

The Tarski Theorems and Elementary Equivalence of Group Rings

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Abstract

The Tarski theorems, proved by Myasnikov and Kharlampovich and independently by Sela say that all nonabelian free groups satisfy the same first-order or elementary theory. Kharlampovich and Myasnikov also prove that the elementary theory of free groups is decidable. For a group ring they have proved that the first-order theory (in the language of ring theory) is not decidable and have studied equations over group rings, especially for torsion-free hyperbolic groups. In this note we examine and survey extensions of Tarski-like results to the collection of group rings and examine relationships between the universal and elementary theories of the corresponding groups and rings and the corresponding universal theory of the formed group ring. To accomplish this we introduce different first-order languages with equality whose model classes are respectively groups, rings and group rings. We prove that if $R[G]$ is elementarily equivalent to $S[H]$ then simultaneously the group G is elementarily equivalent to the group H and the ring R is elementarily equivalent to the ring S with respect to the appropriate languages. Further if G is universally equivalent to a nonabelian free group F and R is universally equivalent to the integers \mathbb{Z} then $R[G]$ is universally equivalent to $\mathbb{Z}[F]$ again with respect to an appropriate language.

Keywords

Group Ring, Elementary Equivalent, Universally Equivalent, Direct Union, Discriminates

1. Introduction

For a general algebraic object A , for example a group, a ring, a field or an

algebra, its **elementary theory** is the set of all first-order sentences in a logical language appropriate for that object, true in A (see Section 2). Hence if F is a given free group, its elementary theory consists of all first-order sentences in a language appropriate for group theory that are true in F . Two algebraic objects are **elementary equivalent** or **elementarily equivalent** if they have the same elementary theory. The Tarski theorems proved by Kharlampovich and Myasnikov and independently by Sela (see [1] and [2] and the references therein) say that all nonabelian free groups satisfy the same elementary theory. Kharlampovich and Myasnikov also showed that the elementary theory of free groups is decidable, that is, there is an algorithm to decide if any elementary sentence is true in all free groups or not. For a group ring they have proved that the first-order theory (in the language of ring theory) is not decidable and have studied equations over group rings especially for torsion-free hyperbolic groups (see (2)).

The set of universal sentences (see Section 2) in an algebraic object A that are true in A is its **universal theory** while two objects are **universally equivalent** if they have the same universal theory. It is straightforward to show that all nonabelian free groups have the same universal theory. As part of the general solution to the Tarski theorems it was shown that a finitely generated nonabelian group is **universally free** (that is has the same universal theory as a nonabelian free group) if and only if it is a limit group.

In this note, following the studies by Kharlampovich and Myasnikov on equations over group rings [3], we examine and survey extensions of Tarski-like results to the collection of group rings and examine relationships between the universal and elementary theories of the corresponding groups and rings and the corresponding universal theory of the formed group ring. To accomplish this we introduce different first-order languages with equality whose model classes are respectively groups, rings and group rings. We prove that if $R[G]$ is elementarily equivalent to $S[H]$ then simultaneously the group G is elementarily equivalent to the group H and the ring R is elementarily equivalent to the ring S with respect to the appropriate languages. Further if G is universally equivalent to a nonabelian free group F and R is universally equivalent to the integers \mathbb{Z} then $R[G]$ is universally equivalent to $\mathbb{Z}[F]$ again with respect to an appropriate language.

The outline of this paper is as follows. In the next section we review the basic material on elementary and universal theory. In Section 2 we introduce three first-order languages upon which the results are based. In Section 3 we describe the Tarski theorems and some important related results. In Section 4 we consider the universal theories of group rings while in Section 5 the elementary theories of group rings.

2. Elementary and First-Order Theories

Alfred Tarski in 1940 made three well-known conjectures concerning non-abelian free groups. We call these the **Tarski Problems** or **Tarski Conjectures**

and they asked, among other things, whether all nonabelian free groups satisfy the same first-order or elementary theory. This was proved in the affirmative by Kharlampovich and Myasnikov ([4] [5] [6] [7] [8]) and independently by Sela ([9] [10] [11] [12] [13])). We now explain the terminology.

In order to discuss universal and elementary theory for an algebraic object A we must first define appropriate logical languages to deal with the object. In the context of this paper we need three languages; one for groups, one for rings and one for group rings. To this end we introduce three first-order languages with equality that we will call L_0, L_1, L_2 ; the first appropriate for group theory, the second for ring theory and the third for the theory of group rings. We then list sets of axioms T_0, T_1, T_2 expressed in L_0, L_1, L_2 respectively and view a group as a model of T_0 a ring as a model of T_1 . Moreover, we view the class of group rings as a subclass of the model class of T_2 .

