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Apropos $1+2+3+4+5+\cdots=-\frac{1}{12}$: Mapping Infinity in Light of the Number Circle (or Cycle), in L. Euler's Footsteps and with the Aid of Two Dimensional Infinite Series, and Replacing Negative Infinity and Positive Infinity with Just Infinity

Leo Depuydt

Egyptology and Assyriology, Brown University, Providence, RI, USA
Email: leo_depuydt@brown.edu

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Abstract

The number circle—that is, the notion that the largest possible positive numbers are followed by infinity and then by the smallest possible negative numbers—is not new. L. Euler defended it in the eighteenth century and, before him, J. Wallis considered something vaguely similar. However, in the nineteenth century, the number circle was for the most part abandoned—even if something similar is on occasion accepted in geometry, in the sense that space is circular. The design of the present paper is to present positive proof of the veracity of the number circle and therefore, at the same time, to falsify the number line. Verifying the number circle implies falsifying negative infinity and positive infinity—infinity instead being neither negative nor positive, just like 0. Part of said proof involves showing that infinity can be defined both as $1+1+1+1+1+\cdots$ and as $-1-1-1-1-1-\cdots$ and that the following Equation applies: $1+1+1+1+1+\cdots=-1-1-1-1-1-\cdots$ The principal mathematical technique that will be used to provide said proof is introduced here for the first time. It is called the two dimensional infinite series. It is an infinite series of infinite series. Some additional observations regarding the geography of infinity will be made. A more detailed description of the geography of infinity will be reserved for other papers. The Equation

$1+2+3+4+5+\cdots=-\frac{1}{12}$ is discussed in this paper only to the extent that the attention that has been paid to it has necessitated the construction of a theory of infinity that, upon closer inspection, makes the Equation more self-evident and intuitively apparent; a fuller discussion will take place in a later paper.

Keywords

Euler, L., Infinite Series, Infinite Series of Infinite, Infinity, Infinity, Geography of, Negative Infinity, Invalidity of, Number Circle, Veracity of, Number Cycle, Veracity of, Number Line, Invalidity of, Positive Infinity, Invalidity of, Two dimensional Infinite Series, Ramanujan, Rational Human Intelligence, Wallis, J.

1. Anticipating the End Design of the Present Study in Subsequent Papers: Rendering the Equation $1+2+3+4+5+\cdots=-\frac{1}{12}$ Intuitively Clear

What triggered this study in the first place is the remarkable Equation

$$1+2+3+4+5+\cdots=-\frac{1}{12}.$$

There is no denying that the Equation does not at first sight—or even upon closer inspection—look intuitively clear in the least to anyone who encounters it. Nor did it look intuitively clear to me when I first encountered it. I am in fact quite confident that there has never been anyone to whom the Equation has been intuitively clear at first sight when they first encountered it.

It should be noted right at the outset that the design of the present paper is not to prove the above Equation, as an anonymous reviewer seemed to suggest. That has been done a long time ago, convincingly and definitively (see below). Nor is the design of the present paper to render this very Equation more intuitive at this time. The design is rather to establish necessary mathematical foundations so that this goal can be achieved in later papers. The matter of infinite series is just too wide-ranging. A principal issue concerns the so-called divergent series and whether they do or do not have sums? It is common to believe that they do not have sums. I personally believe that they do.

The total absence of intuitive clarity evidently pertains to the Equation's two most striking characteristics.

1) The sum to the left of the Equation sign has every appearance of adding up to infinity. And yet it does not. How on earth is that possible?

2) The sum to the left of the Equation sign consists of numbers that are all positive. In common experience, a sum of positive numbers always produces a positive number as its result. And yet, in the present case, the result is a negative number. How on earth is that possible?

There was a time, until three or four centuries ago, when negative numbers were disregarded or aroused suspicion. The metaphor most commonly used to persuade anyone of the validity of negative numbers has always been the notion of debt. Money has this peculiar way of peaking human attention.

Someone who is in debt has a negative sum of money, as it were. Accordingly, someone who is 100 dollars in debt has 200 dollars less than someone how has 100 dollars. Let us assume, then, that the numbers in the above Equation refer to dollars. How can a person who endlessly keeps accumulating money end up in debt? It boggles the mind.

To be absolutely clear, the design of what follows is not to prove the veracity of the Equation

$$1+2+3+4+5+\cdots=-\frac{1}{12}.$$

L. Euler and S. Ramanujan have already provided sufficient proof long ago. There is no doubt that the Equation is true.

S. Ramanujan's mathematical powers were absolutely phenomenal. Then again, in the popular imagination, he has become known as the man "who knew infinity". The reason is the appearance of this expression in the title of a best-selling novel about his life and a popular movie with the same title derived from the novel.

I do not subscribe to "the man who knew infinity" as a characterization of S. Ramanujan. S. Ramanujan knew an enormous amount. But he did not "know infinity". He manipulated infinite series in astonishing ways. But he was unaware, I believe, of certain crucial subtleties of engaging the concept of infinity. Engaging the concept of infinity requires a critical appreciation of exactly what the brain can comprehend and what it cannot comprehend. Again, it seems eminently reasonable that the brain can do certain things and not other things. This distinction is critically relevant when it comes to engaging infinity.

Nor is there any need to prove that sums of all positive numbers can have negative results. The above Equation is one example. There are infinitely many examples. Another example is the following Equation:

$$1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\cdots=-2.$$

But that does not take away that anyone studying the Equation might not be tempted to doubt its veracity. And so did the present writer at some point.

One potential advantage of doubting the veracity of the Equation at some point is that one is only much more convinced of its veracity as a result of moving through, and beyond, doubt than when one had never doubted it in the first place. Such has been the experience of the present writer.

The Equation has been widely used in the popular media to astound and perplex the minds of the uninitiated. But these efforts run counter to the aim of mathematics itself, which is to offer as clear and simple an understanding of reality as possible. There is much that remains to be done in regard to the Equation at hand.

In fact, the final design of the present paper is to put an end to the madness by presenting a simple, clear, and natural understanding of the Equation in question. This understanding perhaps does not make the Equation as intuitively clear as the Equation

$$1+1=2\ .$$

But it comes close.

Then again, in order to achieve this pragmatic aim, it will be necessary to address some issues affecting some of the deepest foundations of number theory and mathematics in general.

To conclude, I wish to express my deep gratitude to two anonymous referees. One was unequivocally positive. The other had concerns and I wish to address them. The reviewer in question saw a lack of "genuine" mathematics in the present paper and repeatedly emphasized the need for rigor in mathematics. The two concerns basically come down to the same thing (see below).

But first, it is true that I am strictly speaking not a mathematician and I have no intention of pretending to be. The reason that I am doing mathematics is that I am in need of mathematics to a certain limited degree in order to formulate a complete theory of rational human intelligence. And hardly anything is more fascinating to the cause of rational human intelligence than how the brain deals with the concept of infinity.

The two concerns of the referee in fact come down to a single concern: lack of rigor implying lack of genuine mathematics is what characterizes the present paper. In reply, I should state that I am not very enthusiastic about the nineteenth century of efforts by A.-L. Cauchy and others to bring rigor to mathematics after the age of L. Euler and J.-L. Lagrange. I also cannot appreciate N. Abel's suspicion, yes even abhorrence, of divergent series. In fact, there is clear evidence from N. Abel's writings that he could not deny that divergent series somehow make sense, in spite of everything. And he intended to make sense of it but died too young to ever do anything about it. It is time to catch up with the matter. I will discuss the tortured understanding of divergent series by such luminaries as E. Borel, T. J. Bromwich, and G. H. Hardy in later papers. G. W. Leibniz already believed that is possible to evaluate all divergent series. I think that Euler did too. And so do I, but not necessarily in the capacity of a mathematician. I quickly add my appreciation for the fact that the referee pointed out the significance of the researches of Terrence Tao on the subject.

2. Core Designs Touching on Some of the Deepest Foundations of Number Theory and Mathematics: Elimination of 1) the Number Line and 2) Negative and Positive Infinity

In order to render the Equation

$$1+2+3+4+5+\cdots=-\frac{1}{12}$$

intuitively more transparent, the need will be for first eliminating two interrelated deeply held beliefs of current mathematics and replacing them.

If that seems audacious, the elimination of the beliefs in question involve in some way nothing less than executing the legacy of the most prolific mathematician of all time, Leonhard Euler (1707-1783). In that regard, the present paper is to some extent a vindication of some of L. Euler's deeply held beliefs, beliefs that were dismissed after him. The time has come to rectify the record.

What are the two interrelated beliefs?

The first is that there is something like the number line. It is time to eliminate the number line and replace it with the number circle or cycle.

The second belief is that there is such a thing as negative infinity and positive infinity. It is time to eliminate negative infinity and positive infinity and replace it with just infinity. The contrast between negative and positive is irrelevant when one reaches infinity.

What is the number circle? And how does it relate to the irrelevance and elimination of negative infinity and positive infinity?

The number circle revolves entirely around the notion that numbers follow one another in the following manner:

If one moves all the way up through the positive numbers to infinity and then proceeds beyond infinity, one ends up at the largest negative numbers.

Conversely, if one moves all the way down through the negative numbers to infinity and then proceeds beyond infinity, one ends up at the largest positive numbers.

In sum, there is no negative infinity and there is no positive infinity because it is one and the same infinity that one encounters at the end and is neither negative nor positive. Therefore, the symbols $+\infty$ *and* $-\infty$ *can be eliminated.*

In the geography of numbers, what lies beyond infinity, as it is approached and reached from the positive numbers, is the negative numbers. Evidently, as one proceeds upward through the negative numbers from infinity, one reaches zero and returns to the positive numbers.

The sequence of all the numbers can be represented by a circle: 0 and infinity are located at both ends of one if its diameters in Figure 1.

As one can see, there is no need for negative infinity and positive infinity in this model.

On this circle, there are two (2) points where one crosses from negative numbers to positive numbers or *vice versa*. They are

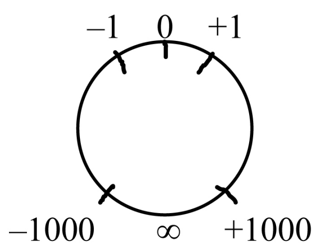

Figure 1. The number circle or cycle.

1) zero (0) and

2) infinity (∞), which is neither negative nor positive.

The sequence of numbers is now universally conceptualized as a line, the so-called number line ending to the left in negative infinity ($-\infty$) and to the right in positive infinity ($+\infty$), with 0 located in the middle. The number line is represented in Figure 2.

