Advances in Pure Mathematics, 2017, 7, 345-351
http://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

The Energy and Operations of Graphs

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How to cite this paper: Ma, H.C. and Liu, X.H. (2017) The Energy and Operations of Graphs. *Advances in Pure Mathematics*, **7**, 345-351.
https://doi.org/10.4236/apm.2017.76021

Received: April 25, 2017
Accepted: June 18, 2017
Published: June 21, 2017

Abstract

Let G be a finite and undirected simple graph on *n* vertices, $A(G)$ is the adjacency matrix of *G*, $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of $A(G)$, then the energy of G is $\varepsilon(\mathbb{G}) = \sum_{i=1}^{n} |\lambda_i|$. In this paper, we determine the energy of graphs obtained from a graph by other unary operations, or graphs obtained from two graphs by other binary operations. In terms of binary operation, we prove that the energy of product graphs $G_1 \times G_2$ is equal to the product of the energy of graphs G_1 and G_2, and give the computational formulas of the energy of Corona graph $G \circ H$, join graph $G \nabla H$ of two regular graphs G and *H*, respectively. In terms of unary operation, we give the computational formulas of the energy of the duplication graph $D_m G$, the line graph $L(G)$, the subdivision graph $S(G)$, and the total graph $T(G)$ of a regular graph *G*, respectively. In particularly, we obtained a lot of graphs pair of equienergetic.

Keywords

Graph, Matrix, Energy, Operation

1. Introduction

Let G be a finite and undirected simple graph, with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of G is *n*, and its vertices are labeled by $v_1, v_2, \cdots, v_n$. The adjacency matrix $A(G)$ of the graph G is a square matrix of order *n*, whose (i, j)-entry is equal to 1 if the vertices v_i and v_j are adjacent and is equal to zero otherwise. The characteristic polynomial of the adjacency matrix, *i.e.*, $det(xI_n - A(G))$, where I_n is the unit matrix of order *n*, is said to be the characteristic polynomial of the graph G and will be denoted by $\phi(G, x)$. The eigenvalues of a graph G are defined as the eigenvalues of its adjacency matrix $A(G)$, and so they are just the roots of the equation $\phi(G, x) = 0$. since $A(G)$ is a real symmetric matrix, so its eigenvalues are all real. Denoting them by $\lambda_1, \lambda_2, \cdots, \lambda_n$ and as a whole, they are called the spectrum of *G*. Spectral properties of graphs, including properties of the characteristic polynomial, have been extensively studied, for details, we refer to [1]. In the 1970s, I. Gutman in [2] introduced the concept of the energy of *G* by

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i| \tag{1}$$

In the Hückel molecular orbital (HMO) theory, the energy approximates the the molecular orbital energy levels of π-electrons in conjugated hydrocarbons (see [3] [4] [5] [6]). Up to now, the energy of *G* has been extensively studied, for details, we refer to [7] [8] [9]. In this paper, we determine the energy of graphs obtained from a graph by other unary operations, or graphs obtained from two graphs by other binary operations. In terms of binary operation, we prove that the energy of product graphs $G_1 \times G_2$ is equal to the product of the energy of graphs G_1 and G_2, and give the computational formulas of the energy of Corona graph $G \circ H$, join graph $G \nabla H$ of two regular graphs *G* and *H*, respectively. In terms of unary operation, we give the computational formulas of the energy of the duplication graph $D_m G$, the line graph $L(G)$, the subdivision graph $S(G)$, and the total graph $T(G)$ of a regular graph *G*, respectively. In particularly, we obtained a lot of graphs pair of equienergetic.

Two nonisomorphic graphs are said to be equienergetic if they have the same energy. Let *G* and *H* be two vertex disjoint graphs, $G \cup H$ denotes the union graph of *G* and *H*. mG denoted the union graph of *m* copies of *G*. K_n denotes the complete graph with *n* vertices. For more notation and terminology, we refer the readers to standard textbooks [10].