We first review the elementary and universal theory of groups. A first-order sentence in group theory has logical symbols $\forall, \exists, \vee, \wedge, \sim$ but no quantification over sets. A first-order theorem in a group G is a theorem that says a first-order sentence is true in G . For a class of groups \mathcal{G} a first-order theorem is a first-order sentence true in all groups in \mathcal{G} . We make this a bit more precise:

We start with a first-order language L_0 appropriate for group theory. L_0 is the first-order language with equality containing a binary operation symbol \cdot a unary operation symbol $^{-1}$ and a constant symbol 1 . A **universal sentence** of L_0 is one of the form $\forall \bar{x} \{ \phi(\bar{x}) \}$ where \bar{x} is a tuple of distinct variables, $\phi(\bar{x})$ is a formula of L_0 containing no quantifiers and containing at most the variables of \bar{x} . Similarly an **existential sentence** is one of the form $\exists \bar{x} \{ \phi(\bar{x}) \}$ where \bar{x} and $\phi(\bar{x})$ are as above. A **universal-existential sentence** is one of the form $\forall \bar{x} \exists \bar{y} \{ \phi(\bar{x}, \bar{y}) \}$. Similarly defined is an **existential-universal sentence**. It is known that every sentence of L_0 is logically equivalent to one of the form $Q_1 x_1 \cdots Q_n x_n \phi(\bar{x})$ where $\bar{x} = (x_1, \dots, x_n)$ is a tuple of distinct variables, each Q_i for $i = 1, \dots, n$ is a quantifier, either \forall or \exists , and $\phi(\bar{x})$ is a formula of L_0 containing no quantifiers and containing free at most the variables x_1, \dots, x_n . Further vacuous quantifications are permitted. Finally a **positive sentence** is one logically equivalent to a sentence constructed using (at most) the connectives $\vee, \wedge, \forall, \exists$.

If G is a group then the **universal theory** of G consists of the set of all universal sentences of L_0 true in G . We denote the universal theory of a group G by $Th_\forall(G)$. Since any universal sentence is equivalent to the negation of an existential sentence it follows that two groups have the same universal theory if and only if they have the same **existential theory**. Two groups G and H that have the same universal equivalently existential theory are said to be **universally equivalent**.

The set of all sentences of L_0 true in G is called the **first-order theory** or the **elementary theory** of G . We denote this by $Th(G)$. We note that being **first-order** or **elementary** means that in the intended interpretation of any formula or sentence all of the variables (free or bound) are assumed to take on as values

only individual group elements—never, for example, subsets of, nor functions on, the group in which they are interpreted.

We say that two groups G and H are **elementarily equivalent** (symbolically $G \equiv H$) if they have the same first-order theory, that is $Th(G) = Th(H)$.

Two very important concepts in the elementary theory of groups, are **completeness** and **decidability**. Given a nonempty class of groups \mathcal{X} closed under isomorphism we say that its first-order theory is **complete** if given a sentence ϕ of L_0 either ϕ is true in every group in \mathcal{X} or ϕ is false in every group in \mathcal{X} . The first-order theory of \mathcal{X} is **decidable** if there exists a recursive algorithm which, given a sentence ϕ of L_0 , decides whether or not ϕ is true in every group in \mathcal{X} . Kharlampovich and Myasnikov (see [1] [2]) proved that the elementary theory of free groups is decidable.

Let L_0 be the first-order language as above and let T_0 be the axioms

$$\begin{aligned} \forall x, y, z ((x \cdot y) \cdot z &= x \cdot (y \cdot z)) \\ \forall x (x \cdot 1 &= x) \\ \forall x (x \cdot x^{-1} &= 1). \end{aligned}$$

Then we view groups as models of T_0 , that is the model class of T_0 is the class of groups.

In order to extend some of these results to group rings we introduce two additional first-order languages L_1 and L_2 . We then give axioms in these languages so that the models of the axiomatic systems are rings and group rings respectively.

