

The Steiner Formula and the Polar Moment of Inertia for the Closed Planar Motions in Complex Plane

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Abstract

In this paper, the Steiner area formula and the polar moments of inertia were expressed during one-parameter closed planar motions in complex plane. The Steiner points or Steiner normal concepts were described according to whether rotation number was different from zero or equal to zero. The moving pole point was given with its components and its relation to the Steiner point or Steiner normal which was specified. The Steiner formula and the polar moments of inertia were expressed for the inverse motion. The fixed pole point was calculated for the inverse motion. The sagittal motion of a telescopic crane was considered as an example. This motion was described by a double hinge consisting of the fixed control panel of the telescopic crane and its moving arm. The results obtained in the first section of this study were applied to this motion.

Keywords

Steiner Formula, Polar Moment of Inertia, Planar Kinematics

1. Introduction

Steiner explained some properties of the area of the path of a point for a geometrical object rolling on a line and making a complete turn [1]. Tutar expressed the Steiner formula and the Holditch theorem during one-parameter closed planar homothetic motions [2]. Müller researched the relation between the Steiner formula and the polar moment of inertia. Then he generalized the Steiner area formula [3] [4]. We calculated the expression of the Steiner formula firstly relative to a moving coordinate system and then a fixed coordinate system during one-parameter closed planar motions in complex plane. If the points of the mov-

ing (or fixed) plane, which enclose the same area lie on a circle, then the center of this circle is called the Steiner point (if these points lie on a line, we use Steiner normal instead of Steiner point). Then we obtained the moving pole point for a closed planar motion. We dealt with the polar moment of inertia of a path generated by closed planar motions. Furthermore, we expressed the relation between the area enclosed by a path and the polar moment of inertia. Moreover, the Steiner formula and the polar moments of inertia were calculated for the inverse motion. The fixed pole point was calculated for the inverse motion. As an example, the Sagittal motion of the telescopic crane which was described by a double hinge being fixed and moving was considered. The Steiner area formula, the moving pole point and the polar moment of inertia were obtained for the direct and inverse motion. Moreover, the relation between the Steiner formula and the polar moment of inertia was expressed. Then the Steiner area formula, the fixed pole point and the polar moment of inertia were calculated for the example.

2. Closed Motions in Complex Plane

We consider one parameter planar motion in complex plane between two reference systems: the fixed E' and the moving E , with their origins (O', O) and orientations. Then, we take into account the motion relative to the fixed system (direct motion), and later the moving system (inverse motion). By taking the displacement vector $O'O = u'$ and the total angle of rotation $\alpha(t)$, the motion was defined by the transformation

$$x'(t) = xe^{i\alpha(t)} + u'(t) \tag{1}$$

where $x'(t)$ is the trajectory of the point x belonging to the moving system with the respect to the fixed system. If we replace $u' = -ue^{i\alpha}$ in Equation (1), the motion can be written as

$$x' = (x - u)e^{i\alpha}. \tag{2}$$

With the coordinates

$$x' = \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3}$$

the components of x' is obtained as

$$\begin{aligned} x'_1 &= (x_1 - u_1)\cos\alpha - (x_2 - u_2)\sin\alpha \\ x'_2 &= (x_1 - u_1)\sin\alpha + (x_2 - u_2)\cos\alpha. \end{aligned} \tag{4}$$

Equation (4) can be written in matrix form

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 - u_1 \\ x_2 - u_2 \end{pmatrix}. \tag{5}$$

The trajectory area formula F of the point x' is given by

$$F = \frac{1}{2} \oint x'_1 dx'_2 - x'_2 dx'_1. \tag{6}$$

If Equation (4) is differentiated,

$$\begin{aligned} dx'_1 &= -(x_1 - u_1) \sin \alpha d\alpha - \cos \alpha du_1 - (x_2 - u_2) \cos \alpha d\alpha + \sin \alpha du_2 \\ dx'_2 &= (x_1 - u_1) \cos \alpha d\alpha - \sin \alpha du_1 - (x_2 - u_2) \sin \alpha d\alpha - \cos \alpha du_2 \end{aligned} \quad (7)$$

is found. If Equations (4) and (7) are placed in Equation (6),

$$\begin{aligned} 2F &= \oint (x_1^2 + x_2^2) d\alpha - 2x_1 \oint u_1 d\alpha - 2x_2 \oint u_2 d\alpha - x_1 \oint du_2 + x_2 \oint du_1 \\ &+ \oint (u_1^2 + u_2^2) d\alpha + \oint u_1 du_2 - u_2 du_1 \end{aligned} \quad (8)$$

is obtained.

