_{02}$$
$$+x_{(-1)}y_{1(-1)} \cdot y_2 \otimes \left(x_{0(-1)} \cdot y_{10}\right)x_{00} - \left(x_{(-1)} \cdot y\right)_{(-1)} \cdot x_{01} \otimes \left(x_{(-1)} \cdot y\right)_0 x_{02}$$
$$+\left(x_{(-1)} \cdot y\right)\left(x_{01(-1)} \cdot x_{02}\right) \otimes x_{010} - \left(x_{(-1)} \cdot y_1\right)x_0 \otimes y_2 + x_{(-1)} \cdot y_1 \otimes x_0 y_2$$
$$-\left(x_{(-1)} \cdot y\right)_{(-1)} x_{01(-1)} \cdot x_{02} \otimes \left(\left(x_{(-1)} \cdot y\right)_{0(-1)} \cdot x_{010}\right)\left(x_{(-1)} \cdot y\right)_{00}$$
$$= \delta[x,y] - B(x,y) + B\left(x_{(-1)} \cdot y, x_0\right)$$
$$= \delta[x,y] - B(x,y) + B \circ \tau(x,y),$$

as desired. We complete the proof. □

Corollary 2.4. Let (A, m, Δ) be a braided infinitesimal bialgebra. Assume that the braiding τ on A is symmetric and the balanceator $B = 0$. Then $(A, [,] = m - m\tau, \delta = \Delta - \tau\Delta)$ is a braided Lie bialgebra.

Proof. Straightforward from Theorem 2.3. □

Example 2.5. Let q be an 2th root of unit of k and G the cyclic group of order 2 generated by g, $H = kG$ be the group algebra in the usual way. We consider the algebra $A_4 = k[x]/(x^4)$. By [8], A_4 is a infinitesimal bialgebra equipped with the comultiplication:

$$\Delta(1) = 0, \Delta(x) = x \otimes x^2 - 1 \otimes x^3, \Delta\left(x^2\right) = x^2 \otimes x^2, \Delta\left(x^3\right) = x^3 \otimes x^2.$$

Define the left-H-module action and the left-H-comodule coaction of A by

$$g^i \cdot x^j = q^{ij} x^j, \rho\left(x^j\right) = g^j \otimes x^j, \ i = 0,1, \ j = 0,1,2,3.$$

It is not hard to check that the multiplication and the comultiplicaition are both H-linear and H-colinear, therefore A_4 is a braided infinitesimal bialgebra. Since $B(x,x) = 2x^2 \otimes x^2 - qx \otimes x^2 - qx^2 \otimes x - qx^3 \otimes x - x \otimes x^3$ and $\tau(x \otimes x) = (x_{(-1)} \cdot x) x_0 = (g \cdot x) x = qx \otimes x$, it is clear that $B(x,x) = B\tau(x,x)$ if and only if $q = 1$. If $q = 1$, it is not hard to check that the balanceator is symmetric on A_4. By Theorem 2.3, $(A_4, [,] = m - m\tau, \delta = \Delta - \tau\Delta)$ is a braided Lie bialgebra.

Example 2.6. Let q be a 4th root of unit of k. Consider the Hopf algebra $H = kG$, where G is a cyclic group of order 4 generated by g. Recall from [1] that $A = M_2(k)$ is a braided infinitesimal bialgebra in ${}_H^H\mathcal{YD}$ equipped with the comultiplication:

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$$

and the H-module action, the H-comodule coaction:

$$g^k \cdot E_{ij} = q^{2k(i+j)} E_{ij}, \rho(E_{ij}) = g^{2(i+j)} \otimes E_{ij}, k = 0,1,2,3, i, j = 1,2.$$

Since

$$B(E_{11}, E_{21}) = 2(E_{12} \otimes E_{22} - E_{11} \otimes E_{12}),$$

$$B\left(E_{11_{(-1)}} \cdot E_{21}, E_{11_0}\right) = B(E_{21}, E_{11}) = 2(E_{22} \otimes E_{12} - E_{11} \otimes E_{11}),$$

we claim that the balanceator is not symmetric. By Theorem 2.3, $(M_2(k), [,] = m - m\tau, \delta = \Delta - \tau\Delta)$ is not a braided Lie bialgebra, where m is the multiplication of A.