Let L_1 be the first-order language with equality appropriate for ring theory. The language L_1 contains two binary operation symbols $+$ and \cdot , two unary operation symbols $-$ and $^{-1}$ and two constant symbols 0 and 1 . A **ring with unity** will then be a model of the following set T_1 of axioms.

$$\begin{aligned} \forall x, y (x + y &= y + x) \\ \forall x, y, z ((x + y) + z &= x + (y + z)) \\ \forall x (x + 0 &= x) \\ \forall x (x + (-x) &= 0) \\ \forall x, y, z ((x \cdot y) \cdot z &= x \cdot (y \cdot z)) \\ \forall x (x \cdot 1 &= x) \\ \forall x (1 \cdot x &= x) \\ \forall x, y (((x \cdot y = 1) \wedge (y \cdot x = 1)) &\rightarrow (x^{-1} = y)) \\ \forall x (\forall y ((x \cdot y \neq 1) \vee (y \cdot x \neq 1)) &\rightarrow (x^{-1} = 0)) \\ \forall x, y, z (x \cdot (y + z) &= x \cdot y + x \cdot z) \\ \forall x, y, z ((x + y) \cdot z &= x \cdot z + y \cdot z) \end{aligned}$$

$$1 \neq 0$$

Thus, we view rings with multiplicative identity $1 \neq 0$ as models of T_1 .

We now let L_2 be the first-order language with equality containing two binary operation symbols $+$ and \cdot , two unary operation symbols $-$ and $^{-1}$, two constant symbols 0 and 1 and two unary relation symbols Γ and P .

To clarify the relation symbols we wish to model a group ring $R[G]$. The relation symbol Γ will identify elements in G and the relation symbol P identifies elements in R . Hence $\Gamma(x)$ is true if $x \in G$ and $P(x)$ is true if $x \in R$.

Now we let T_2 be the union of T_1 and the following additional axioms.

$$\begin{aligned} &\Gamma(1) \\ &\forall x, y ((\Gamma(x) \wedge \Gamma(y)) \rightarrow \Gamma(x \cdot y)) \\ &\forall x (\Gamma(x) \rightarrow \exists y (\Gamma(y) \wedge (x \cdot y = 1) \wedge (y \cdot x = 1))) \\ &P(0) \\ &P(1) \\ &\forall x, y ((P(x) \wedge P(y)) \rightarrow (P(x + y))) \\ &\forall x, y ((P(x) \wedge P(y)) \rightarrow P(x \cdot y)) \\ &\forall x (P(x) \rightarrow P(-x)) \\ &\forall x (P(x) \rightarrow P(x^{-1})) \\ &\forall x, y (P(x) \rightarrow x \cdot y = y \cdot x) \end{aligned}$$

If A is a model of T_2 , we let $\Gamma(A) = \{a \in A : \Gamma(a)\}$ and $P(A) = \{a \in A : P(a)\}$.

Thus, $\Gamma(A)$ is a group and $P(A)$ is a commutative ring with identity $1 \neq 0$ which is central in A . We call a model A of T_2 **standard** provided every element $a \in A$ is uniquely expressible in the form $\sum_{g \in \Gamma(A)} r(g) \cdot g$ where

$r(g) \in P(A)$ for all $g \in \Gamma(A)$ and at most finitely many $r(g)$ are nonzero. Thus, we view group rings as standard models of T_2 . Henceforth we omit the binary operation symbol \cdot in favor of juxtaposition.

With these languages and model classes in place we can speak of universal equivalence and elementary equivalence of rings and group rings. Hence two rings R, S are universally equivalent if they satisfy the same universal sentences relative to L_1 and elementary equivalent if they satisfy the same first-order sentences relative to L_1 . Analogously two group rings $R[G], S[H]$ are universally equivalent if they satisfy the same universal sentences relative to L_2 and elementary equivalent if they satisfy the same first-order sentences relative to L_2 .

3. The Tarski Theorems in Groups

Alfred Tarski made three well-known conjectures concerning the first-order

theory of nonabelian free groups. The purpose of this paper is to extend if possible parts of these to group rings. Here we recall these basic results. Complete discussions can be found in the book [1] and in the paper [2].

Tarski first asked the general question whether all nonabelian free groups share the same elementary theory. Vaught, a student of Tarski's, proved almost immediately that all free groups of infinite rank do have the same elementary theory, and thus reduced the question to the class of nonabelian free groups of finite rank. After this, Tarski's question was formalized into the following conjectures.

Tarski Conjecture 1. *Any two nonabelian free groups are elementarily equivalent. That is any two nonabelian free groups satisfy exactly the same first-order theory.*

Tarski Conjecture 2. *If the nonabelian free group H is a free factor in the free group G then the inclusion map $H \rightarrow G$ is an elementary embedding.*

An **elementary embedding** is a group monomorphism which preserves the truth of first-order formulas. Specifically, if H and G are groups and

$$f : H \rightarrow G$$

is a monomorphism then f is an **elementary embedding** provided whenever $\phi(x_0, \dots, x_n)$ is a formula of L_0 containing free at most the distinct variables x_0, \dots, x_n and $(h_0, \dots, h_n) \in H^{n+1}$ then $\phi(h_0, \dots, h_n)$ is true in H if and only if

$$\phi(f(h_0), \dots, f(h_n))$$

is true in G . If H is a subgroup of G and the inclusion map $i : H \rightarrow G$ is an elementary embedding then we say that G is an **elementary extension** of H . It follows that the second conjecture is stronger than the first and in fact implies the first. If true, then the theory of the nonabelian free groups would be **complete**, that is given a sentence ϕ of L_0 then either ϕ is true in every nonabelian free group or ϕ is false in every nonabelian free group.