By contrast, I am convinced that the two ends are connected at infinity, which is neither negative nor positive. Proof of this fact will be provided. For example it will be shown that

$$1+1+1+1+1+\cdots=-1-1-1-1-1-\cdots$$

The number circle exhibits a remarkable property that the number line does not. If one keeps moving or rotating in one direction, clockwise, the numbers perpetually increase in quantity. If one moves or rotates in the opposite direction, counterclockwise, they perpetually decrease. The continuity is perfect. Perpetual rising occurs in one direction and perpetual descending in another direction.

In addition to the number circle and infinity, it will also be necessary divergent series as a name and as a concept.

3. Excursus: J. Wallis, L. Euler, and the Number Circle

The conception of the sequence of numbers as a number circle rather than as a number line may seem peculiar. And yet, as was said above, I am not the first to accept the veracity of the number circle. L. Euler was absolutely convinced of it. Evidently, I have been deeply influenced and inspired by L. Euler in accepting it myself.

I cannot readily find any reflection of anything like the number circle in more recent mathematics. But the mathematical literature is vast and I have not undertaken a large scale search. The number line and the concepts of negative and positive infinity seem so universally accepted that they would presumably inevitably stand in the way of anyone accepting the conception of the sequence of numbers as a circle rather than as a line. The conception involves completely abandoning the notions of negative infinity and positive infinity.

In order to remove all doubt that L. Euler was firmly convinced that one encounters the negative numbers after rising to infinity through the positive numbers, it will be useful to quote him in full. He wrote about the matter in an article written in Latin published in 1760 in the acts of the Saint Petersburg academy when he was still at the Berlin academy [1]. To simplify the bibliographical trail, I take the liberty of simply quoting the English translation of the Latin original presented by E. J. Barbeau and P. J. Leah in a most helpful article [2].

The following statement by L. Euler leaves no doubt about his firm conviction:

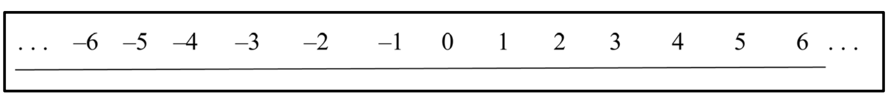

Figure 2. The number line.

However, it seems in accord with the truth if we say that the same quantities which are less than zero can be considered to be greater than infinity. For not only from algebra but also from geometry, we learn that there are two jumps from positive quantities to negative ones, one through nought or zero, the other through infinity, and that quantities whether increasing from zero or decreasing come back on themselves and return to the same destination 0, *so that quantities greater than infinity are thereby less than zero and quantities less than infinity coincide with quantities greater than zero.*

Dixit Euler. But L. Euler does not provide proof at this juncture. I am not sufficiently familiar with L. Euler's massive output to locate a proof anywhere else in his work. I have not undertaken any systematic search. Perhaps someone else might be able locate one somewhere in his works. I therefore believe that I will do nothing superfluous in presenting further below what I believe to be positive proof that one arrives at the negative numbers after rising to infinity through the positive numbers and leave it to speculation what Euler would have thought of the proof or what proofs he himself had in mind.

The question arises: If someone of the caliber of L. Euler, whose great strengths specifically included number theory, was so convinced that one passes from the negative numbers to the positive numbers and *vice versa* at infinity, then why is it so difficult to find any reflections of this conviction in later mathematics?

The answer seems fairly obvious. There is this universally accepted notion about the history of mathematics that a lot of order and rigor was added to mathematics in the nineteenth and early twentieth century. By this notion, up to that time, mathematicians had raced forward headlong making countless discoveries. But they had never stopped to bring their house in order and provide rigorous definitions of all their concepts. By this same notion, there is an impression that L. Euler was sometimes a little methodically loose in how he did mathematics and that nineteenth century mathematicians cleaned up after him, as it were. I do not subscribe to this view.

Notions such as infinity and the infinitesimally small came under special scrutiny in the nineteenth century. The result was the creation of the concept of the limit. I have elsewhere voiced my own opinion about the concept of the limit in calculus [3]. I personally do not see a need for it. I approach the whole matter from an entirely different angle, namely from the angle of rational human intelligence.

This is not the place to get into any detail. Suffice it to note that all my efforts directed towards the study of rational human intelligence are entirely subordinated to the notion that, as with any mechanical tool, there are certain things that the brain can do and there are certain things that the brain cannot do. Where is the line between the two?

In that regard, the concept of the limit involves an imperfect effort to make sense of notions that are simply beyond human intelligence. Limits imply the false implication that it is possible to grasp infinity. But enough about limits here.

In my understanding of infinity, I find myself completely aligned with L. Euler's views, as contrasted with dominant modern views—except in one respect. I am constantly explicitly in search of the very real line that separates what the brain can do from what the brain cannot do. It cannot be entirely excluded that L. Euler believed that a certain level of understanding of mathematical reality transcends the abilities of the brain. But he does not give explicit expression to this distinction. The acceptance of the line and any efforts to try to find it and clearly define it make it easier to accept something that may seem perplexing. It seems so easy to accept that there are things that the brain can do and things that the brain cannot do. But where is the line between the two?

Before proceeding to a proof of the number circle, it is useful to note that Euler was not entirely alone in stumbling upon the negative numbers after rising up to infinity through the positive numbers.

Before L. Euler, John Wallis (1616-1703) had already run into negative numbers when going beyond infinity after passing through the positive numbers. In that regard, he was in my opinion 100% correct. J. Wallis concluded that the negative numbers in question are "greater than infinity (*plusquam infinitam*)" [4].

But J. Wallis never reconciled this notion with the fact that negative numbers can also be viewed as smaller than positive numbers. Who could possibly disagree that −1 is smaller than +1?

Moreover, if there are numbers that are larger than positive infinity, should there not also be numbers that are smaller than negative infinity? Negative numbers are often metaphorically presented as debt, money that is owed. I personally believe that, if one could run up debts that are larger than infinity, one would in fact be impossibly wealthy.

As an indispensable precursor to the works of the mighty I. Newton and so many others, J. Wallis simply does not occupy the place in the history of science that he deserves. Who among the wider public has heard of J. Wallis?

I have not examined the matter in detail, but it is possible that J. Wallis was the first ever to use the representation of the number line, though perhaps only with the positive numbers. This is not the place to investigate the matter in more detail. But it was not R. Descartes who was first.

The most important study of the mathematical work of J. Wallis is, as far as I know, a book by J. F. Scott already referenced in note 4 [5]. I find it symptomatic of the current state of mathematics that J. Wallis's view that one encounters the negative numbers when rising through the positive numbers to infinity and moving beyond is deemed to be an "error" in this book [6]. This is a clear indication that the view propounded in the present article radically departs from any current mathematical orthodoxy. And yet, it is a view endorsed by L. Euler himself.

Consequently, it is all the more important to prove positively that the sequence of numbers does not behave like a line but rather like a circle. Providing proof is what is next.

In addition, various interesting facets of the geography of infinity and of the number circle will be described. It is these facets that will do much to fulfil the main design of the present paper, that is, to make the Equation

$$1+2+3+4+5+\cdots=-\frac{1}{12}$$

much more intuitively clear. These facets will involve more than one new critical insight into the sequence of the numbers as it relates to infinity.

From an intuitive perspective, there is a distinct kind of harmony to the number circle that the number line does not have. As one moves in one direction on the circle, one *always* rises in quantity; in the opposite direction, one *always* descends in quantity.

Throughout what follows, an effort will be made to carefully distinguish between what is accessible to rational human intelligence and what is not. It is not possible, I believe, to attain an adequate understanding of the concept of infinity and the number line without making this distinction. The distinction is otherwise never made in mathematics. But mathematics is something done by the human brain. The distinction between what the brain can do and what it cannot do means everything to mathematics, I believe. Where exactly is the line between what the brain can do and what it cannot do?

4. The Tool Needed 1) to Prove the Veracity of the Number Circle and 2) to Eliminate Negative Infinity and Positive Infinity: Two Dimensional Infinite Series or Infinite Series of Infinite Series

The need and desire is for a single mathematical tool or technique that demonstrates—not only strictly mathematically but, if possible, also fully intuitively—that as one

1) proceeds upward through the positive numbers,

2) continues all the way to infinity,

and

3) then proceeds onward beyond infinity,

one will end up at the largest possible negative numbers and gradually move up through them to zero and rise again through to the positive numbers to infinity. The progression applies in reverse in the opposite direction.

Demonstrating the veracity of such a progression would also eliminate the need for negative infinity and positive infinity.

As was noted above, this progression can be represented as a circle, the number circle. It may seem awkward that one ends up in negative territory at the end of positive territory. The two seem like opposites. But the opposition vanishes if one considers the following remarkable circumstance, also noted above.

If one keeps moving in clockwise direction on the number circle, one never ceases rising in terms of quantity. That is because, when one ends at infinity after traversing the positive numbers, one has done nothing but rising. But as one proceeds through the negative numbers after having moved beyond infinity, one just keeps rising.

That is the beauty and consistency of the number circle: the infinite progression of always rising in one direction and the infinite progression of always descending in the opposite direction. It makes for a certain kind of remarkable harmony.

But is there a mathematical technique to demonstrate this infinite progression and eliminate the need for negative infinity and positive infinity.

Such a mathematical technique exists and is introduced below. I call it the *two dimensional infinite series*.

The infinite series is one of the most important mathematical discoveries of all time, dating to the seventeenth century with traces of its use going all the way back to Archimedes and antiquity. In fact, the sum featuring in the title of the present article is an infinite series.

It is not easy to impress Euler when it comes to mathematical inventions. But he was mightily impressed by the potential of infinite series and in a rare expression of genuine admiration for a mathematical phenomenon states the following [7]:

It should be… observed, that, from this branch of mathematics [that is, infinite series] *inventions of the utmost importance have been derived*; *on which account the subject deserves to be studied with the greatest attention.*

It is infinite series that make it possible to prove the veracity of the number circle. But one has to deploy them in not one, as has been done until now, but in two dimensions.

A two dimensional infinite series is an infinite series of infinite series.

Infinite series in one dimension is one thing. Taking them into two dimensions is another. How does one take infinite series into two dimensions? The matter is discussed and clarified next.

In sum, finite series have been exploited with enormous success in all kinds of contexts in the past three to four centuries or so. Euler thought the world of them. However, they have been used in only one dimension. It is time to use them into a second dimension to perform the critically important function of proving the existence and veracity of the number circle and eliminate negative infinity and positive infinity once and for all.

5. The One Dimensional Infinite Series

A well-known example of a one dimensional infinite series is as follows:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots. \quad (1)$$

In fact, all the infinite series that have ever been studied by mathematicians are one dimensional. What a two dimensional infinite series looks like is clarified below.

The present paper is strictly limited to the contemplation of infinite series that are *equal to a rational number*. The reason is simply that this type of infinite series amply provides what is needed for the designs of the present paper, which is

1) to prove the existence of the number circle or cycle and

2) eliminate negative infinity and positive infinity in favor of infinity *tout court*,

all this with the ulterior design of making the Equation

$$1+2+3+4+5+\cdots=-\frac{1}{12}$$

seem like the most normal thing on earth—or close to it.