2. The Binary Operations of Graphs

Let G_1 and G_2 be two graphs with vertex set $V(G_1)$ and $V(G_2)$ respectively. the product $G_1 \times G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$, in which two vertices, say (x_1, y_1) and (x_2, y_2), are adjacent if and only if x_1 is adjacent to x_2 in G_1 and y_1 is adjacent to y_2 in G_2. Let $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{p \times q}$ be two matrices, the Kronecker product $A \otimes B$ of *A* and *B* is the $mp \times nq$ matrix obtained from *A* by replacing each element a_{ij} with the block $a_{ij} B$.

Lemma 2.1. [1] Let $A(G_1)$, $A(G_2)$ be adjacency matrices of graphs G_1, G_2, respectively. Then the product graph $G_1 \times G_2$ has as adjacency matrix $A(G_1) \otimes A(G_2)$.

Lemma 2.2. [11] Let *A*, *B*, *C*, *D* be matrices and the products *AC*, *BD* exist. Then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \tag{2}$$

Theorem 2.1. Let *G*, *H* be two graphs. Then

$$\varepsilon(G \times H) = \varepsilon(G) \times \varepsilon(H). \tag{3}$$

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\mu_1, \mu_2, \cdots, \mu_m$ be the eigenvalues of *G* and *H*, respectively, suppose $x_i\ (i = 1, 2, \cdots, n)$ are linearly independent eigenvectors of

$A(G)$ corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively, and $y_i\,(i = 1, 2, \cdots, m)$ are linearly independent eigenvectors of $A(H)$ corresponding to $\mu_1, \mu_2, \cdots, \mu_m$ respectively, Consider the vector $z_{ij} = x_i \otimes y_j\,(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$.
Using Lemma 2.1, we see that

$$\left(A(G) \otimes A(H)\right) z_{ij} = \left(A(G) x_i\right) \otimes \left(A(H) y_j\right) = \lambda_i \mu_j x_i \otimes y_j = \lambda_i \mu_j z_{ij}.$$

In this way ,we find mn linearly independent eigenvectors, and hence $\lambda_i \mu_j\,(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$ are the eigenvalues of $G \times H$.

And so

$$\varepsilon(G \times H) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left|\lambda_i \mu_j\right| = \sum_{i=1}^{n} \left|\lambda_i\right| \sum_{j=1}^{m} \left|\mu_j\right| = \varepsilon(G) \varepsilon(H).$$

□

Corollary 2.1. Let $G_1, G_2, \cdots, G_k$ be k graphs. Then

$$\varepsilon(G_1 \times G_2 \times \cdots \times G_k) = \varepsilon(G_1) \varepsilon(G_2) \cdots \varepsilon(G_k). \tag{4}$$

Let G be a graph with n vertices, and let H be a graph with m vertices. The Corona $G \circ H$ is the graph with $n + mn$ vertices obtained from G and n copies of H by joining the i-th vertex of G to each vertex in i-th copy of $H\,(i = 1, 2, \cdots, n)$.

Lemma 2.3. [1] Let G be a graph with n vertices, and let H be an r-regular graph with m vertices. Then the characteristic polynomial of the corona $G \circ H$ is given by

$$\phi(G \circ H, x) = \phi\left(G, x - \frac{m}{x - r}\right)\left(\phi(H, x)\right)^n. \tag{5}$$

Theorem 2.2. Let G be a graph with n vertices, and let H be an r-regular graph with m vertices. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $r, \mu_2, \cdots, \mu_m$ be the eigenvalues of G and H, respectively. then

$$\varepsilon(G \circ H) = \frac{1}{2} \sum_{i=1}^{n} \left(\left| r + \lambda_i + \sqrt{(r - \lambda_i)^2 + 4m} \right| + \left| r + \lambda_i - \sqrt{(r - \lambda_i)^2 + 4m} \right| \right) + n\left(\varepsilon(H) - r\right). \tag{6}$$