Since u_1 and u_2 are periodic functions, $\oint du_1 = 0$ and $\oint du_2 = 0$. If $x = 0 (x_1 = 0, x_2 = 0)$ is taken, then we have

$$2F_o = \oint (u_1^2 + u_2^2) d\alpha + \oint u_1 du_2 - u_2 du_1 \quad (9)$$

for the trajectory area of the initial point. So we can rewrite Equation (8) as

$$2(F - F_o) = (x_1^2 + x_2^2) \oint d\alpha - 2x_1 \oint u_1 d\alpha - 2x_2 \oint u_2 d\alpha.$$

Using the abbreviations

$$a := -2 \oint u_1 d\alpha, \quad (10)$$

$$b := -2 \oint u_2 d\alpha, \quad (11)$$

we obtain

$$2(F - F_o) = (x_1^2 + x_2^2) \oint d\alpha + x_1 a + x_2 b. \quad (12)$$

F is the quadratic form of the coordinates x_1 and x_2 . The surface $F(x_1, x_2)$ describes either a cone or a plane, so it follows that the sections of the constant area describe either concentric circles or parallel lines.

The coefficient m

$$m := \oint d\alpha = 2\pi \nu \quad (13)$$

with the rotation number ν determines whether the lines with $F = \text{const}$ describe circles or straight lines. If $\nu \neq 0$, then we have circles. If $\nu = 0$, the circles are reduced to straight lines. If Equation (13) is replaced in Equation (12), then we have

$$2(F - F_o) = (x_1^2 + x_2^2) m + x_1 a + x_2 b. \quad (14)$$

2.1. Steiner Formula for the Inverse Motion

In order to obtain the Steiner formula relative to the inverse motion, we begin by exchanging the fixed system with the moving system. If Equation (2) is solved according to x , we obtained the motion for the inverse motion as

$$x = x' e^{-i\alpha} + u. \quad (15)$$

If $u = -u' e^{-i\alpha}$ is replaced in Equation (15), we have

$$x = (x' - u') e^{-i\alpha}. \quad (16)$$

Moreover, for the components of x , from

$$x_1 + ix_2 = ((x'_1 - u'_1) + i(x'_2 - u'_2))(\cos \alpha - i \sin \alpha),$$

we have

$$x_1 = (x'_1 - u'_1)\cos \alpha + (x'_2 - u'_2)\sin \alpha, \tag{17}$$

$$x_2 = -(x'_1 - u'_1)\sin \alpha + (x'_2 - u'_2)\cos \alpha. \tag{18}$$

Furthermore, if the coordinates is derived, we obtain

$$dx_1 = -(x'_1 - u'_1)\sin \alpha d\alpha - \cos \alpha du'_1 + (x'_2 - u'_2)\cos \alpha d\alpha - \sin \alpha du'_2, \tag{19}$$

$$dx_2 = -(x'_1 - u'_1)\cos \alpha d\alpha + \sin \alpha du'_1 - (x'_2 - u'_2)\sin \alpha d\alpha - \cos \alpha du'_2. \tag{20}$$

If Equations (17), (18), (19) and (20) are inserted into the formula for the area formula of inverse motion

$$F' = \frac{1}{2} \oint x_1 dx_2 - x_2 dx_1, \tag{21}$$

we find

$$2F' = -\left((x'_1)^2 + (x'_2)^2\right) \oint d\alpha + 2x'_1 \oint u'_1 d\alpha + 2x'_2 \oint u'_2 d\alpha - x'_1 \oint du'_2 + x'_2 \oint du'_1 - \oint \left((u'_1)^2 + (u'_2)^2\right) d\alpha + \oint u'_1 du'_2 - u'_2 du'_1. \tag{22}$$