There is so much more to infinite series than the little that is exploited here. But none of is needed for the purpose at hand.

The infinite series (1) above is equal to a rational number. The number is 2.

It is not all that difficult to establish that the infinite series approaches ever closer to 2 as one keeps adding terms. It is also intuitively clear that one can always keep adding terms. One can always double 16, 32, 64, and so on. There is no reason to ever stop.

What is not intuitively clear is that the result of the sum is actually 2 *if one never stops*, or if one stops after having gone on forever as it were. In other words, the following Equation applies:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=2. \qquad (2)$$

Here is where the line is located between what the brain can do and what it cannot do. It can easily imagine never stopping. Why stop if you can just keep going on? But arriving somewhere specific, like at the number 2, and yet going on forever, that is what the brain cannot imagine. As the brain experiences reality, arriving somewhere always involves the end of a journey and stopping. But in the case of the infinite series cited above, one arrives somewhere and kind of stops without the journey ever coming to an end. How can one arrive somewhere if the journey keeps on going?

In any event, in spite of the inability of the brain to wrap itself around the notion of infinity, no one has ever doubted the veracity of Equation (2).

The veracity of Equation (2) can easily be established inductively by adding up a sufficiently large number of terms and seeing the instances of the number 9 accumulating as the number takes on a form such as 1.9999… The absence of any reduction in the instances of the number 9 makes it clear where this is going. Clearly, it is going to 2, even it is impossible to get there by keeping adding terms. It would evidently be possible to get to 2 if one could keep adding forever. That is obvious, even if it is not possible to understand what it means to keep adding up forever.

6. One Way of Proving or Verifying Infinite Series Equal Rational Numbers: Algebra

As is well-known, there are ways other than induction to verify, or prove, that the sum of Equation (2) is 2, at least two.

A first way is algebraic, namely by means of multiplying and subtracting Equations. It begins by assuming that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=x. \quad (3)$$

If one multiplies this Equation by 2, then the result is as follows:

$$2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=2x. \quad (4)$$

Evidently, if one subtracts the right-hand side of Equation (3) from the right-hand side of Equation (4), the result is as follows:

$$2x-x=x.$$

Moreover, if one subtracts the left-hand side of Equation (3) from the left-hand side of Equation (4), one obtains 2. Consequently, x equals 2. QED.

This technique makes it possible to control, and feel confident about, the exact value of the sum of an infinite series. The technique cannot be applied to all infinite series.

For completeness' sake, I mention here another obvious way of making clear that Equation (1) equals 2. It was noted that the Equation gradually takes on the form 1.9999… The focus is on the component 0.9999… Consider the following sequence:

$$\frac{1}{9}=0.1111\cdots;$$

$$\frac{2}{9}=0.2222\cdots;$$

$$\cdots$$

$$\frac{7}{9}=0.7777\cdots;$$

$$\frac{8}{9}=0.8888\cdots;$$

$$\frac{9}{9}=0.9999\cdots=1.$$

The sequence suggests an equivalence between 0.9999… and 1 and therefore between 1.9999… and 2.

7. Converting Rational Numbers into One Dimensional Infinite Series

Another way of proving or verifying that an infinite series is indeed equal to an exact number is by converting a rational number into an infinite series. The resulting infinite series is obviously equal to a rational number because it was derived from it.

It so happens that the technique of converting a rational number into one or more infinite series will play an absolutely critical role below in proving the veracity of the number circle and finally eliminating negative infinity and positive infinity once and for all from mathematics.

The theory of infinite series has been enormously successful in both mathematics and physics, as was already noted above. There is so much more to the theory of infinite series than what is taken from it here, including the resolution of irrational numbers into infinite series involving I. Newton's binomial theorem. L. Euler really ran with infinite series for the benefit of physics.

For the present purposes, infinite series that can be expressed by rational numbers will suffice to prove the veracity of the number circle and the futility of the concepts of negative infinity and positive infinity. There will be no need for infinite series featuring irrational or even transcendental results.

8. A Well-Known Way of Converting Rational Numbers into One Dimensional Infinite Series: Infinite Decimal Fractions

There is in fact more than one way of converting a rational number into an infinite series. The best known way has been practiced by all who have even just some notion of elementary division.

An example. It is obvious that dividing 2 by 3 results in 0.6666… This is high school, if not elementary school, arithmetic.

Still, this expression by itself does not readily encourage one to contemplate infinite series. And yet, the following Equation applies:

$$\frac{2}{3}=\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}+\cdots.$$

Clearly, a rational number is resolved into an infinite series. The series is what L. Euler called "infinite decimal fractions". He describes "infinite decimal fractions" with his usual unsurpassed lucidity in his *Elements of Algebra* [8].

L. Euler does not make a connection between "infinite decimal fractions" and what he describes elsewhere in his *Elements of Algebra* as "the resolutions of fractions into infinite series" [9]. I will make the connection explicitly further below.

If one carries on the right-hand side of the Equation *without ever stopping*, one will arrive at the rational number found at the left-hand side of the Equation. It is safe to assume that elementary school students or high school students are hardly ever invited to reflect on what it means to go on forever without stopping and yet getting somewhere.

9. Other Ways of Representing Rational Numbers by Means of One Dimensional Infinite Series

There is an infinite number of ways of representing rational numbers, including all the rational fractions, by means of infinite series that consist themselves of rational numbers. As it happens, only one specific member of this infinite set will be needed to prove the veracity of the number circle and to eliminate negative infinity and positive infinity. Nor will it be necessary to consider anything else but rational numbers in the present paper.

L. Euler presents a lucid and delightfully concise description of this phenomenon in the chapter entitled "the resolution of fractions into infinite series" in his *Elements of Algebra*, which has been described as the second most popular mathematics book of all time, after Euclid's *Elements* [10]. It has much inspired the following account. Then again, the following account will approach the phenomenon from a different angle in order to place it in wider context and to clarify what exactly happens when one turns any rational number into an infinite number of infinite series. I believe, in fact, that L. Euler did not capture the phenomenon in its entire scope from first principles in the afore-mentioned account.

It should be noted that any rational number can be resolved into an infinite series, not only fractions, but also round numbers such as 1, 2, 3, 4, etc. But according to the technique described by Euler and discovered before him, the round number needs to be represented as a fraction. For example, the integer 2 needs to be in a fractional form such as

$$\frac{1}{1-\frac{1}{2}}\left(=\frac{1}{\frac{1}{2}}=2\right)$$

in order to be converted into an infinite series.

In the previous section, the fraction $\frac{2}{3}$ was converted into the infinite series

$$\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}+\cdots,$$

involving infinite decimal fractions.

In infinite series involving decimal fractions, the numerators evidently do not all need to be the same. For example,

$$\frac{1}{7}=\frac{1}{10}+\frac{4}{100}+\frac{2}{1000}+\frac{8}{10000}+\frac{5}{100000}+\frac{7}{1000000}+\cdots,$$

1, 4, 2, 8, 5, and 7 then repeat themselves in the numerator. Or

$$\frac{1}{7}=0.\overline{142857}.$$

Infinite series involving decimal fractions result from dividing the numerator directly by the nominator according to the standard basic rules of long division in the decimal notation system. However, there are other ways of converting a rational fraction such as $\frac{2}{3}$ into infinite series, an infinite number of other ways in fact.

In order to find other series, one needs to represent $\frac{2}{3}$ into equivalent but different fractional forms. Four such fractional forms are as follows:

$$\frac{1}{1+\frac{1}{2}}, \tag{5}$$

$$\frac{1}{2-\frac{1}{2}}, \tag{6}$$

$$\frac{2}{1+2}, \tag{7}$$

and

$$\frac{2}{2+1}. \tag{8}$$

Expression (5) yields the infinite sum

$$\frac{2}{3}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}+\cdots. \tag{5'}$$

Expression (6) yields the infinite sum

$$\frac{2}{3}=\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\frac{1}{512}+\cdots. \tag{6'}$$

Expression (7) yields the infinite sum

$$\frac{2}{3}=2-4+8-16+32-\cdots. \tag{7'}$$

Expression (8) yields the infinite sum

$$\frac{2}{3}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots. \tag{8'}$$

The infinite sum in (7') looks peculiar at first sight. A sum of integers produces a fraction. However, it will be shown below that this is altogether expected. After all, the sum featured in this paper's title is also a sum of integers resulting in a fraction, and a negative one at that.

How are the infinite sums (5'), (6'), (7'), and (8') obtained? L. Euler explains the matter clearly.

Generalized algebraic equivalents of the arithmetic fractions (5), 6), (7), and (8) are as follows:

$$\frac{1}{1+a}, \tag{5''}$$

$$\frac{1}{2-a}, \tag{6''}$$

$$\frac{2}{1+a}, \tag{7''}$$

and

$$\frac{2}{2+a}. \tag{8''}$$

For the purposes of the present paper, only fraction

$$\frac{1}{1+a} \tag{5''}$$

and its negative counterpart

$$\frac{1}{1-a}$$

will be needed.

L. Euler discusses fraction (5'') and its negative counterpart separately. But in what follows, it will be crucial to reduce the two to just one single fraction either by taking (5'') as

$$\frac{1}{1-(-a)}$$

or by taking its negated counterpart as

$$\frac{1}{1+(-a)}.$$

The reason is that it will be important to observe what happens when (−)*a* assumes the form of any of the integers, negative or positive, in a continuous series.

Again, there are an infinite number of fractions like (5''). But only (5'') and its negated counterpart will be needed in order to prove what needs to be proven.

The fractions (5'), (6'), (7'), and (8') can be converted into infinite sums as follows:

$$\frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 - \cdots; \tag{5'''}$$

$$\frac{1}{2-a} = \frac{1}{2} + \frac{a}{4} + \frac{a^2}{8} + \frac{a^3}{16} + \frac{a^4}{32} + \cdots; \tag{6'''}$$

$$\frac{2}{1+a} = 2 - 2a + 2a^2 - 2a^3 + 2a^4 - \cdots; \tag{7'''}$$

and

$$\frac{2}{2+a} = 1 - \frac{a}{2} + \frac{a^2}{4} - \frac{a^3}{8} + \frac{a^4}{16} - \cdots; \tag{8'''}$$

The matter is clarified further below.

Incidentally, the negative counterpart of fraction (5) yields the following result:

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + a^4 + \cdots$$

The next step is to evaluate the four infinite sums if *a* has the values listed in fractions (5), (6), (7), and (8). These values of *a* are as follows.

Fraction (5): $a = \frac{1}{2}$;

fraction (6): $a = \frac{1}{2}$;

fraction (7): $a = 2$; and

fraction (8): $a = 1$.