Proof. By Lemma 2.3, we have

$$\begin{aligned} \phi(G \circ H, x) &= (x - r)^n (x - \mu_2)^n \cdots (x - \mu_m)^n \prod_{i=1}^{n} \left(x - \frac{m}{x - r} - \lambda_i \right) \\ &= (x - \mu_2)^n \cdots (x - \mu_m)^n \prod_{i=1}^{n} \left(x^2 - (r + \lambda_i) x + r\lambda_i - m \right). \end{aligned}$$

And so

$$\begin{aligned} \varepsilon(G \circ H) &= \frac{1}{2} \sum_{i=1}^{n} \left(\left| r + \lambda_i + \sqrt{(r - \lambda_i)^2 + 4m} \right| + \left| r + \lambda_i - \sqrt{(r - \lambda_i)^2 + 4m} \right| \right) + n\left(\sum_{j=2}^{m} \left|\mu_j\right| \right) \\ &= \frac{1}{2} \sum_{i=1}^{n} \left(\left| r + \lambda_i + \sqrt{(r - \lambda_i)^2 + 4m} \right| + \left| r + \lambda_i - \sqrt{(r - \lambda_i)^2 + 4m} \right| \right) + n\left(\varepsilon(H) - r\right). \end{aligned}$$

□

Corollary 2.2. Let H_1 and H_2 be two equienergetic r-regular graph with m vertices, and let G be a graph with n vertices. Then $G \circ H_1$ and $G \circ H_2$ are equienergetic.

Corollary 2.3. Let $m \geq 2, n \geq 3$. Then $\varepsilon(K_n \circ K_m) = mn + m - 2 + (n-1)\sqrt{m^2 + 4m}.$

Proof. K_m has spectrum $n-1, -1$ ($n-1$ times). Since $(m-1) - 1 - \sqrt{(m-1+1)^2 + 4m} \leq 0$, and $m \geq 2, n \geq 3$ means $(m-1) + (n-1) - \sqrt{(m-n)^2 + 4m} \geq 0$. Hence

$$\begin{aligned}\varepsilon(K_n \circ K_m) &= \frac{1}{2}\sum_{i=1}^{n}\left(\left|(m-1) + \lambda_i + \sqrt{(m-1-\lambda_i)^2 + 4m}\right|\right.\\&\quad \left.+\left|m-1+\lambda_i - \sqrt{(m-1-\lambda_i)^2 + 4m}\right|\right) + n(\varepsilon(H) - (m-1))\\&= m + n - 2 + (n-1)\sqrt{m^2 + 4m} + n(m-1)\\&= mn + m - 2 + (n-1)\sqrt{m^2 + 4m}.\end{aligned}$$

Let G and H be two graphs, The join $G \nabla H$ of (disjoint) grapgs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

Lemma 2.4. [1] If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \nabla G_2$ is given by

$$\phi(G_1 \nabla G_2, x) = \frac{\phi(G_1, x)\phi(G_2, x)}{(x-r_1)(x-r_2)}\left((x-r_1)(x-r_2) - n_1 n_2\right). \tag{7}$$

Corollary 2.4. Let G_i be r_i-regular graph with n_i vertices, $i = 1, 2.$ Then

$$\varepsilon(G_1 \nabla G_2) = \varepsilon(G_1) + \varepsilon(G_2) - (r_1 + r_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)}. \tag{8}$$

Corollary 2.5. Let G_1 and H_1 be two equienergetic r_1-regular graph with n_1 vertices, and let G_2 and H_2 be two equienergetic r_2-regular graph with n_2 vertices, then $G_1 \nabla G_2$ and $H_1 \nabla H_2$ are equienergetic.