Two very important concepts in the elementary theory of groups, are **completeness** and **decidability**. Given a non-empty class of groups \mathcal{X} closed under isomorphism then we say its first-order theory is **complete** if given a sentence ϕ of L_0 then either ϕ is true in every group in \mathcal{X} or ϕ is false in every group in \mathcal{X} . The first-order theory of \mathcal{X} is **decidable** if there exists a recursive algorithm which, given a sentence ϕ of L_0 decides whether or not ϕ is true in every group in \mathcal{X} . The third Tarski conjecture was about decidability.

Tarski Conjecture 3. *The elementary theory of the nonabelian free groups is decidable.*

After a long series of partial results (see [1] [2]) the positive solution to the Tarski conjectures (1) and (2) was given by O. Kharlampovich and A. Myasnikov ([3] [4] [5] [6] [7]) and independently by Z. Sela ([8] [9] [10] [11] [12]). The proofs by both sets of authors involved the development of whole new areas of mathematics, in particular an algebraic geometry (Sela calls this diophantine

geometry) over free groups. In addition Kharlampovich and Myasnikov proved Tarski Conjecture (3).

Theorem 3.1. (1) *Any two nonabelian free groups are elementarily equivalent. That is any two nonabelian free groups satisfy exactly the same first-order theory.*

(2) If the nonabelian free group H is a free factor in the free group G then the inclusion map $H \rightarrow G$ is an elementary embedding.

(3) The elementary theory of the nonabelian free groups is decidable.

Although Tarski was never explicit on the origin of the basic question, it is motivated by several basic results, and concepts, in the theory of free groups (see [1] and [14] for complete discussions of free groups). First is the observation that most free group properties, involving elements, are rank independent, that is, true for all free groups independent of rank. For example all nonabelian free groups are torsion-free and all abelian subgroups of nonabelian free groups are cyclic.

A second possible motivation, which also shows that all nonabelian free groups have the same universal theory, is the following. Let F_2 be a free group of rank 2. It is a straightforward consequence of the Reidemeister-Schreier process (see [MKS]) that the commutator subgroup of F_2 is free of infinite rank. This implies that if we let F_ω denote a free group of countably infinite rank, then $F_\omega \leq F_2$. That is, F_ω can be embedded isomorphically as a subgroup of F_2 . It follows that for any $m, n \geq 2$ with $m < n$ we have

$$\cdots \leq F_\omega \leq F_2 \leq F_m \leq F_n \leq \cdots \leq F_\omega \leq \cdots \quad (3.1)$$

This shows that F_n can be embedded isomorphically in F_m and F_m can be embedded isomorphically in F_n . Its like a snake eating its tail.

If $G \subset H$ then any universal sentence in H must also be true in G , that is $Th_\forall(H) \subset Th_\forall(G)$. This observation combined with the observations above prove that all nonabelian free groups have the same universal theory and hence are universally equivalent.

Theorem 3.2. *All nonabelian free groups are universally equivalent.*

Gaglione and Spellman [15] and independently Remeslennikov [16] gave a characterization of the universally free groups. This characterization became one of the cornerstones of the proof of the Tarski theorems. Recall group G is **residually free** provided, for every $g \in G \setminus \{1\}$, there is a homomorphism $\varphi: G \rightarrow F$, where F is a free group, such that $\varphi(g) \neq 1$. G **fully residually free** provided, for every finite set of nontrivial elements $g_1, \dots, g_n \in G \setminus \{1\}$, there is a homomorphism $\varphi: G \rightarrow F$, where F is a free group, such that $\varphi(g_i) \neq 1$. We also say that G is **discriminated** by a free group. Finitely generated fully residually free groups are also known as **limit groups** since they arise as limits of homomorphisms into free groups (see [1]).

The following is the result due to Gaglione and Spellman and independently to Remeslennikov that ties together fully residual freeness and universal freeness.

Theorem 3.3. ([15] [16]) *Suppose G is residually free. Then the following*

are equivalent:

- (1) G is fully residually free,
- (2) G is commutative transitive,
- (3) G is universally free if nonabelian.

Parts (1) and (2) of the theorem were due originally to Benjamin Baumslag [17].