Inserting them into Equations (5'''), (6'''), (7'''), and (8''') yields the following results:

$$\frac{2}{3} = \frac{1}{1+\frac{1}{2}} = 1 - \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 - \cdots = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots; \quad (5'''')$$

$$\frac{2}{3} = \frac{1}{2-\frac{1}{2}} = \frac{1}{2} + \frac{\frac{1}{2}}{4} + \frac{\left(\frac{1}{2}\right)^2}{8} + \frac{\left(\frac{1}{2}\right)^3}{16} + \frac{\left(\frac{1}{2}\right)^4}{32} + \cdots = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \frac{1}{256} + \cdots; \quad (6'''')$$

$$\frac{2}{3} = \frac{2}{1+2} = 2 - 2\times 2 + 2\times 2^2 - 2\times 2^3 + 2\times 2^4 - \cdots = 2 - 4 + 8 - 16 + 32 - \cdots; \quad (7'''')$$

and

$$\frac{2}{3} = \frac{2}{2+1} = 1 - \frac{1}{2} + \frac{(1)^2}{4} - \frac{(1)^3}{8} + \frac{(1)^4}{16} - \cdots = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots. \quad (8'''')$$

Again, Equation (7'''') may at first sight seem peculiar. A sum of integers has a fraction as its result. If one keeps adding up integers, how can one end up with a fraction? This result is in fact normal, all things considered. After all, in the Equation featured in the title of the present article, the sum of all the natural numbers is also a fraction. How is this possible? Upon closer inspection, it will appear that the result is hardly unexpected, however peculiar it may seem at first sight.

All this leaves Equations (5''''), (6''''), (7''''), and (8'''') without an explanation. How are they obtained?

The manner of dividing a fraction such as

$$\frac{1}{1+a},$$

in which the divider does not divide the dividend, was presumably discovered some time in the seventeenth century, not too long before Euler. I have not engaged in any detailed historical investigations as to its origins.

How does one divide 1 by $1-a$? In describing the matter, I will be much more explicit than is usual in a mathematics article, even more explicit than L. Euler's description of the procedure, in order to make the proof of the number circle and the need for eliminating negative and positive infinity as transparent as possible.

$\frac{1}{1+a}$ is an algebraic expression. For the sake of simplicity, I prefer to reason at first arithmetically in an inductive manner, and then move from arithmetic specificity to algebraic generality. I believe that, what an arithmetic argument loses in sophistication or the like (in comparison to an algebraic argument), it much gains in clarity. It should not be the case that only mathematicians can read mathematics. This was not the case in the time of the great Euler and the great Lagrange.

The general context of the present argument is division.

If the dividend is divisible by the divisor, matters are quite simple. The quotient is an integer. An example is as follows:

$$\frac{10}{5} = 2.$$

But there are other ways of making the same division, with an infinite number of infinite sums as the result. L. Euler describes the phenomenon as the "resolution of fractions into infinite series" and he applies it "[w]hen the dividend is not divisible by the divisor" [9]. But this description does not cast the net wide enough. Any rational number can be converted into an infinite sum. That includes all the rational fractions. And it includes even those fractions in which the divisor is divisible by the dividend, such as

$$\frac{10}{5}.$$

I exclude infinite sums involving irrational numbers and transcendental numbers from the present discussion because they are not needed to meet the core designs of the present article, which are to prove the existence of the number circle, to eliminate negative infinity and positive infinity from mathematics, and to achieve a better appreciation of the nature of what is called divergent series.

I believe that I cannot do anything better than—and will not do anything superfluous by—developing the eminently simple fraction

$$\frac{10}{5} = 2.$$

into an infinite sum and demonstrating that this simple fraction can be converted into an infinite number of infinite sums. By not taking the conversion of numbers into infinite series down to this most elementary level, L. Euler may have missed an opportunity to evidence the truly vast scope of the phenomenon (even if he attached the highest importance to the theory of infinite series and developed it in unsurpassed fashion). Infinity is everywhere! Presenting the phenomenon in the following way might make it easier to introduce infinite series earlier into the mathematical curriculum.

How does one turn the above fraction into an infinite sum?

The most fundamental principle that underlies the development of rational numbers into an infinite number of infinite sums, not quite identified or articulated as such by L. Euler or by anyone else as far as I know, is

to divide the dividend by only PART of the divisor.

Consider again the fraction

$$\frac{10}{5} = 2.$$

What would happen if one divided the divisor 5 into two partial divisors 3 and

2 and divided the dividend 10 only by the first partial divisor 5, namely 3? The division can be presented as follows, in line with how divisions are always represented:

$$\begin{array}{r|l} 10 & 3+2 \\ \hline & x \end{array}$$

In divisions, the quotient multiplied by the divisor yields the dividend. If the dividend is 10 and the divisor is 3, then the quotient is x in the Equation

$$x \times 3 = 10.$$

The result is evidently as follows:

$$x = \frac{10}{3}.$$

Consequently, the division by just 3 instead of all of 5 may now be presented as follows:

$$\begin{array}{r|l} 10 & 3+2 \\ \hline & \frac{10}{3} \end{array}$$

The next step is to multiply the quotient with the divisor to check the resulting dividend. The result is 10, as follows.

$$\frac{10}{3} \times 3 = 10.$$

The division can now be presented as follows:

$$\begin{array}{r|l} 10 & 3+2 \\ \hline 10 & \frac{10}{3} \\ - & \\ \hline 0 & \end{array}$$

The remainder is 0. However, the actual divisor is not 3 but 5; 3 is only the partial divisor. Therefore, whereas it is possible to divide the dividend by a partial divisor and obtain a result that multiplied with the divisor yields the dividend, one still ought to multiply the quotient by the full divisor in order to evaluate how the resulting dividend differs from the actual dividend.

$$\begin{array}{l|l} 10 & 3+2 \\ \hline 10+\frac{20}{3} & \frac{10}{3} \\ - & \\ \hline 0-\frac{20}{3} & \end{array}$$

The remainder is $-\frac{20}{3}$. As a rule, in divisions, the final quotient is obtained by dividing the remainder by the divisor and adding the result, which is in this case negative, to the temporary quotient. The result of dividing the remainder by the divisor is as follows:

$$-\frac{\frac{20}{3}}{3+2}.$$

The division now looks as follows:

$$\begin{array}{l|l}
10 & 3+2 \\
\hline
10+\dfrac{20}{3} & \dfrac{10}{3}-\dfrac{\frac{20}{3}}{3+2} \\
- & \\
\hline
0-\dfrac{20}{3} &
\end{array}$$

A more detailed description of how $-\frac{\frac{20}{3}}{3+2}$ is obtained is as follows. In order to obtain the dividend of a division, one multiplies the quotient by the divisor. The above division is by 5. In other words, the divisor is 5. However, since the initial quotient $\frac{10}{3}$ was obtained by division by 3, multiplying the quotient $\frac{10}{3}$ by the divisor 5 does not yield the correct dividend 10. Still, eliminating the initial quotient $\frac{10}{3}$ is not an option, since dividing by part of the divisor, namely by 3, is the whole intent of the operation. Consequently, the initial quotient $\frac{10}{3}$ needs to be modified by an additional term. How is this additional term determined?

One first multiplies the initial and incomplete quotient $\frac{20}{3}$ by the full divisor 5. The result is an incorrect dividend, namely $\frac{50}{3}$. This result is too large by $\frac{20}{3}$. It is obvious that the correct dividend should be exactly 5 times larger than the correct quotient because the divisor is 5. It is also obvious that the incorrect dividend $\frac{50}{3}$ is 5 times larger than the incomplete quotient $\frac{10}{3}$ because it was obtained by multiplying the latter by 5.

It follows that the difference between the incorrect dividend and the correct dividend, namely the quotient $\frac{20}{3}$, will be 5 times larger than the difference between the incomplete quotient and the final quotient. Therefore, $\frac{\frac{20}{3}}{5}$ needs to be subtracted from the incomplete quotient to obtain the final quotient.

At this point, there are three possibilities: 1) one can stop; or 2) one can go on dividing by the partial divisor and then stop; or 3) one can go on forever.

1) If one stops, the quotient is

$$\frac{10}{3}-\frac{\frac{20}{3}}{3+2}=\frac{10}{3}-\frac{20}{3\times(3+2)}=\frac{10}{3}-\frac{20}{15}=\frac{10}{3}-\frac{4}{3}=\frac{6}{3}=2.$$

The result is 2, as it should be.

2) Suppose that one goes on with one additional division and then stops. The need is for dividing $\frac{20}{3}$. If one divides it by 3 + 2 or 5, the result is $\frac{4}{3}$ (see 1) above) and one reaches an endpoint. If one divides again by the partial divisor 3, then the result is as follows:

$$\begin{array}{l|l} \frac{20}{3} & 3+2 \\ \hline \frac{20}{3}+\frac{40}{9} & \frac{20}{9}-\frac{\frac{40}{9}}{3+2} \\ - & \\ \hline 0-\frac{40}{9} & \end{array}$$

If one inserts this result for $\frac{\frac{20}{3}}{3+2}$ into the quotient on page 94 top above, the resulting quotient is as follows:

$$\frac{10}{3}-\left(\frac{20}{9}-\frac{\frac{40}{9}}{3+2}\right)=\frac{10}{3}-\frac{20}{9}+\frac{\frac{40}{9}}{3+2}.$$

If one stops here, the result is again 2, as follows:

$$\frac{10}{3}-\frac{20}{9}+\frac{\frac{40}{9}}{3+2}=\frac{30}{9}-\frac{20}{9}+\frac{40}{45}=\frac{10}{9}+\frac{8}{9}=\frac{18}{9}=2.$$

One can keep dividing the remainder in this fashion as long as one wants. Whenever one stops, the result will be 2.

3) There is also the possibility of *never* stopping dividing by the partial divisor 3. The result is the following infinite sum:

$$\frac{10}{3}-\frac{20}{9}+\frac{40}{27}-\frac{80}{81}+\frac{160}{243}-\cdots=2.$$

I agree with Euler that, if one does not stop, there is no need for a final fraction. In this connection, L. Euler makes the following statements, which appear to have been universally misunderstood or disregarded:

Were we to continue the series without intermission, the fraction would be no longer considered; but, in that case, the series would still go on. [11]

For once a series is said to be continued to infinity, it is contrary to [the afore-mentioned] *idea if some term of the same series is thought of as last even if* *it is infinitesimal. Therefore, the above-noted objection concerning the addition or subtraction of a remainder after the ultimate term disappears of its own accord. Since, therefore, we never reach the end of an infinite series, we never get besides to such a place where it is necessary to add that remainder; accordingly, this same remainder not only can be neglected, but also should be, because nowhere is a place for it found.* [12]

How misunderstood or disregarded? A footnote is added to L. Euler's *Elements*, probably by the French translator, and directly contradicts the first statement quoted above.