Since u'_1 and u'_2 are periodic functions, $\oint du'_1 = 0$ and $\oint du'_2 = 0$. If $x' = 0 (x'_1 = 0, x'_2 = 0)$ is taken, then we have

$$2F'_0 = -\oint \left((u'_1)^2 + (u'_2)^2\right) d\alpha + \oint u'_1 du'_2 - u'_2 du'_1. \tag{23}$$

Therefore, we can write Equation (22) as

$$2(F' - F'_0) = -\left((x'_1)^2 + (x'_2)^2\right) \oint d\alpha + 2x'_1 \oint u'_1 d\alpha + 2x'_2 \oint u'_2 d\alpha.$$

Using the abbreviations

$$a' := \oint u'_1 d\alpha, \tag{24}$$

$$b' := \oint u'_2 d\alpha, \tag{25}$$

$$m' := -\oint d\alpha = -m, \tag{26}$$

we have

$$2(F' - F'_0) = -\left((x'_1)^2 + (x'_2)^2\right) m' + x'_1 a' + x'_2 b'.. \tag{27}$$

2.2. Steiner Point or Steiner Normal

We begin by rewriting Equation (14) for the case $m \neq 0$,

$$(x_1^2 + x_2^2)m + x_1 a + x_2 b = 2(F - F_0). \tag{28}$$

By dividing this equation by m and by completing the squares, we obtain the equation of a circle

$$\left(x_1 + \frac{a}{2m}\right)^2 + \left(x_2 + \frac{b}{2m}\right)^2 = \frac{2(F - F_0)}{m} + \frac{a^2}{4m^2} + \frac{b^2}{4m^2}.. \tag{29}$$

From Equation (29), we can relate the radius r of the circle with the area of the trajectories by

$$r^2 = \frac{2(F - F_0)}{m} + \frac{a^2}{4m^2} + \frac{b^2}{4m^2} \dots$$

For $m \neq 0$, the centre of the circles in the moving plane whose trajectories have the same area is called the Steiner point

$$s = -\frac{1}{2m} \begin{pmatrix} a \\ b \end{pmatrix}. \quad (30)$$

In the case of $m = 0$, Equation (28) can be written

$$2(F - F_0) = x_1 a + x_2 b. \quad (31)$$

The Steiner circles are reduced to straight lines and then the Steiner point lies at infinity. The normal to the lines of the equal areas in Equation (31) is given by

$$n = \begin{pmatrix} a \\ b \end{pmatrix} \quad (32)$$

which is called the Steiner normal [5].

Inverse Motion

The expressions for the inverse Steiner point and normal can be deduced in the same way as in the previous subsection. From the Equation (27), we can write the final results

$$s' = -\frac{1}{2m'} \begin{pmatrix} a' \\ b' \end{pmatrix}, \quad (33)$$

$$n' = \begin{pmatrix} a' \\ b' \end{pmatrix}. \quad (34)$$

2.3. The Moving Pole Point

The pole point is the point whose trajectories are instantaneously constant. For the motion

$$x' = (x - u)e^{i\alpha},$$

if we calculate the determining equation

$$dx' = 0,$$

we can obtain the moving pole point. The motion can be written in the matrix form as

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x_1 - u_1 \\ x_2 - u_2 \end{pmatrix}.$$

If we differentiate the matrix,

$$\begin{pmatrix} dx'_1 \\ dx'_2 \end{pmatrix} = \begin{pmatrix} -\sin \alpha & -\cos \alpha \\ \cos \alpha & -\sin \alpha \end{pmatrix} d\alpha \begin{pmatrix} x_1 - u_1 \\ x_2 - u_2 \end{pmatrix} + \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} -du_1 \\ -du_2 \end{pmatrix} = 0 \quad (35)$$

is obtained. Then we have

$$\begin{pmatrix} -\sin \alpha & -\cos \alpha \\ \cos \alpha & -\sin \alpha \end{pmatrix} \begin{pmatrix} x_1 - u_1 \\ x_2 - u_2 \end{pmatrix} d\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \end{pmatrix}. \quad (36)$$