Prior to the solution of the Tarski problems, it was asked whether there exist finitely generated non-free **elementary free groups**. By this it was meant that if all countable nonabelian free groups do have the same first-order theory do there exist finitely generated non-free groups with exactly the same first-order theory as the class of nonabelian free groups.

In the finitely generated case, the answer is yes, and both the Kharlampovich-Myasnikov solution and the Sela solution provide a complete characterization of the finitely generated elementary free groups. In the Kharlampovich-Myasnikov formulation these are given as a special class of what are termed NTQ groups (see [1] or [2]). The primary examples of non-free elementary free groups are the orientable surface groups of genus $g \geq 2$ and the nonorientable surface groups of genus $g \geq 4$ (see [1]). Recall that a **surface group** is the fundamental group of a compact surface. If the surface is orientable it is an orientable surface group otherwise a nonorientable surface group.

4. Universal Equivalence of Group Rings

We now attempt to extend the Tarski results to group rings, in particular to group rings of free groups. We start by considering the universal theory of a group ring $R[G]$ where R is a commutative ring with an identity. Let F be a nonabelian finitely generated free group and \mathbb{Z} the integers. If F_1 is any other nonabelian countable free group then we have the same snake eating its tail situation $F_1 \leq F \leq F_1$ as before. Since every subring of \mathbb{Z} with an identity is \mathbb{Z} itself the same argument as for groups shows that for any language we use

$$Th_{\mathbb{Z}}[F] \subset Th_{\mathbb{Z}}[F_1] \subset Th_{\mathbb{Z}}[F] \tag{4.1}$$

It follows as for groups that all integral group rings of finitely generated nonabelian free groups are universally equivalent. Here we will use the universal theory with the axioms T_2 and language L_2 for group rings.

Theorem 4.1. *All integral group rings for nonabelian countable free groups are universally equivalent.*

In this context we call any group ring universally equivalent to $\mathbb{Z}[F]$ for a nonabelian countable free group a **universally free group ring**. We now consider the question of classifying the universally free group rings in a manner similar to the Gaglione-Spellman-Remeslennikov theorem.

In [18] the following more general results were proved answering this. The proofs are short so they are included. The proofs depend on material in the book by Bell and Slomson [19].

Theorem 4.2. *Let R be a commutative ring with identity $1 \neq 0$ and let G*

and H be groups. If G and H are universally equivalent with respect to L_0 , then $R[G]$ and $R[H]$ are universally equivalent with respect to L_1 .

Proof. H is a model of $Th_{\forall}(G)$. Thus, by Lemma 3.8 of Chapter 9 of [19], there is a nonempty set I and an ultrafilter D on I such that H embeds in ${}^*G = G^I/D$. Hence, $R[H]$ embeds in ${}^*(R[G]) \cong R[G]$. Hence, $R[H]$ is a model of $Th_{\forall}(R[G])$ with respect to L_1 . Interchanging, the roles of G and H in the above, we get $R[G]$ is a model of $Th_{\forall}(R[H])$ with respect to L_1 . Therefore, $Th_{\forall}(R[G]) = Th_{\forall}(R[H])$ with respect to L_1 .

Theorem 4.3. *Let R and S be commutative rings with identity $1 \neq 0$ and let G be a group. If R and S are universally equivalent with respect to L_1 , then $R[G]$ and $S[G]$ are universally equivalent with respect to L_1 .*

The proof of the second theorem is similar to that of the first (see [18]).

Combining these two theorems and using the transitivity of universal equivalence with respect to L_1 , we immediately deduce.

Theorem 4.4. *Let G and H be groups and let R and S be commutative rings with identity $1 \neq 0$. If G and H are universally equivalent with respect to L_0 and R and S are universally equivalent with respect to L_1 , then $R[G]$ and $S[H]$ are universally equivalent with respect to L_1 .*

These two theorems can be utilized to handle the classification of universally free group rings which is stated in the following corollary.

Corollary 1. *Let $F = \langle a_1, a_2 \rangle$ be the rank 2 free group and let \mathbb{Z} be the ring of integers. Let G be a group and R be a commutative ring with identity $1 \neq 0$. If G is universally equivalent to F with respect to L_0 and R is universally equivalent to \mathbb{Z} with respect to L_1 , then $R[G]$ is universally equivalent to $\mathbb{Z}[F]$ with respect to L_1 .*

In [18] a second independent proof of the above result was given. We want next to extend to group rings an analog of the Gaglione-Spellman-Remeslennikov result tying universal freeness to fully residual freeness. First we need the following idea which is the appropriate analog of residual freeness for rings.