It may be observed, that no infinite series is in reality equal to the fraction from which it is derived, unless the remainder is considered. [13]

The footnote is not in the original German edition of 1770.

Seeking corroboration, the footnote then refers to the paragraph in which L. Euler makes the first statement quoted above. But L. Euler there states exactly the opposite (see above).

It was noted above that there are an infinite number of ways of converting the fraction

$$\frac{10}{5}$$

into infinite sums.

Some of those infinite sums are as follows, all obtained by the method described above:

$$\frac{10}{4+1}=\frac{10}{4}-\frac{10}{16}+\frac{10}{64}-\frac{10}{256}+\frac{10}{1024}-\cdots;$$

$$\frac{10}{2+3}=\frac{10}{2}-\frac{30}{4}+\frac{90}{8}-\frac{270}{16}+\frac{810}{32}-\cdots;$$

and

$$\frac{10}{1+4}=2=10-40+160-640+2560-\cdots$$

This last Equation provides an interesting definition of the number 2.

The number of ways of dividing a positive integer into two positive integers is finite and the subject of indeterminate algebra. In the case of 5, there are only two ways: 1 + 4 and 2 + 3. But since order matters presently, there are four ways: 1 + 4, 2 + 3, 3 + 2, and 4 + 1. All are converted into infinite series above.

It is not necessary to divide the divisor into a sum of two positive integers. One can also use negative integers. There are an infinite number of ways of dividing a divisor in this manner. Two examples may suffice:

$$\frac{10}{6-1}=2=\frac{10}{6}+\frac{10}{36}+\frac{10}{216}+\frac{10}{648}+\frac{10}{1952}+\cdots;$$

$$\frac{10}{-1+6}=2=-10-60-360-2160-12960-\cdots$$

The second example looks peculiar at first sight. But the result makes perfect sense in a wider context. More on this follows below.

Nor is it necessary to divide the divisor into integers. It may also divided into rational fractions. Again, the ways of division are infinite in number. The following example close to $\frac{10}{3+2}$ may suffice:

$$\frac{10}{\frac{31}{10}+\frac{19}{10}}=\frac{10}{4}-\frac{10}{16}+\frac{10}{64}-\frac{10}{256}+\frac{10}{1024}-\cdots$$

And so on into infinity. I refrain from pursuing irrational and transcendental numbers.

In his chapter on the resolution of fractions into infinite series, L. Euler does not use arithmetical examples as has been done above in the present paper. He uses algebraic examples. Except for the number 1, all numbers are represented by symbols such as *a*, *b*, *c*, and *x*.

Much of his chapter is taken up by what are the two most elementary cases, which he styles as follows:

$$\frac{1}{1-a}$$

and

$$\frac{1}{1+a}.$$

Indeed, in any scientific endeavor, it is much preferable to move from the simple to the more complex. Beginning with the *most* simple is therefore by far the preferred *modus operandi*.

L. Euler does not use the notations

$$\frac{1}{1-x}$$

and

$$\frac{1}{1+x}.$$

Indeed, the symbols *x* and *y* are used for entities whose value is being sought, as derived from an Equation. For example, if

$$ax^2=1,$$

then what is *x*? The answer is

$$x=\frac{\sqrt{a}}{a}.$$

Then again, in the exercises at the end of L. Euler's chapter added by someone else and describing the resolution of fractions into infinite series, the student is asked to resolve the fractions

$$\frac{ax}{a-x},\frac{b}{a+x},\frac{a^2}{x+b},\frac{1+x}{1-x},\ \text{and}\ \frac{a^2}{(a+x)^2}$$

into infinite sums [13]. The following representations would have been preferable:

$$\frac{ab}{a-b}, \frac{a}{b+c}, \frac{a}{b+c}, \frac{1+a}{1-a}, \quad \text{and} \quad \frac{a^2}{(a+b)^2}.$$

The gist of L. Euler's description is that alternative division is still possible whenever ordinary division is not because the dividend cannot be divided by the divisor. L. Euler focuses exclusively on algebraic expressions such as

$$\frac{1}{1-a}.$$

There are three facts to which L. Euler's account does not draw the attention.

First, one should not forget that alternative division is also always possible when ordinary division is as well. Instances have been adduced above.

Second, one should not forget that not only algebraic but also purely arithmetic fractions that cannot be further divided by a divisor can be subjected to alternative division. An example is

$$\frac{1}{5}.$$

As was noted above, a fraction in which the dividend cannot be divided by the divisor can still be divided by part of the divisor, with an infinite sum as the result. The fraction above can be represented as

$$\frac{1}{2+3}.$$

Division by 2 results in the following infinite sum:

$$1-\frac{3}{4}+\frac{9}{8}-\frac{27}{16}+\frac{81}{32}-\cdots.$$

Third, one should not forget arithmetic fractions can be subdivided in an infinite number of ways. Examples have been adduced above.

Fourth, one should not forget that all rational numbers, that means also integers, and not only rational fractions, can be divided in an infinite number of ways. Quite a few conversions of 2 into infinite sums have been illustrated above.

It will not be superfluous to describe in more detail how one divides the dividend by the divisor in the expression

$$\frac{1}{1-a}.$$

L. Euler does not explicitly state that the key concept is to divide a dividend by part, and not all, of a divider. Instead, he simply states that the indivisibility of the dividend by the divisor

does not prevent us from attempting the division according to the rules that have been given, nor from continuing it as far as we please, and we shall not fail thus to find the true quotient, though under different forms. [9]

It might have aided the cause of clarity if the technique had been described more explicitly. I believe that using arithmetical examples, as has been done, also promotes clarity.

If the design is to divide the divisor by *part* of the divisor in the expression

$$\frac{1}{1-a},$$

then there are evidently two options:

1) dividing by just 1;

2) dividing by just $-a$.

If just 1 is the divisor, the division may be presented as follows.

$$\begin{array}{c|c} 1 & 1-a \\ \hline & \end{array}$$

If $-a$ is the divisor, then the division may be presented as follows.

$$\begin{array}{c|c} 1 & -a+1 \\ \hline & \end{array}$$

First 1 as divisor. The first step is to find the quotient that, if multiplied by the divisor 1, yields the dividend 1. In other words, the quotient is x in the Equation

$$x \times 1 = 1.$$

The quotient is evidently 1. The result is as follows:

$$\begin{array}{c|c} 1 & 1-a \\ \hline & 1 \end{array}$$

The next step is to multiply the initial quotient by the divisor, with an incorrect dividend as the result because an initial and incomplete quotient has been used. One next subtracts the resulting incorrect dividend from the correct dividend in order to determine the difference between the two, as follows.

$$\begin{array}{c|c} 1 & 1-a \\ \hline 1-a & 1 \\ - & \\ \hline 0+a & \end{array}$$

The difference between the correct dividend and the incorrect dividend is $+a$. The difference between the initial temporary coefficient and the final correct coefficient is therefore that very difference of $+a$ divided by the divisor, or $\frac{a}{1-a}$. The matter has been described above in arithmetical fashion in relation to the division of 10 by 5 by the mediation of the partial divisor 3. All it takes is to add $\frac{a}{1-a}$ to the coefficient and the correct coefficient is obtained, as follows:

$$\begin{array}{c|c} 1 & 1-a \\ \hline 1-a & 1+\frac{a}{1-a} \\ - & \\ \hline 0+a & \end{array}$$

To leave no stone unturned and be as explicit as possible, it may be verified that multiplying the coefficient by the divisor indeed produces the dividend. The multiplication of the two is as follows:

$$\left(1+\frac{a}{1-a}\right)(1-a).$$

And therefore also as follows:

$$1(1-a)+\frac{a(1-a)}{1-a}.$$

And also as follows:

$$1-a+a=1.$$

In developing sums from a fraction in which the dividend cannot be divided by the divisor one can keep going and stop at any time. But one can also go on forever. In all cases, the result is the same.

Taking the division one step further will need to suffice here. It means to resolve the division

$$\frac{a}{1-a},$$

which can also be presented as follows:

$$\begin{array}{c|c} a & 1-a \\ \hline & \end{array}$$

I refrain from explaining every single step in detail. Suffice it to note that the operation begins with dividing the dividend by part of the divider, in this case 1. The quotient that, if multiplied by 1, yields *a* is evidently *a*. The division may therefore be presented as follows:

$$\begin{array}{c|c} a & 1-a \\ \hline & a \end{array}$$

This expression is then readily developed into the following:

$$\begin{array}{c|c} a & 1-a \\ \hline a-a^2 & a^2+\dfrac{a^2}{1-a} \\ - & \\ \hline 0+a^2 & \end{array}$$

The division of 1 by 1 − *a* produced the result

$$1+\frac{a}{1-a}. \tag{9}$$

The division of *a* by 1 − *a* produced the result

$$a+\frac{a^2}{1-a}. \tag{10}$$

Inserting (10) into (9) yields the following result:

$$1+a+\frac{a^2}{1-a}.$$

It is quite justified to generalize the division as follows if the design is to stop it at any time:

$$1+a+a^2+a^3+\cdots+\frac{a^n}{1-a}.$$

If there is no desire to stop it and the division is carried on into all infinity, then the result is as follows:

$$1+a+a^2+a^3+\cdots.$$

In dividing 1 by part of the divisor 1 − *a*, the component 1 has been chosen above. But it is also possible to choose the component $-a$. The division may then conveniently be represented as follows:

$$\begin{array}{c|c} 1 & -a+1 \\ \hline & \end{array}$$

All it takes at this point is to ask what needs to be multiplied by $-a$ to yield 1. The answer is evidently $-\frac{1}{a}$, as follows:

$$\begin{array}{c|c} 1 & -a+1 \\ \hline & -\frac{1}{a} \end{array}$$

I refrain from detailing all the steps. The resulting infinite sum is as follows:

$$-\frac{1}{a}-\frac{1}{a^2}-\frac{1}{a^3}-\frac{1}{a^4}-\frac{1}{a^5}-\cdots.$$

10. Taking Infinity into a Second Dimension: Two Dimensional Infinite Series

Rational fractions can be resolved into infinite sums. And some of these infinite sums can be converted into an infinite series in a second, additional, dimension.

Consider, for example, the expression $\frac{1}{1-a}$. One way of converting this fraction into an infinite sum is as follows:

$$\frac{1}{1-a}=1+a+a^2+a^3+a^4+\cdots.$$

Let us assume that

$$a=\frac{1}{2}.$$

The result is as follows:

$$2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots.$$

It is possible, however, to expand this Equation into infinity in a second dimension by progressively changing the value of *a* in the following manner:

$$a=\frac{1}{2};$$

$$a=\frac{1}{2^2}=\frac{1}{4};$$

$$a=\frac{1}{2^3}=\frac{1}{8};$$

$$a=\frac{1}{2^4}=\frac{1}{16};$$

$$a=\frac{1}{2^5}=\frac{1}{32};$$

and so on.