Let $A_1 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in k \right\} \subset M_2(k)$. It is clear that A_1 is both H-stable and H-costable, hence A_1 is also a braided infinitesimal bialgebra contained in A. One can check easily that the balanceator $B = 0$ on A_1. By Corollary 2.4, $(A_1, [,] = m - m\tau, \delta = \Delta - \tau\Delta)$ is a braided Lie bialgebra.

Acknowledgements

The paper is partially supported by the China Postdoctoral Science Foundation (No. 2015M571725), the Key University Science Research Project of Anhui Province (Nos. KJ2015A294 and KJ2016A545), the outstanding top-notch talent cultivation project of Anhui Province (No. gxfx2017123) and the NSF of Chuzhou University (No. 2015qd01).

References

[1] Wang, S.X. and Wang, S.H. (2014) Drinfeld Double for Braided Infinitesimal Hopf Algebras. *Communications in Algebra*, **42**, 2196-2213. https://doi.org/10.1080/00927872.2013.766796

[2] Joni, S.A. and Rota, G.C. (1979) Coalgebras and Bialgebras in Combinatorics. *Studies in Applied Mathematics*, **61**, 93-139. https://doi.org/10.1002/sapm197961293

[3] Hirschhorn, P.S. and Raphael, L.A. (1992) Coalgebraic Foundations of the Method of Divided Differences. *Advances in Mathematics*, **91**, 75-135.

[4] Aguiar, M. (2002) Infinitesimal Hopf Algebras and the Cd-Index of Polytopes. *Discrete & Computational Geometry*, **27**, 3-28. https://doi.org/10.1007/s00454-001-0049-4

[5] Ehrenborg, R. and Readdy, M. (1998) Coproducts and the Cd-Index. *Journal of Algebraic Combinatorics*, **8**, 273-299. https://doi.org/10.1023/A:1008614816374

[6] Foissy, L. (2009) The Infinitesimal Hopf Algebra and the Poset of Planar Forests. *Journal of Algebraic Combinatorics*, **30**, 277-309. https://doi.org/10.1007/s10801-008-0163-y

[7] Aguiar, M. (2000) Infinitesimal Hopf Algebras. *Contemporary Mathematics*, **267**, 1-30. https://doi.org/10.1090/conm/267/04262

[8] Aguiar, M. (2001) On the Associative Analog of Lie Bialgebras. *Journal of Algebra*, **244**, 492-532. https://doi.org/10.1006/jabr.2001.8877

[9] Yau, D. (2010) Infinitesimal Hom-Bialgebras and Hom-Lie Bialgebras. arXiv. 1001.5000v1.

[10] Drinfeld, V.G. (1986) Quantum Groups. *Proceedings International Congress of Mathematicians*, Berkeley, 798-820.

[11] Wang, S.H. (2002) On the Generalized H-Lie Structure of Associative Algebras in Yetter-Drinfeld Categories. *Communications in Algebra*, **30**, 307-325. https://doi.org/10.1081/AGB-120006492

[12] Wang, S.H. and Chen, H.X. (2002) Hopf-Galois Coextensions and Braided Lie Coalgebras. *Progress in Natural Science*, **12**, 264-270.

[13] Sweedle, M.E. (1969) Hopf Algebras. Benjamin, New York.

[14] Joyal, A. and Street, R. (1993) Braided Tensor Category. *Advances in Mathematics*, **102**, 20-78. https://doi.org/10.1006/aima.1993.1055

[15] Radford, D.E. (1985) The Structure of a Hopf Algebra with a Projection. *Journal of Algebra*, **92**, 332-347. https://doi.org/10.1016/0021-8693(85)90124-3

[16] Kassel, C. (1995) Quantum Groups. GTM 155. Springer-Verlag, Berlin. https://doi.org/10.1007/978-1-4612-0783-2

[17] Radford, D.E. and Towber, J. (1993) Yetter-Drinfeld Categories Associated to an Arbitrary Bialgebra. *Journal of Pure and Applied Algebra*, **87**, 259-279. https://doi.org/10.1016/0022-4049(93)90114-9

[18] Zhang, S.C. and Zhang, T. (2007) Double Bicrosssum of Braided Lie Algebras. Cornell University, New York.