By following the necessary operations

$$\begin{pmatrix} x_1 - u_1 \\ x_2 - u_2 \end{pmatrix} = \frac{1}{d\alpha} \begin{pmatrix} -\sin \alpha & \cos \alpha \\ -\cos \alpha & -\sin \alpha \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \end{pmatrix} \tag{37}$$

$$\begin{pmatrix} x_1 - u_1 \\ x_2 - u_2 \end{pmatrix} = \frac{1}{d\alpha} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \end{pmatrix} \tag{38}$$

$$\begin{pmatrix} x_1 - u_1 \\ x_2 - u_2 \end{pmatrix} = \frac{1}{d\alpha} \begin{pmatrix} du_2 \\ -du_1 \end{pmatrix} \tag{39}$$

is obtained. From the Equation (39), we have the moving pole point

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{d\alpha} \begin{pmatrix} du_2 \\ -du_1 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{40}$$

We can write the moving pole point

$$P_M = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} u_1 + \frac{du_2}{d\alpha} \\ u_2 - \frac{du_1}{d\alpha} \end{pmatrix}. \tag{41}$$

If integral is taken over the total angle, we have

$$\oint P_M d\alpha = \oint \begin{pmatrix} u_1 + \frac{du_2}{d\alpha} \\ u_2 - \frac{du_1}{d\alpha} \end{pmatrix} d\alpha = \oint \begin{pmatrix} u_1 d\alpha + du_2 \\ u_2 d\alpha - du_1 \end{pmatrix} = \oint \begin{pmatrix} u_1 d\alpha \\ u_2 d\alpha \end{pmatrix}. \tag{42}$$

For $m \neq 0$ we arrive at the relation

$$\oint P_M d\alpha = sm = s \oint d\alpha$$

between the Steiner point and the pole point. Then we can write

$$s = \frac{\oint P_M d\alpha}{\oint d\alpha}. \tag{43}$$

For $m = 0$, we have

$$\oint P_M d\alpha = -\frac{1}{2} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2} n. \tag{44}$$

2.4. The Fixed Pole Point

We begin by differentiating the equation $x = (x' - u')e^{-i\alpha}$. And then for $dx = 0$, we can obtain the fixed pole point. The motion can be written in the matrix form as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x'_1 - u'_1 \\ x'_2 - u'_2 \end{pmatrix}.$$

If we differentiate the matrix above,

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} -\sin \alpha & \cos \alpha \\ -\cos \alpha & -\sin \alpha \end{pmatrix} d\alpha \begin{pmatrix} x'_1 - u'_1 \\ x'_2 - u'_2 \end{pmatrix} + \begin{pmatrix} -\cos \alpha & -\sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix} \begin{pmatrix} du'_1 \\ du'_2 \end{pmatrix} = 0 \tag{45}$$

is obtained. Then we have

$$\begin{pmatrix} -\sin \alpha & \cos \alpha \\ -\cos \alpha & -\sin \alpha \end{pmatrix} \begin{pmatrix} x'_1 - u'_2 \\ x'_2 - u'_2 \end{pmatrix} d\alpha = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} du'_1 \\ du'_2 \end{pmatrix}. \quad (46)$$

By following the necessary operations

$$\begin{pmatrix} x'_1 - u'_1 \\ x'_2 - u'_2 \end{pmatrix} = \frac{1}{d\alpha} \begin{pmatrix} -\sin \alpha & -\cos \alpha \\ \cos \alpha & -\sin \alpha \end{pmatrix} \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} du'_1 \\ du'_2 \end{pmatrix} \quad (47)$$

$$\begin{pmatrix} x'_1 - u'_1 \\ x'_2 - u'_2 \end{pmatrix} = \frac{1}{d\alpha} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} du'_1 \\ du'_2 \end{pmatrix} \quad (48)$$

and finally

$$\begin{pmatrix} x'_1 - u'_1 \\ x'_2 - u'_2 \end{pmatrix} = \frac{1}{d\alpha} \begin{pmatrix} -du'_2 \\ du'_1 \end{pmatrix} \quad (49)$$

is found. From the Equation (50), we have the fixed pole point

$$\begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix} = \frac{1}{d\alpha} \begin{pmatrix} -du'_2 \\ du'_1 \end{pmatrix} + \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix}. \quad (50)$$