Accordingly, the value of

$$\frac{1}{1-a}$$

successively becomes as follows:

$$\frac{1}{1-\frac{1}{2}}=2;$$

$$\frac{1}{1-\frac{1}{4}}=\frac{4}{3};$$

$$\frac{1}{1-\frac{1}{8}}=\frac{8}{7};$$

$$\frac{1}{1-\frac{1}{16}}=\frac{16}{15};$$

$$\frac{1}{1-\frac{1}{32}}=\frac{32}{31};$$

and so on.

Let us see where this is going. Take for example

$$\frac{1}{1-\frac{1}{2^{20}}}=\frac{1}{1-\frac{1}{1048576}}=\frac{1048576}{1048575}.$$

It is easy to see that the value is tending towards 1. The fact is that x in the expression

$$\frac{1}{1-\frac{1}{x}}$$

is becoming ever larger.

In the end, the expression becomes

$$\frac{1}{1-\frac{1}{\infty}}.$$

It is now common in mathematics to describe the expression

$$\frac{1}{\infty}$$

as undefined. But L. Euler had no difficulty in simply declaring

$$\frac{1}{\infty}=0. \tag{11}$$

I definitely side with L. Euler. And I do so by bringing into the picture the critical dimension of rational human intelligence. After L. Euler, in the nineteenth century, mathematicians were uncomfortable with never stopping when it comes to making x in Equation (11) bigger. This discomfort implies the assumption that everything should be intelligible to human reason. I firmly believe that it is not and that it is imperative to establish where the line is located between what is intelligible and what is not. In other words, I do not accept the notion of "undefined" in mathematics. The discussion of "undefined" needs to be taken to a whole different level, that of rational human intelligence. The critical distinction ought not to be between what is defined and what is not defined. It ought to be between what is accessible to rational human intelligence and what is not accessible.

Where to begin? All mathematicians accept that, if one carries on the infinite sum

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots,$$

one will end at 2. This acceptance implies acceptance of something else, at least two matters, that is as a rule left unsaid. First, nobody ever mentions stopping. The unspoken implication seems to be that one goes on forever. Second, if one wants to reach 2, one has to stop adding. To stop adding means the same as adding zero. This is never said, but it is clearly implied.

There is something totally inevitable about the following desired scenario. It needs to include two critical facts:

1) one never stops;

2) one ends up by adding nothing (how can one not overshoot 2 if one keeps adding?).

The concept of infinity perfectly takes care of these two conditions at the same time. If one never stops, one will definitely end up adding

$$\frac{1}{\infty}.$$

After all, never stopping brings one to infinity. What else could infinity be?

Then again, after never stopping, one also wants to end up by adding nothing or zero. Defining the expression

$$\frac{1}{\infty}$$

as 0, as L. Euler does and I do too (and as apparently few if anyone after L. Euler has), makes full allowance for the remarkable dual requirement that

1) one has to keep increasing the denominator forever and

2) end up by adding nothing.

Indeed, 1), the expression ∞ fully guarantees that one has gone on forever. And, 2), dividing 1 by the infinitely large ∞ guarantees that one has reached 0.

From about the nineteenth century onward, mathematicians generally felt very uncomfortable with all this. I impute this discomfort to the desire of mathematicians to consider nothing off limits to rational human intelligence. There was this implied hope—so it seems to me—that, if one thought about the matter a little harder, one would understand it a little better. And if one kept doing this, one would reach a complete understanding. I personally believe that there are certain matters that, however hard one thinks about them, remain completely inaccessible to rational human intelligence. The alternative is to assume that there are no limits to what the brain can do. How could anyone think that this alternative conclusion is acceptable?

The following inductive procedure is usually applied to infinite sums such as

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots.$$

As one adds ever more terms, it soon becomes very apparent that the infinite sum assumes the value

$$1.99999\cdots$$

It is then only a small step—a key step universally accepted by all—to conclude that the result must be 2.

A welcome confirmation of this fact is that the fraction

$$\frac{1}{1-\frac{1}{2}},$$

from which the above infinite sum was derived in the first place, is equivalent to 2, as follows:

$$\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2.$$

Since 1.9999… and 2 result from equivalent mathematical expressions, they too must be equivalent. It remains a fact that there is something inaccessible to rational human intelligence about this equivalence. The reason is that *one can never stop* the infinite sum in order to arrive at 2. The notion of not stopping is accessible to rational human intelligence. Why stop if you can go on? But what does it mean to never ever stop? One thing is certain, if one never ever stops, one definitely arrives at 2. What is more, the denominator of the fractions keeps rising. When the denominator becomes infinitely large, the fraction becomes zero and one stops adding.

The inductive approach can also be applied to the second dimension. In the second dimension, an infinite series of Equations is produced, with each Equation resolving a fraction into an infinite series of the first dimension, as follows:

$$\frac{1}{1-\frac{1}{2}} = 2 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots;$$

$$\frac{1}{1-\frac{1}{4}} = \frac{4}{3} = 1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{1024}+\cdots;$$

$$\frac{1}{1-\frac{1}{8}} = \frac{8}{7} = 1+\frac{1}{8}+\frac{1}{64}+\frac{1}{512}+\frac{1}{4096}+\cdots;$$

$$\frac{1}{1-\frac{1}{16}} = \frac{16}{15} = 1+\frac{1}{16}+\frac{1}{256}+\frac{1}{4096}+\frac{1}{65536}+\cdots;$$

$$\frac{1}{1-\frac{1}{32}} = \frac{32}{31} = 1+\frac{1}{32}+\frac{1}{1024}+\frac{1}{32768}+\frac{1}{1048576}+\cdots;$$

....

$$\frac{1}{1-\frac{1}{2^{20}}} = \frac{1}{1-\frac{1}{1048576}} = \frac{1048576}{1048575}$$
$$= 1+\frac{1}{1048575}+\left(\frac{1}{1048575}\right)^2+\left(\frac{1}{1048575}\right)^3+\left(\frac{1}{1048575}\right)^4+\left(\frac{1}{1048575}\right)^5+\cdots.$$

And so on.

And finally, when one reaches infinity.

$$\frac{1}{1-\frac{1}{\infty}} = 1+\frac{1}{\infty}+\left(\frac{1}{\infty}\right)^2+\left(\frac{1}{\infty}\right)^3+\left(\frac{1}{\infty}\right)^4+\left(\frac{1}{\infty}\right)^5+\cdots.$$

This infinite series does not end at 2 like the one dimensional infinite sum from which it is derived, but rather at 1.

I personally have no problems with rewriting

$$\frac{1}{1-\frac{1}{\infty}}$$

as

$$\frac{1}{1-0} = 1.$$

One might look at it this way. There is no doubt that the two dimensional infinite series ends in 1 if one goes on forever. But the fraction at hand can only turn to 1 if one accepts the following Equation as real:

$$\frac{1}{\infty} = 0.$$

I am not denying that handling infinity is a very delicate matter. But that should not lead mathematicians to drop it like some kind of hot potato. It should encourage them to establish through accepted mathematical procedures what rational human intelligence can know about infinity and what it cannot know.

11. J. Wallis, L. Euler, and Infinity

It seems to me that there is so much more to be known about the constitution of infinity in a systematic way than is generally assumed to be the case in the field of mathematics. Then again, at the same time, there is no doubt that infinity is in some fundamental way inaccessible to human intelligence. The quest of the human brain that is endowed with rational intelligence must therefore always be to be aware of the line that separates what can be known about infinity and what cannot be known about infinity and explicitly keep pointing to the distinction between the two in order to gain a proper appreciation of the phenomenon.

The design of this paper is to definitively prove a number of facts all relating to infinity. The present section is concerned with what may well concern the most important fact about infinity, that is, the abolishment of the number line once and for all in favor of the number circle.

It needs to be demonstrated that there is an undeniable undisrupted continuity that runs from infinitely large positive numbers to infinitely large negative numbers through infinity and *vice versa*. If this is the case, then it is impossible to characterize infinity as either negative or positive because it can be demonstrated that the very same infinity is undoubtedly bordered on one side by extremely large positive numbers and on the other side by extremely large negative numbers.

In this conviction, I am hardly alone. L. Euler, the most prolific mathematician of all time, subscribed to the same view. And the ideas of J. Wallis, without whose work the epochal achievements of I. Newton would have been impossible, lead to the same conclusion. However, all kinds of discomfort regarding dealing with infinity led mathematicians of the nineteenth century to reject, if not condemn, such concepts as divergent series with a precise result, and therefore the notion that there is no such thing as positive and negative infinity.

To be sure, J. Wallis himself had not quite abandoned the notions of positive infinity and negative infinity. That is why he believed that there are numbers that are larger than infinity. How can anything be larger than infinity? Nothing is larger than infinity. That is why it is infinity. In fact, if one moves from extremely large positive numbers to infinity itself and then goes on, one ends up in extremely large negative numbers. How can extremely large negative numbers be larger than infinity? J. Wallis does not provide an answer to this question. But L. Euler did, more or less along the following lines.

The key notion is that of rising in quantity if one moves to extremely large positive numbers to infinity and beyond. There is no doubt that, if extremely large negative numbers lie beyond infinity, then—as one continues from ever larger positive numbers to infinity to the largest possible negative numbers and moves towards zero through the negative numbers—one just keeps rising in quantity.

L. Euler very cleverly states that "the same quantities that are less than zero can be considered to be greater than infinity" [2]. Modern mathematicians would define the infinity in question as negative infinity. Still, it is infinitely smaller than all the negative numbers because, if one starts from it, one rises in quantity through the negative numbers.

There is something very satisfyingly coherent about the number circle. There are two ways of moving on a circle, clockwise and counterclockwise. If the number circle is real, as I believe that I can prove that it is, then there is something eminently consistent about movement in one direction always involving an increase in quantity and movement in the other direction always involving a decrease in quantity. This is exactly what happens in the number circle. By contrast, in the number line, one ascends to positive infinity and all ends there and one descends to negative infinity and all ends there. What happens when one tries to go on? The answer to this question cannot be found.

There are two models at stake. One is the number line. The other is the number circle. The number line is now universally accepted by all. Does the number circle even stand a chance? I believe that, on closer inspection, the mathematical facts all support the veracity of the number circle as opposed to the number line. These facts will be laid out below. Then why is the number line so completely dominant in the history of mathematics? Dealing with the difference between the number circle and the number line involves addressing the matter of infinity. Again, I have the impression that mathematicians across the ages exhibit a certain aversion in dealing with infinity. They prefer to identify it as undefined or the like. Then again, L. Euler had no problem with confronting infinity head-on. His approach was completely abandoned soon after him. Then again, later on, there was a sense that he may have been far ahead of his time. One of the subsidiary designs of the present paper is to show that he indeed was.