Lemma 2.5. [1] Let $G_1, G_2, \cdots, G_k$ be regular graphs, let G_i have degree r_i and n_i vertices $(i = 1, 2, \cdots, k)$. where the relations $n_1 - r_1 = n_2 - r_2 = \cdots = n_k - r_k = s$ hold. Then the graph $G = G_1 \nabla G_2 \nabla \cdots \nabla G_k$ has $n = n_1 + n_2 + \cdots + n_k$ vertices and is regular of degree $r = n - s$, the characteristic polynomial of the join G is given by

$$\phi(G, x) = (x-r)(x+n-r)^{k-1}\prod_{i=1}^{k}\frac{\phi(G_i, x)}{x - r_i}. \tag{9}$$

By Lemma 2.5, we have following Corollary.

Corollary 2.6. Let $G_1, G_2, \cdots, G_k$ be regular graphs, let G_i have degree r_i and n_i vertices $(i = 1, 2, \cdots, k)$. where the relations $n_1 - r_1 = n_2 - r_2 = \cdots = n_k - r_k = s$ hold. Then

$$\varepsilon(G_1 \nabla G_2 \nabla \cdots \nabla G_k) = 2(k-1)s + \sum_{i=1}^{k}\varepsilon(G_i). \tag{10}$$

3. The Unary Operations of Graphs

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$, the duplication graph

D_mG is the graph with mn vertices obtained from mG by joining v_i to each neighbors of v_i in the *j*-th copy of $G(j = 1, 2, \cdots, m, i = 1, 2, \cdots, n)$.

Theorem 3.1. Let G be a graph. Then

$$\varepsilon(D_mG) = m\varepsilon(G). \tag{11}$$

Proof. If $A(G)$ is the adjacency matrix of graph *G*, then, it is obviously that the adjacency matrix of the duplication graph D_mG is $J_m \otimes A(G)$, where J_m is all 1 matrix of order *m*. the spectrum of J_m is $m, 0(m-1)$ times, similar to the proof of Theorem 2.1, we have $\varepsilon(D_mG) = m\varepsilon(G)$.

Corollary 3.1. Let G and H be two equienergetic graph, then D_mG and D_mH are equienergetic.

Let G be a graph, the line graph $L(G)$ of graph G is the graph whose vertices are the edges of *G*, with two vertices in $L(G)$ adjacent whenever the corresponding edge in G have exactly one vertex in common.

Lemma 3.1 [1] If G is a regular graph of degree *r*, with *n* vertices and $m\left(=\frac{1}{2}nr\right)$ edges, then

$$\phi(L(G), x) = (x+2)^{m-n}\phi(G, x-r+2). \tag{12}$$

Corollary 3.2. Let G be a regular graph of degree *r*, with *n* vertices and $m\left(=\frac{1}{2}nr\right)$ edges, If $\lambda_1(=r), \lambda_2, \cdots, \lambda_n$ is the eigenvalues of *G*, then

$$\varepsilon(L(G)) = 2(m-n) + \sum_{i=1}^{n}|r + \lambda_i - 2|. \tag{13}$$

Corollary 3.3.

$$\varepsilon(L(K_n)) = \begin{cases} 2n^2 - 6n & 4 \le n, \\ 4(n-2) & 2 \le n \le 3. \end{cases} \tag{14}$$

Let G be a graph, the subdivision graph $S(G)$ of graph G is the graph obtained by inserting a new vertex into every edge of *G*. The graph $R(G)$ of graph G is the graph obtained from G by adding, for each edge *uv*, a new vertex whose neighbours are *u* and *v*. The graph $Q(G)$ of graph G is the graph obtained from G by inserting a new vertex into every edge of *G*, and joining by edges those pairs of new vertices which lie on adjacent edges of *G*. The total graph $T(G)$ of graph G is the graph whose vertices are the vertices and edges of *G*, with two vertices of $T(G)$ adjacent if and only if the corresponding element of G are adjacent or incident.

Lemma 3.2. [1] If G is a regular graph of degree *r*, with *n* vertices and $m\left(=\frac{1}{2}nr\right)$ edges, then

1) $\phi(S(G), x) = x^{m-n}\phi(G, x^2 - r)$,

2) $\phi(R(G), x) = x^{m-n}(x+1)^n \phi\left(G, \frac{x^2 - r}{x+1}\right)$,

3) $\phi(Q(G), x) = (x+2)^{m-n}(x+1)^n \phi\left(G, \frac{x^2 - (r-2)x - r}{x+1}\right)$.