We can write the fixed pole point

$$P_F = \begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix} = \begin{pmatrix} u'_1 - \frac{du'_2}{d\alpha} \\ u'_2 + \frac{du'_1}{d\alpha} \end{pmatrix}. \quad (51)$$

For $m' \neq 0$, we arrive at the relation

$$\oint P_F d\alpha = s'm' = s' \oint d\alpha \quad (52)$$

between the Steiner point and the fixed pole point.

For $m' = 0$, we have

$$\oint P_F d\alpha = -\frac{1}{2} \begin{pmatrix} a' \\ b' \end{pmatrix} = -\frac{1}{2} n'. \quad (53)$$

2.5. The Polar Moments of Inertia

Blaschke and Müller gave a relation between the Steiner formula and the polar moment of inertia around the pole for a moment [6]. A relation to the polar moment of inertia around the origin is demonstrated by Müller [3]. Also the same relation for closed functions is inspected by Tölke [7]. Furthermore Kuruoğlu, Düldül and Tutar [8] generalized Müller's results for homothetic motion.

In this section we find a formula for the polar moment of inertia and we arrive at the relation between the polar moments of inertia "T" and the formula of area "F".

2.5.1. Direct Motion

If we use α as a parameter, we need to calculate

$$T = \oint \left((x'_1)^2 + (x'_2)^2 \right) d\alpha. \quad (54)$$

Then by using Equation (4) in Equation (54),

$$\begin{aligned}
 T &= \oint \left[((x_1 - u_1) \cos \alpha - (x_2 - u_2) \sin \alpha)^2 + ((x_1 - u_1) \sin \alpha + (x_2 - u_2) \cos \alpha)^2 \right] d\alpha \\
 &= \oint \left((x_1 - u_1)^2 (\cos^2 \alpha + \sin^2 \alpha) + (x_2 - u_2)^2 (\cos^2 \alpha + \sin^2 \alpha) \right) d\alpha \\
 &= \oint \left((x_1 - u_1)^2 + (x_2 - u_2)^2 \right) d\alpha \\
 &= (x_1^2 + x_2^2) \oint d\alpha - 2x_1 \oint u_1 d\alpha - 2x_2 \oint u_2 d\alpha + \oint (u_1^2 + u_2^2) d\alpha
 \end{aligned}
 \tag{55}$$

is found.

If we calculate the polar moments of inertia for the origin of the moving system, for $T_o = T(x_1 = 0, x_2 = 0)$, we have

$$T_o = \oint (u_1^2 + u_2^2) d\alpha. \tag{56}$$

If Equation (56) is replaced in Equation (55),

$$T - T_o = (x_1^2 + x_2^2) \oint d\alpha - 2x_1 \oint u_1 d\alpha - 2x_2 \oint u_2 d\alpha \tag{57}$$

is obtained. If Equations (10), (11) and (13) is replaced in Equation (57),

$$T - T_o = (x_1^2 + x_2^2) m + x_1 a + x_2 b \tag{58}$$

can be written.

As a result, we arrive at the relation between the polar moments of inertia and the formula for the area,

$$T - T_o = 2(F - F_o). \tag{59}$$

2.5.2. Inverse Motion

If we use α as a parameter, we need to calculate

$$T' = \oint (x_1^2 + x_2^2) d\alpha. \tag{60}$$

Then by using Equations (17) and (18) in Equation (60),

$$\begin{aligned}
 T' &= \oint \left[((x'_1 - u'_1) \cos \alpha + (x'_2 - u'_2) \sin \alpha)^2 + (-(x'_1 - u'_1) \sin \alpha + (x'_2 - u'_2) \cos \alpha)^2 \right] d\alpha \\
 &= \oint \left[(x'_1 - u'_1)^2 (\cos^2 \alpha + \sin^2 \alpha) + (x'_2 - u'_2)^2 (\cos^2 \alpha + \sin^2 \alpha) \right] d\alpha \\
 &= \oint [(x'_1 - u'_1)^2 + (x'_2 - u'_2)^2] d\alpha \\
 &= -((x'_1)^2 + (x'_2)^2) \oint d\alpha + 2x'_1 \oint u'_1 d\alpha + 2x'_2 \oint u'_2 d\alpha + \oint ((u'_1)^2 + (u'_2)^2) d\alpha
 \end{aligned}
 \tag{61}$$

is found.