A ring R with identity $1 \neq 0$ which is discriminated by \mathbb{Z} is said to be ω -**residually** \mathbb{Z} . Such rings have characteristic zero so $\mathbb{Z} \leq R$. A ring all of whose finitely generated subrings are ω -residually \mathbb{Z} is said to be **locally** ω -**residually** \mathbb{Z} . Of course, every ring is the direct union of its finitely generated subrings. Hence, every locally ω -residually \mathbb{Z} ring is universally equivalent to \mathbb{Z} . The converse is also known to be true.

Using these concepts we may paraphrase Corollary 1 thusly. Let $F = \langle a_1, a_2 \rangle$ and let R be a locally ω -residually \mathbb{Z} ring. If G is a nonabelian locally fully residually free group, then $R[G]$ is universally equivalent to $\mathbb{Z}[F]$ with respect to L_1 . Now $R[G]$ is the direct union of the family of subrings of the form $R_0[G_0]$ where R_0 is a finitely generated subring of R and G_0 is a nonabelian finitely generated subgroup of G . If $x, y \in G_0$ and do not commute, then $\langle x, y \rangle \cong F$. By Lemm 3.5 it suffices to show that such subrings are discriminated by $\mathbb{Z}[F]$.

If R is a ring with identity $1 \neq 0$ we let $U(R)$ be its group of units. If R

is a commutative ring with identity $1 \neq 0$ and G a group then an element of $U(R[G])$ is said to be a **trivial unit** provided it has the form ug where $u \in U(R)$ and $g \in G$.

Definition 4.5. A group G is **orderable** provided it admits a linear order $<$ satisfying the conditions that $hg_1 < hg_2$ whenever $g_1 < g_2$ and $g_1h < g_2h$ whenever $g_1 < g_2$ as g_1, g_2 and h vary over G .

It is well known that free groups are orderable. See e.g. [20]. One can find, for example in Passman’s book [21], that, if K is a field and G is an orderable group, then $K[G]$ has trivial units only. From that it easily follows that if D is an integral domain and G is an orderable group, then $D[G]$ has trivial units only. Now suppose that R is any locally ω -residually \mathbb{Z} ring. Then R is universally equivalent to \mathbb{Z} and thus the following four sentences hold in R .

- (i) $\forall x, y (xy = yx)$.
- (ii) $\forall x, y ((xy = 0) \rightarrow ((x = 0) \vee (y = 0)))$.
- (iii) $\forall xy ((xy = 1) \rightarrow ((x = 1) \vee (x = -1)))$.
- (iv) $1 \neq -1$.

It follows that any locally ω -residually \mathbb{Z} ring R is an integral domain with group of units $U(R) = \{1, -1\} = U(\mathbb{Z})$. Note that since $F = \langle a_1, a_2 \rangle$ is orderable and \mathbb{Z} is an integral domain, $\mathbb{Z}[F]$ has trivial units only and thus

$$U(\mathbb{Z}[F]) = \langle -1 \rangle \times F.$$

Theorem 4.6. Let $F = \langle a_1, a_2 \rangle$, R be a locally ω -residually \mathbb{Z} ring and G be a group. If $R[G]$ is universally equivalent to $\mathbb{Z}[F]$ with respect to L_1 , then G is a universally free group.

Lemma 4.7. $U(R[G])$ is universally equivalent to $U(\mathbb{Z}[F])$ with respect to L_0 .

Proof. Suppose that $\forall x_1, \dots, x_n \varphi(x_1, \dots, x_n)$ is a universal sentence of L_0 true in $U(\mathbb{Z}[F])$. Then

$$\forall x_1, \dots, x_n, y_1, \dots, y_n \left(\bigwedge_{i=1}^n ((x_i y_i = 1) \wedge (y_i x_i = 1)) \rightarrow \varphi(x_1, \dots, x_n) \right) \tag{4.1}$$

is true in $\mathbb{Z}[F]$; hence, it is true in $R[G]$ and so $\forall x_1, \dots, x_n \varphi(x_1, \dots, x_n)$ holds in $U(R[G])$.

Suppose $\exists x_1, \dots, x_n \psi(x_1, \dots, x_n)$ is an existential sentence of L_0 true in $U(\mathbb{Z}[F])$. Then

$$\exists x_1, \dots, x_n, y_1, \dots, y_n \left(\bigwedge_{i=1}^n ((x_i y_i = 1) \wedge (y_i x_i = 1)) \wedge \psi(x_1, \dots, x_n) \right) \tag{4.2}$$

is true in $U(\mathbb{Z}[F])$; hence, it is true in $R[G]$ and so $\exists x_1, \dots, x_n \psi(x_1, \dots, x_n)$ holds in $U(R[G])$. It follows that $U(\mathbb{Z}[F])$ and $U(R[G])$ are universally equivalent with respect to L_0 .