12. A First Application of Two Dimensional Infinite Series Making the Equation $\frac{1}{2}=1-1+1-1+1-1\cdots$. Fully Intuitively Apparent

The great G. W. Leibniz already accepted "without any hesitation"—to use L. Euler's expression [14]—that the Equation

$$\frac{1}{2}=1-1+1-1+1-1+\cdots.$$

is valid.

I have not tried to find the original sources documenting G. W. Leibniz's view. The task would take some effort. I instead rely on L. Euler's testimony. Clearly, one could rely on less trustworthy testimonies.

There was much discussion in the eighteenth century and presumably beyond as to whether the above Equation is valid or not. G. W. Leibniz was convinced that it is. So was L. Euler. And so am I, beyond a doubt.

One does not easily find any reflections of this controversy after the eighteenth century. Engaging infinity was more or less conceived of as dealing with the devil in nineteenth century mathematics. I have not been able to find engagement with the Equation at hand after the eighteenth century. Then again, I have not performed a systematic search.

But let there be no mistake. The Equation

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

is the truth and nothing but the truth, however improbable it may look at first sight. The design of the present section is to demonstrate once and for all the veracity of this truth by making the Equation intuitively more transparent.

L. Euler noted [14], as G. W. Leibniz already did, that the Equation is evident if one considers it to be an expansion of the fraction

$$\frac{1}{1+1}.$$

This fraction is but one manifestation of the more general expression

$$\frac{1}{1+a}.$$

It was noted above that this expression can be resolved into an infinite series, with the following Equation as the result:

$$\frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 - \cdots. \quad (12)$$

The technique of obtaining the series is described in detail above. If $a = 1$, then the fraction becomes

$$\frac{1}{1+1} = \frac{1}{2}.$$

This fraction can be resolved into an infinite series, with the following Equation as the result:

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots. \quad (13)$$

There are various ways in which the veracity of this Equation has been defended. But what is needed is an explanation that not only renders intuitive, but also serves to prove, the fact that the Equation must be true. I believe that two dimensional infinite series make such an explanation possible.

As regards one dimensional series, it seems intuitively more than obvious that the following Equation is true:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2. \quad (2)$$

The matter has been discussed above. The way in which the sum gradually turns into 1.9999999… can be considered an inductive proof of sorts. Then again, it needs to be acknowledged that there is something elusive about an addition that never stops. Indeed, as has been suggested more than once above, fully comprehending infinity is ultimately inaccessible to rational human intelligence.

I believe that two dimensional infinite series can be used to make Equation (13) as transparent as—with the exception of a full understanding of infinity—Equation (2). How so?

Consider the fraction

$$\frac{1}{2}=\frac{1}{1+1}. \tag{14}$$

As was noted above, (14) can be converted into an infinite series in a first dimension by resolving the fraction according to the technique described above with the following Equation as the result:

$$1-1+1-1+1-1+1-\cdots. \tag{15}$$

How to introduce infinity in a second dimension into Equation (15)? The value of the individual terms of the infinite series in Equation (15) is determined by the component *a* in the fraction

$$\frac{1}{1+a}.$$

In fraction (14), this component is 1.

It is possible to manipulate *a* so that it is transformed into an infinite series in a second, additional, dimension. In what way?

If *a* is 1, then the infinite series adopts the peculiar form $1-1+1-1+1-1+\cdots$.

The design of what follows is to establish by means of infinite series what happens

1) as *a* comes ever infinitely closer to 1 from what is smaller than 1 by ever increasing and

2) as *a* comes ever infinitely closer to 1 from what is bigger than 1 by ever decreasing.

For the present purpose, it will suffice to determine what happens when *a*

1) approaches 1 by *rising* from $\frac{1}{2}$

or

2) approaches 1 by *descending* from $\frac{3}{2}$.

At the same time, $\frac{1}{1+a}$

1) approaches $\frac{1}{2}$ by *descending* from $\frac{2}{3}$

or

2) approaches $\frac{1}{2}$ by *rising* from $\frac{2}{5}$.

Evidently, when *a* rises, $\frac{1}{1+a}$ descends and *vice versa*.

As regards 1), four somewhat randomly selected values will be assigned to *a*, namely $\frac{1}{2}$, $\frac{3}{4}$, $\frac{999999}{1000000}$, and $\frac{999999999}{1000000000}$. It is easy to imagine how *a* infinitely keeps growing bigger while ever approaching 1 if one keeps adding an equal number of nines to the numerator as one adds zeros to the denominator of the fourth fraction. This is the dimension of infinity.

As *a increases* in quantity towards 1 from $\frac{1}{2}$, the fraction

$$\frac{1}{1+a}$$

decreases in quantity towards $\frac{1}{2}$. The fraction assumes the following forms:

$$\frac{1}{1+\frac{1}{2}};\frac{1}{1+\frac{3}{4}};\frac{1}{1+\frac{999999}{1000000}};\frac{1}{1+\frac{999999999}{1000000000}}.$$

Equivalent values of these four fractions are as follows:

$$\frac{2}{3};\frac{4}{7};\frac{1000000}{1999999};\frac{1000000000}{1999999999}.$$

As one can see, the infinite series of fractions of the type $\frac{1}{1+a}$ gradually decreases—starting from $\frac{2}{3}$—and infinitely approaches $\frac{1}{2}$, while *a* at the same time gradually increases starting from $\frac{1}{2}$ and infinitely approaches 1.

As regards 2), four somewhat randomly selected values will be assigned to *a*, namely $\frac{3}{2}$, $\frac{5}{4}$, $\frac{1000001}{1000000}$, and $\frac{1000000001}{1000000000}$. It is easy to imagine how *a* infinitely keeps growing smaller while ever approaching 1 if one keeps adding an equal number of zeros to the numerator as one adds zeros to the denominator of the fourth fraction.

Since *a* starts by a quantity of $\frac{1}{2}$ *below* 1 in 1), namely at $\frac{1}{2}$, it is only harmonious or congruous or the like for *a* to start by a quantity of $\frac{1}{2}$ *above* 1 in 2), namely at $\frac{3}{2}$.

It is easy to imagine how *a* infinitely keeps growing smaller while ever approaching 1 if one keeps adding an equal number of zeros to the numerator and the denominator of the fourth fraction.

As *a decreases* in quantity towards 1, the fraction

$$\frac{1}{1+a}$$

increases in quantity towards $\frac{1}{2}$. The fraction assumes the following forms:

$$\frac{1}{1+\frac{3}{2}};\frac{1}{1+\frac{5}{4}};\frac{1}{1+\frac{1000001}{1000000}};\frac{1}{1+\frac{1000000001}{1000000000}}.$$

Equivalent values of these four fractions are as follows:

$$\frac{2}{5};\frac{4}{9};\frac{1000000}{2000001};\frac{1000000000}{2000000001}.$$

As one can see, the infinite series of fractions of the type $\frac{1}{1+a}$ gradually increases—starting from $\frac{2}{5}$—and gradually and infinitely approaches $\frac{1}{2}$, while a at the same time gradually decreases starting from $\frac{3}{2}$ and infinitely approaches 1.

What does all this mean for what happens at $a = 1$?

It is first necessary to observe what happens when a infinitely approaches 1. But in order to make this truly possible, there is need for another dimension of infinity, which is obtained by converting the fractions defined above into infinite series, more specifically infinite sums. These conversions, to which the conversion for $a = 1$ has been added, are as follows:

$$a=\frac{1}{2}:$$

$$\frac{1}{1+\frac{1}{2}}=\frac{2}{3}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\cdots;$$

$$a=\frac{3}{4}:$$

$$\frac{1}{1+\frac{3}{4}}=\frac{4}{7}=1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\frac{81}{256}-\frac{243}{1024}+\cdots;$$

$$a=\frac{999999}{1000000}:$$

$$\frac{1}{1+\frac{999999}{1000000}}=\frac{1000000}{1999999}$$
$$=1-\frac{999999}{1000000}+\left(\frac{999999}{1000000}\right)^2-\left(\frac{999999}{1000000}\right)^3+\left(\frac{999999}{1000000}\right)^4-\cdots;$$

$$a=\frac{999999999}{100000000}:$$

$$\frac{1}{1+\frac{999999999}{1000000000}}=\frac{1000000000}{1999999999}$$
$$=1-\frac{999999999}{1000000000}+\left(\frac{999999999}{1000000000}\right)^2-\left(\frac{999999999}{1000000000}\right)^3+\left(\frac{999999999}{1000000000}\right)^4-\cdots;$$

$$a = 1:$$

$$\frac{1}{1+1} = \frac{1}{2} = 1-1+1-1+1-1+\cdots;$$

$$a = \frac{1000000001}{1000000000}:$$

$$\frac{1}{1+\frac{1000000001}{1000000000}} = \frac{1000000000}{2000000001}$$

$$= 1 - \frac{1000000001}{1000000000} + \left(\frac{1000000001}{1000000000}\right)^2 - \left(\frac{1000000001}{1000000000}\right)^3 + \left(\frac{1000000001}{1000000000}\right)^4 - \cdots;$$

$$a = \frac{1000001}{1000000}:$$

$$\frac{1}{1+\frac{1000001}{1000000}} = \frac{1000000}{2000001}$$

$$= 1 - \frac{1000001}{1000000} + \left(\frac{1000001}{1000000}\right)^2 - \left(\frac{1000001}{1000000}\right)^3 + \left(\frac{1000001}{1000000}\right)^4 - \cdots;$$

$$a = \frac{5}{4}:$$

$$\frac{1}{1+\frac{5}{4}} = \frac{4}{9} = 1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \frac{625}{256} - \cdots;$$

$$a = \frac{3}{2}:$$

$$\frac{1}{1+\frac{3}{2}} = \frac{2}{5} = 1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \cdots.$$

One could further increase the value of *a*, say to 2. The result is the following Equation:

$$\frac{1}{1+2} = \frac{1}{3} = 1-2+4-8+16-\cdots.$$

This result seems quite peculiar at first sight. It will be explained further below. One is reminded of the following true Equation, equally peculiar:

$$-1 = 1+2+4+8+16+\cdots.$$

The true nature of these unusual Equations will be clarified further below.

At this juncture, the task at hand is to make sense of the sequence of Equations listed above. I have the impression that the sequence makes fully transparent, or brings to the surface, what is happening in the seemingly elusive Equation

$$\frac{1}{1+(a=1)} = \frac{1}{2} = 1-1+1-1+1-1+\cdots.$$

Three empirical observations are possible.