If we calculate the polar moments of inertia for the origin of the moving system, for $T'_o = T'(x'_1 = 0, x'_2 = 0)$, we have

$$T'_o = \oint ((u'_1)^2 + (u'_2)^2) d\alpha. \tag{62}$$

If Equation (62) is replaced in Equation (61)

$$T' - T'_o = -((x'_1)^2 + (x'_2)^2) \oint d\alpha + 2x'_1 \oint u'_1 d\alpha + 2x'_2 \oint u'_2 d\alpha \tag{63}$$

is obtained. If Equations (24), (25) and (26) is replaced in Equation (63)

$$T' - T'_o = -((x'_1)^2 + (x'_2)^2) m' - x'_1 a' - x'_2 b' \tag{64}$$

can be written.

Finally, we arrive at the relation between the polar moments of inertia and the formula for the area,

$$T' - T'_o = -2(F' - F'_o). \tag{65}$$

3. Application: The Motion of the Telescopic Crane

In the previous sections geometrical objects as the Steiner point or the Steiner normal, the pole point and the polar moments of inertia for closed motions are emphasized in a complex plane. In this section, we want to visualize the experimentally measured motion with these objects.

We choose the sagittal part of the movement of the telescopic crane as an example. The motion of the telescopic crane has a double hinge. The double hinge means that it has two systems, a fixed arm and a moving arm (Figure 1). There is a control panel of the telescopic crane at the origin of the fixed system.

3.1. Direct Motion

By taking

$$R(t) = \begin{pmatrix} \cos(\ell(t) - k(t)) & -\sin(\ell(t) - k(t)) \\ \sin(\ell(t) - k(t)) & \cos(\ell(t) - k(t)) \end{pmatrix}, \quad U'(t) = \begin{pmatrix} L \cos(\ell(t)) \\ L \sin(\ell(t)) \end{pmatrix}, \tag{66}$$

we have Equation (1) namely,

$$X'(t) = R(t)X + U'(t).$$

Also we have $U' = -RU$ and

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -L \cos(k(t)) \\ -L \sin(k(t)) \end{pmatrix}. \tag{67}$$

Then the double hinge motion can be written as

$$\begin{aligned} x'_1(t) &= \cos(\ell(t) - k(t))(x_1 + L \cos(k)) - \sin(\ell(t) - k(t))(x_2 + L \sin(k)) \\ x'_2(t) &= \sin(\ell(t) - k(t))(x_1 + L \cos(k)) + \cos(\ell(t) - k(t))(x_2 + L \sin(k)) \end{aligned} \tag{68}$$

where $\alpha = \ell - k$ is the resulting total angle.

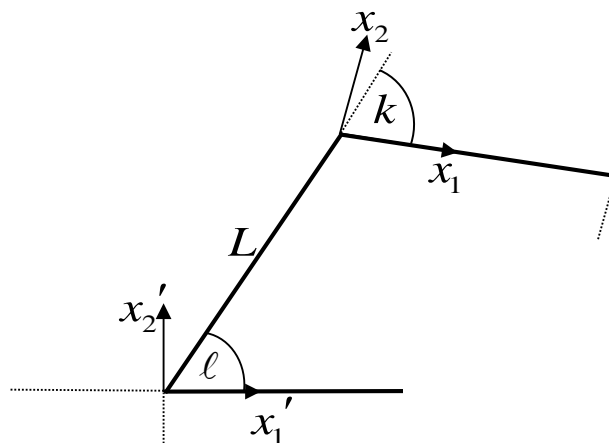


Figure 1. The arms of the telescopic crane as a double hinge.