4) The total graph $T(G)$ has $m-n$ eigenvalues equal to -2 and the following $2n$ eigenvalues:

$$\frac{1}{2}\left(2\lambda_i + r - 2 \pm \sqrt{4\lambda_i + r^2 + 4}\right),\ (i = 1, 2, \cdots, n).$$

Theorem 3.2. Let G be a regular graph of degree r, with n vertices and $m\left(=\frac{1}{2}nr\right)$ edges, If $\lambda_1(=r), \lambda_2, \cdots, \lambda_n$ is the eigenvalues of G, then

1) $\varepsilon(S(G)) = 2\sum_{i=1}^{n}\sqrt{r+\lambda_i}$,

2) $\varepsilon(R(G)) = \sum_{i=1}^{n}\sqrt{\lambda_i^2 + 4(r+\lambda_i)}$,

3) $\varepsilon(Q(G)) = 2(m-n) + \sum_{i=1}^{n}\sqrt{(r+\lambda_i)^2 + 4}$,

4) $$\varepsilon(T(G)) = 2(m-n) + \frac{1}{2}\sum_{i=1}^{n}\left(\left|2\lambda_i + r - 2 + \sqrt{4\lambda_i + r^2 + 4}\right| + \left|2\lambda_i + r - 2 - \sqrt{4\lambda_i + r^2 + 4}\right|\right).$$

Proof. (1) By Lemma 3.2 (1), we know that the spectrum of $S(G)$ is $\left\{0(m-n \text{ times}), \pm\sqrt{r+\lambda_i}\ (i = 1, 2, \cdots, n)\right\}$. So $\varepsilon(S(G)) = 2\sum_{i=1}^{n}\sqrt{r+\lambda_i}$.

(2) By Lemma 3.2 (2), we know that the spectrum of $R(G)$ is

$$\left\{0(m-n \text{ times}), \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4(r+\lambda_i)}}{2}(i = 1, 2, \cdots, n)\right\}.$$ So

$$\varepsilon(R(G)) = \sum_{i=1}^{n}\sqrt{\lambda_i^2 + 4(r+\lambda_i)}.$$

(3), (4) Proof is similar to (1).

Corollary 3.4. 1) If $n \geq 2$, then $\varepsilon(S(K_n)) = 2\left(\sqrt{2n-2} + (n-1)\sqrt{n-2}\right)$.

2) If $n \geq 2$, then $\varepsilon(R(K_n)) = \sqrt{n^2 + 6n - 7} + (n-1)\sqrt{4n-7}$.

3) $\varepsilon(Q(K_n)) = n^2 - 3n + 2\sqrt{n^2 - 2n + 2} + (n-1)\sqrt{n^2 - 4n + 8}$.

4) If $n \geq 2$, then $\varepsilon(T(K_n)) = \begin{cases} 2n^2 - 2n - 4 & n \geq 3, \\ 4 & n = 2. \end{cases}$

4. Conclusion

In this paper, we prove that $\varepsilon(G \times H) = \varepsilon(G) \times \varepsilon(H)$, $\varepsilon(D_m G) = m\varepsilon(G)$. For regular graphs G and H, we give the computational formulas of $\varepsilon(G \nabla H)$, $\varepsilon(G \circ H)$, $\varepsilon(L(G))$, $\varepsilon(S(G))$, $\varepsilon(R(G))$, $\varepsilon(Q(G))$, and $\varepsilon(T(G))$ respectively. In particularly, we obtained a lot of graphs pair of equienergetic.

Acknowledgements

This work is supported by National Natural Science Foundation of China (11561056, 11661066), National Natural Science Foundation of Qinghai Provence (2016-ZJ-914), and Scientific Research Fund of Qinghai University for Nationalities (2015G02).

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