Lemma 4.8. -1 is the unique element of $U(R[G])$ having order 2. Furthermore, -1 is central in $U(R[G])$.

Proof. The following sentences of L_0 are true in $\langle -1 \rangle \times F = U(\mathbb{Z}[F])$.

$$\begin{aligned} & \exists z \left((z^2 = 1) \wedge (z \neq 1) \right) \\ & \forall z_1 z_2 \left(\left((z_1^2 = 1) \wedge (z_2^2 = 1) \wedge (z_1 \neq 1) \wedge (z_2 \neq 1) \right) \rightarrow (z_1 = z_2) \right) \\ & \forall x, z \left((z^2 = 1) \rightarrow (xz = zx) \right). \end{aligned}$$

Hence, they are true in $U(R[G])$ and the conclusions follow. \square

Lemma 4.9. *If $R[G]$ is universally equivalent to $\mathbb{Z}[\mathbb{F}]$ with respect to L_1 , then G is an orderable group. In particular, $R[G]$ has trivial units only and so*

$$U(R[G]) = \langle -1 \rangle \times G.$$

Proof. $\langle -1 \rangle \times G \leq U(R[G])$ from which we get an embedding

$$G \rightarrow U(R[G]) / \langle -1 \rangle$$

by factoring out by the central subgroup $\langle -1 \rangle$. It is shown in [22] that orderability is captured by a set S of universal sentences of L_0 . Hence,

$$F \cong U(\mathbb{Z}[F]) / \langle -1 \rangle$$

is a model of S .

Let $\sigma \in S$. Suppose σ is $\forall x_1, \dots, x_n \varphi(x_1, \dots, x_n)$. Let σ^* be the sentence

$$\forall x_1, \dots, x_n, z \left((z^2 = 1) \wedge (z \neq 1) \rightarrow \hat{\varphi}(x_1, \dots, x_n, z) \right) \tag{4.3}$$

where $\hat{\varphi}(x_1, \dots, x_n, z)$ is obtained from $\varphi(x_1, \dots, x_n)$ by replacing each equation $u(\bar{x}) = 1$ with the disjunction $(u(\bar{x}) = 1) \vee (u(\bar{x}) = z)$ and each inequation $w(\bar{x}) \neq 1$ with the conjunction $(w(\bar{x}) \neq 1) \wedge (w(\bar{x}) \neq z)$. Then σ^* holds in $U(\mathbb{Z}[F])$ since it asserts that σ holds in $U(\mathbb{Z}[\mathbb{F}])$ modulo $\langle -1 \rangle$. Therefore, σ^* holds in $U(R[G])$ and so σ holds in $U(R[G])$ modulo $\langle -1 \rangle$. Therefore, $U(R[G]) / \langle -1 \rangle$ is orderable. Since orderability is preserved in subgroups and G embeds in $U(R[G]) / \langle -1 \rangle$, G is orderable. \square

This far, we have viewed group rings as instances of models of T_1 in L_1 . If we view group rings as standard models of T_2 in L_2 and assert the stronger condition of universal equivalence with respect to L_2 , then the converse to Theorem 4.3 is true (see [18]).

Theorem 4.10. *Let G and H be groups and let R and S be commutative rings with identity $1 \neq 0$. View the group rings $R[G]$ and $S[H]$ as standard models of T_2 , A and B respectively. If A and B are universally equivalent with respect to L_2 , then G is universally equivalent to H with respect to L_0 and R is universally equivalent to S with respect to L_1 .*

Proof. Suppose $\forall x_1, \dots, x_n \varphi(x_1, \dots, x_n)$ is a universal sentence of L_0 true in $G = \Gamma(A)$. Then

$$\forall x_1, \dots, x_n \left(\bigwedge_{i=1}^n \Gamma(x_i) \rightarrow \varphi(x_1, \dots, x_n) \right) \tag{4.4}$$

is true in $R[G] = A$. Therefore it is true in $S[H] = B$ and so

$\forall x_1, \dots, x_n \varphi(x_1, \dots, x_n)$ is true in $H = \Gamma(B)$.

Suppose $\exists x_1, \dots, x_n \psi(x_1, \dots, x_n)$ is an existential sentence of L_0 true in $G = \Gamma(A)$. Then

$$\exists x_1, \dots, x_n \left(\bigwedge_{i=1}^n \Gamma(x_i) \wedge \psi(x_1, \dots, x_n) \right) \tag{4.5}$$

is true in $R[G] = A$. Hence it is true in $S[H] = B$ and so

$\exists x_1, \dots, x_n \psi(x_1, \dots, x_n)$ is true in $H = \Gamma(B)$. It follows that G and H are universally equivalent with respect to L_0 .