If Equation (68) is derived, we obtain the velocities $\dot{x}'_1(t)$ and $\dot{x}'_2(t)$ and so we find

$$\begin{aligned} x'_1 \dot{x}'_2 - x'_2 \dot{x}'_1 &= (x_1^2 + x_2^2) \left(\dot{\ell}(t) - \dot{k}(t) \right) + x_1 \left(2L \cos(k(t)) \left(\dot{\ell}(t) - \dot{k}(t) \right) + L \cos(k(t)) \dot{k}(t) \right) \\ &+ x_2 \left(2L \sin(k(t)) \left(\dot{\ell}(t) - \dot{k}(t) \right) + L \sin(k(t)) \dot{k}(t) \right) + L^2 \dot{l}(t). \end{aligned} \tag{69}$$

We now integrate the previous equation using periodic boundary conditions while assuming that the integrands are periodic functions. The periodicity of f implies that the integrals of the following types vanish $\oint df = \int_1^F \dot{f} dt = f|_1^F = 0$.

As a result, only the integrals appearing in the second row of Equation (69) do not become equal to zero and we finally obtain a simplified expression for the area namely,

$$\begin{aligned} F &= \frac{1}{2} \int_{t_1}^{t_2} \left(x'_1 \dot{x}'_2 - x'_2 \dot{x}'_1 \right) dt \\ 2F &= x_1 \left(\int_{t_1}^{t_2} 2L \cos k \dot{l} dt \right) + x_2 \left(\int_{t_1}^{t_2} 2L \sin k \dot{l} dt \right). \end{aligned} \tag{70}$$

In the last equation, by taking

$$\int_{t_1}^{t_2} 2L \cos k \dot{l} dt = a \tag{71}$$

$$\int_{t_1}^{t_2} 2L \sin k \dot{l} dt = b, \tag{72}$$

we can write

$$2F = ax_1 + bx_2. \tag{73}$$

In this case, we have the steiner normal

$$n = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{74}$$

3.1.1. The Moving Pole Point of the Telescopic Cranemotion

If Equation (67) is replaced in Equation (41), we obtain the pole point

$P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ with the components

$$p_1 = -L \cos k \frac{\dot{\ell}}{\dot{\ell} - \dot{k}} \tag{75}$$

$$p_2 = -L \sin k \frac{\dot{\ell}}{\dot{\ell} - \dot{k}} \tag{76}$$

is obtained. If integral is taken over the total angle, we have

$$\oint P d\alpha = \oint P \begin{pmatrix} \dot{\ell} \\ \dot{\ell} - \dot{k} \end{pmatrix} dt = -L \int \begin{pmatrix} \cos k \\ \sin k \end{pmatrix} \dot{l} dt.$$

If we consider Equations (71) and (72) in the last equation, we obtain

$$\oint P d\alpha = -\frac{1}{2} \left(\frac{a}{b} \right). \quad (77)$$

3.1.2. The Polar Moments of Inertia of the Motion of the Telescopic Crane

If we consider Equations (54) and (68), then Equation (67) is replaced in Equation (55)

$$T = x_1 \oint 2L \cos k \left(\dot{\ell} - \dot{k} \right) dt + x_2 \oint 2L \sin k \left(\dot{\ell} - \dot{k} \right) dt$$

and finally

$$T = x_1 \oint 2L \cos k \dot{\ell} dt + x_2 \oint 2L \sin k \dot{\ell} dt \quad (78)$$

is obtained. If we consider Equations (71), (72) and (73) together, we arrive at the relation between the polar moments of inertia and the formula for the area namely,

$$T = 2F. \quad (79)$$

3.2. Inverse Motion

By taking

$$R(t) = \begin{pmatrix} \cos(\ell(t) - k(t)) & -\sin(\ell(t) - k(t)) \\ \sin(\ell(t) - k(t)) & \cos(\ell(t) - k(t)) \end{pmatrix}, U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} -L \cos(k(t)) \\ -L \sin(k(t)) \end{pmatrix}, \quad (80)$$

we have

$$X(t) = (R(t))^T (X' - U'(t)).$$

Also we have $U' = -RU$ and

$$U'(t) = \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} L \cos(\ell(t)) \\ L \sin(\ell(t)) \end{pmatrix}. \quad (81)$$

So the double hinge may be written as

$$\begin{aligned} x_1(t) &= \cos(\ell(t) - k(t))(x_1' - L \cos(\ell)) + \sin(\ell(t) - k(t))(x_2' - L \sin(\ell)) \\ x_2(t) &= -\sin(\ell(t) - k(t))(x_1' - L \cos(\ell)) + \cos(\ell(t) - k(t))(x_2' - L \sin(\ell)) \end{aligned} \quad (82)$$

where $\alpha = \ell - k$ is the resulting total angle.

By following the same operations similar to the direct motion, we finally obtain the Steiner formula for the inverse motion

$$2F' = a'x_1' + b'x_2' \quad (83)$$

where

$$-\int_{t_1}^{t_2} 2L \cos l \dot{k} dt = a' \quad (84)$$

$$-\int_{t_1}^{t_2} 2L \sin l \dot{k} dt = b'. \quad (85)$$

In this case, we have the Steiner normal

$$n' = \begin{pmatrix} a' \\ b' \end{pmatrix}. \quad (86)$$

3.2.1. The Fixed Pole Point of the Inverse Telescopic Crane Motion

If Equation (81) is replaced in Equation (51), we obtain the pole point

$$P' = \begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix} \text{ with the components}$$

$$\begin{aligned} p'_1 &= -L \cos \ell \frac{\dot{k}}{\ell - k} \\ p'_2 &= -L \sin \ell \frac{\dot{k}}{\ell - k}. \end{aligned} \quad (87)$$

3.2.2. The Polar Moments of Inertia of the Inverse Telescopic Crane Motion

If we consider Equations (60) and (82), then Equation (81) is replaced in Equation (78)

$$T' = x'_1 \oint \left(2L \cos \ell \dot{k} \right) dt + x'_2 \oint \left(2L \sin \ell \dot{k} \right) dt \quad (88)$$

is obtained. If we consider Equations (83), (84), (85) and (88) together, we arrive at the relation between the polar moments of inertia and the formula for the area below:

$$T' = -2F'. \quad (89)$$

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References

- [1] Steiner, J. (1840) Von dem Krümmungs-Schwerpunkte ebener Curven. *Journal für die reine und angewandte Mathematik*, **21**, 33-63.
<https://doi.org/10.1515/crll.1840.21.33>
- [2] Tutar, A. and Kuruoğlu, N. (1999) The Steiner Formula and the Holditch Theorem for the Homothetic Motions on the Planar Kinematics. *Mechanism and Machine Theory*, **34**, 1-6.
- [3] Müller, H.R. (1978) Verallgemeinerung einer Formel von Steiner. *Abh. Braunschweig. Wiss. Ges.*, **29**, 107-113.
- [4] Müller, H.R. (1978) Über Trägheitsmomente bei Steinerscher Massenbelegung. *Abh. Braunschweig. Wiss. Ges.*, **29**, 115-119.
- [5] Dathe, H. and Gezzi, R. (2012) Characteristic Directions of Closed Planar Motions. *Zeitschrift für Angewandte Mathematik und Mechanik*, **92**, 731-748.
<https://doi.org/10.1002/zamm.201100178>
- [6] Blaschke, W. and Müller, H.R. (1956) *Ebene Kinematik*, R. Oldenbourg, München.
- [7] Tölke, J. (1978) Steiner-Formeln für die Bahnflächen geschlossener Aquiaffin-

bewegungen. *Sitzungsber. Österr. Akad. Wiss, Math.-Nat. Klasse*, **187**, 325-337.

- [8] Kuruoğlu, N., Düldül, M. and Tutar, A. (2003) Generalization of Steiner Formula for the Homothetic Motions on the Planar Kinematics. *Applied Mathematics and Mechanics*, **24**, 945-949. <https://doi.org/10.1007/BF02446500>



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