Suppose $\forall x_1, \dots, x_n \alpha(x_1, \dots, x_n)$ is a universal sentence of L_1 true in $R = P(A)$. Then

$$\forall x_1, \dots, x_n \left(\bigwedge_{i=1}^n P(x_i) \rightarrow \alpha(x_1, \dots, x_n) \right) \tag{4.6}$$

is true in $R[G] = A$. Hence it is true in $S[H] = B$ and so

$\forall x_1, \dots, x_n \alpha(x_1, \dots, x_n)$ is true in $S = P(B)$.

Suppose $\exists x_1, \dots, x_n \beta(x_1, \dots, x_n)$ is an existential sentence of L_1 true in $R = P(A)$. Then

$$\exists x_1, \dots, x_n \left(\bigwedge_{i=1}^n \Gamma(x_i) \wedge \beta(x_1, \dots, x_n) \right) \tag{4.7}$$

is true in $R[G] = A$. Hence, it is true in $S[H] = B$ and so

$\exists x_1, \dots, x_n \beta(x_1, \dots, x_n)$ is true in $S = P(B)$. It follows that R and S are universally equivalent with respect to L_1 .

5. Elementary Equivalence of Group Rings

We now examine the elementary equivalence of group rings. We need the following proposition that can be found in the book of Chang and Keisler ([23]).

Proposition 1. (Keisler-Shelah [24]) Let L be a first-order language with equality and A and B be L -structures. If $A \equiv_L B$, then there is a nonempty set I and an ultrafilter D in I such that the ultrapowers ${}^*A = A^I/D$ and ${}^*B = B^I/D$ are isomorphic.

This proposition was first proven by Keisler using the Generalized Continuum Hypothesis and subsequently reproven by Shelah without need of that assumption.

Using this we get the following which says that if two group rings are elementary equivalent with respect to L_2 then the groups are elementary equivalent with respect to L_0 and the rings are elementary equivalent with respect to L_1 .

Theorem 5.1. Let G and H be groups and R and S be commutative rings with $1 \neq 0$. View the group rings $R[G]$ and $S[H]$ as standard models of T_2 , A and B respectively. If $A \equiv_{L_2} B$, then $G \equiv_{L_0} H$ and $R \equiv_{L_1} S$.

Proof. By Keisler-Shelah there is a nonempty set I and an ultrafilter D on I such that the ultrapowers ${}^*A = A^I/D$ and ${}^*B = B^I/D$ are isomorphic. Let $\varphi: {}^*A \rightarrow {}^*B$ be an isomorphism. Then φ induces a group isomorphism φ_Γ from $\Gamma({}^*A) = {}^*G = G^I/D$ onto $\Gamma({}^*B) = {}^*H = H^I/D$ as well as a ring isomorphism φ_P from $P({}^*A) = {}^*R = R^I/D$ onto $P({}^*B) = {}^*S = S^I/D$: Now,

$G \equiv_{L_0} *G \cong *H \equiv_{L_0} *H \equiv_{L_0} H$. Hence, $G \equiv_{L_0} H$. Similarly, $R \equiv_{L_1} *R \cong *S \equiv_{L_1} S$. Hence $R \equiv_{L_1} S$. \square

In [18] the converse was considered, that is whether or not $G \equiv_{L_0} H$ and $R \equiv_{L_1} S$ imply that $R[G] \equiv_{L_2} S[H]$. The total answer was not settled in that paper but a sufficient condition on group rings was introduced for this to occur (see [18]). This is a finiteness condition since the question is answered affirmatively for finite groups G and H .

6. Summary and Conclusion

In this note we extended the Tarski theorems for groups to the collection of group rings. In particular we examined the relationships between the universal and elementary theories of the corresponding groups and rings and the corresponding universal and elementary theory of the formed group ring. In order to do this we considered three first-order languages with equality whose model classes are respectively groups, rings and group rings. Our main results are that if $R[G]$ is elementarily equivalent to $S[H]$ then simultaneously the group G is elementarily equivalent to the group H and the ring R is elementarily equivalent to the ring S with respect to the appropriate languages. This yields the analog of the basic Tarski theorem for groups, that is, that all integral groups rings $Z[F]$ where F is a nonabelian free groups are elementarily equivalent. Further if G is universally equivalent to a nonabelian free group F and R is universally equivalent to the integers \mathbb{Z} then $R[G]$ is universally equivalent to $\mathbb{Z}[F]$ again with respect to an appropriate language.

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