First Empirical Observation

As the value of *a* rises towards 1 on an infinite trajectory, so do the terms of the infinite sum.

Second Empirical Observation

As the value of *a* rises towards 1 on an infinite trajectory, the individual terms of the infinite sum oscillate down and up—away from the initial term 1—by an infinitely growing quantity and the infinite series reaches the rational number that is its final sum by ever more densely distributed terms.

Third Empirical Observation

As the individual terms of the infinite sum oscillate down and up—away from the initial term 1—by an infinitely growing quantity while themselves approaching 1, the sum of the Equation grows ever more closely to $\frac{1}{2}$. How is this possible? One has the impression of two contradictory movements. In one respect, the individual terms are clearly getting farther away from $\frac{1}{2}$. In another respect, the *sum* of the individual terms is clearly approaching $\frac{1}{2}$. How can an evolving sum at the same get farther away and get closer to a certain number? The answer is as follows. As the individual terms get farther away from $\frac{1}{2}$, there are ever more of them at ever smaller intervals. This dynamic clearly overcomes the dynamic of the individual terms growing away from $\frac{1}{2}$. But how can we be certain of the fact that the removal of the individual terms from $\frac{1}{2}$ goes hand in hand with their sum approaching $\frac{1}{2}$? The value of the fractions from which the infinite sums have been derived clearly proves it. One might object and state that there is something elusive about this proof. This is exactly where the main point of the present paper comes into play. All the numerical dynamics described above happen in the dimension of infinity, which is ultimately inaccessible to rational human intelligence. All positive indications are that the individual terms can gradually grow closer to 1 as their sum approaches 1/2. It may therefore be very easily anticipated already now that, in the dimension of infinity, the individual terms reach the value 1 as their sum reaches 1/2 beyond a doubt, even if a complete and final understanding of this critical fact is ultimately beyond rational human intelligence because it involves an understanding of infinity. And comprehending infinity is clearly beyond rational human intelligence. The crucial search remains, as before, the line that divides what rational human intelligence can know from what it cannot know.

To illustrate and confirm these three empirical observations, I produce again one of the Equations listed above:

$$\frac{1}{1+\frac{999999}{1000000}}=\frac{1000000}{1999999}=1-\frac{999999}{1000000}+\left(\frac{999999}{1000000}\right)^2-\left(\frac{999999}{1000000}\right)^3+\left(\frac{999999}{1000000}\right)^4-\cdots.$$

There is no doubt about the veracity of this Equation. It is easy to verify inductively that the infinite series comes ever closer to $\frac{1000000}{1999999}$ as one keeps adding and subtracting ever more terms in the infinite series. The fraction $\frac{1000000}{1999999}$ corresponds to the decimal notation

$$0.500000250000125000062500003125002\ldots$$

The notation shows an interesting multiplication by 5 and a gradual reduction of the number of zeros. I have not investigated how the decimal notation proceeds.

The end scenario seems obvious. Two things happen as *a* rises towards—and reaches—1:

1) every individual term of the infinite series turns into 1 and

2) the fraction from which the infinite series is derived turns into $\frac{1}{2}$,

as can be seen in the Equation

$$\frac{1}{1+1}=\frac{1}{2}=1-1+1-1+1-1+\cdots.$$

It is fairly easy to accept that the following expression equals 1:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\cdots,$$

even if it remains elusive how a sum that never stops results in an integer in the dimension of infinity.

It seems just as easy to accept that

$$\frac{1000000000}{1999999999}$$

turns into

$$\frac{1}{2}$$

as

$$\frac{999999999}{1000000000}$$

turns into

$$1$$

in the Equation

$$\frac{1}{1+\frac{999999999}{1000000000}}=\frac{1000000000}{1999999999}$$
$$=1-\frac{999999999}{1000000000}+\left(\frac{999999999}{1000000000}\right)^2$$
$$-\left(\frac{999999999}{1000000000}\right)^3+\left(\frac{999999999}{1000000000}\right)^4-\cdots.$$

It is also clear that the end results 1 and $\frac{1}{2}$ are obtained in the dimension of infinity. Comprehending them is therefore inaccessible to rational human intelligence.

So far, the rise of a from $\frac{1}{2}$ to 1 has been documented in detail to prove—and at the same time to render fully transparent and intuitive (excluding a comprehension of infinity, which cannot be transparent or intuitive)—that the following Equation is the most natural thing in the world:

$$\frac{1}{1+(a=1)}=\frac{1}{2}=1-1+1-1+1-1+\cdots.$$

So far, a has risen from $\frac{1}{2}$ to 1. At the same time, the value of the fraction $\frac{1}{1+a}$ has fallen from $\frac{3}{2}$ to $\frac{1}{2}$. As a result, the individual terms of the infinite sum following the initial 1 have all risen to 1. One therefore expects all the individual terms of the infinite sum following the initial 1 to rise above 1 when a rises above 1.

In the following Equation, a rises to $\frac{1000000001}{1000000000}$, just above 1:

$$\begin{aligned}\frac{1}{1+\frac{1000000001}{1000000000}}&=\frac{1000000000}{2000000001}\\&=1-\frac{1000000001}{1000000000}+\left(\frac{1000000001}{1000000000}\right)^2\\&\quad-\left(\frac{1000000001}{1000000000}\right)^3+\left(\frac{1000000001}{1000000000}\right)^4-\cdots.\end{aligned}$$

As expected, the value of the fraction $\frac{1}{1+a}$ drops just below $\frac{1}{2}$. And the individual terms of the infinite sum following the initial 1 have all risen above 1. The terms grow infinitely. But the infinite series of additions is counteracted by an infinite series of subtractions. In the end, the term $\frac{1}{2}$ is reduced by just $\frac{1}{400000002}$.

Additional proof of the veracity of

$$\frac{1}{2}=1-1+1-1+1-1+\cdots$$

involves the notion of continuity.

It cannot be denied that there is perfect continuity in the transition from

$$\frac{1}{1+\frac{999999999}{1000000000}}=\frac{1000000000}{1999999999}$$

to

$$\frac{1}{1+1}=\frac{1}{2}$$

and on to

$$\frac{1}{1+\frac{1000000001}{1000000000}}=\frac{1000000000}{2000000001}.$$

Then why would there not be perfect continuity between the undeniable mathematical equivalents of these three expressions, the following three infinite series:

$$1-\frac{999999999}{1000000000}+\left(\frac{999999999}{1000000000}\right)^2-\left(\frac{999999999}{1000000000}\right)^3+\left(\frac{999999999}{1000000000}\right)^4-\cdots;$$

$$1-1+1-1+1-1+\cdots;$$

and

$$1-\frac{1000000001}{1000000000}+\left(\frac{1000000001}{1000000000}\right)^2-\left(\frac{1000000001}{1000000000}\right)^3+\left(\frac{1000000001}{1000000000}\right)^4-\cdots.$$

There are infinite Equations of the same type as

$$\frac{1}{2}=1-1+1-1+1-1+\cdots. \qquad (16)$$

By dividing the dividend −1 by part of the divisor 1 + 1, namely just 1, according to the method described above, one obtains

$$-\frac{1}{2}=-1+1-1+1-1+1-\cdots.$$

By multiplying (16) by any number, say by 2 or by $\frac{1}{2}$, one obtains

$$1=2-2+2-2+2-2+\cdots$$

and

$$\frac{1}{4}=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\cdots.$$

And so on.

13. Sequences Suggesting the Veracity of the Number Circle

Consider the following sequence of fractions.

$$\frac{1}{3}\frac{1}{2}\frac{1}{1}(=1)\frac{1}{\frac{1}{2}}(=2)\frac{1}{\frac{1}{1000}}(=1000)\frac{1}{\frac{1}{1000000}}(=1000000)$$

$$\frac{1}{0}(=\infty?+\infty?)$$

$$\frac{1}{\frac{1}{-1000000}}(=-1000000)\frac{1}{\frac{1}{-1000}}(=-1000)\frac{1}{\frac{1}{-2}}(=-2)\frac{1}{-1}(=-1)\frac{1}{-2}\frac{1}{-3} \qquad (17)$$

Any high school student will easily notice that the numerator of these fractions continuously decreases. The numerator is located on an uninterrupted continuous downward path.

But what happens at the same time to the fraction as whole? In the first half of the sequence, the fractions are positive and their value keeps rising towards infinity. Then the numerator turns to 0. Immediately after it has turned to 0, the fractions are hugely negative and keep rising towards 0.

But what happens at $\frac{1}{0}$ itself? The crucial question is as follows: Is there continuity at the fraction $\frac{1}{0}$ just as there is continuity at the numerator 0? If it is certain that there is continuity at $\frac{1}{0}$, then there is an uninterrupted continuum from infinitely large positive numbers to $\frac{1}{0}$ and then on to infinitely large negative numbers.

As it happens, there are various issues with the matter of continuity at $\frac{1}{0}$. These issues are discussed below.

But let us assume for the sake of the argument that there is indeed continuity at $\frac{1}{0}$. What does this mean?

If there is continuity at $\frac{1}{0}$, then $\frac{1}{0}$ is bordered on one side by infinitely large positive numbers and on the other side by infinitely large negative numbers. The fraction $\frac{1}{0}$ borders both infinitely large positive numbers on one side and infinitely large negative numbers on the other side.

At the same time, it is easy to think of $\frac{1}{0}$ as infinitely large even if mathematicians—in the wake of nineteenth century mathematics—would label it as "undefined". L. Euler had no difficulty whatsoever in equating $\frac{1}{0}$ with infinity. Nor did J. Wallis. Nor do I.

To anyone accepting that $\frac{1}{0}$ equals infinity, as L. Euler and I do, it becomes extremely tempting to assume—as L. Euler does and I do too—that one passes from infinitely large positive numbers through infinity to infinitely large positive numbers, and *vice versa*.

That means that infinity is neither negative nor positive. It is rather a point at which one passes from the positive numbers to the negative numbers and *vice versa*. In the same way, there is no negative zero and no positive zero. In that regard, infinity and zero are two points at which one passes from the positive numbers to the negative numbers and *vice versa*.

Again, the beautiful harmony of all this is as follows. If one moves in one direction on the number circle, one *always* rises in quantity. If one moves in the other direction, one *always* falls in quantity. Quantity behaves perfectly as a kind of clockwork.

L. Euler—and apparently no one after him as far as I know (until now)—stated very clear that

not only from algebra but also from geometry, we learn that there are two jumps from positive quantities to negative ones, one through nought or zero, the other through infinity, and that quantities greater than infinity are thereby less than zero and quantities less than infinity coincide with quantities greater than zero.

To prove the veracity of the number circle, one wishes that two conditions were met, the following:

1) one wished that one could prove that there is continuity at $\frac{1}{0}$ in sequence (17);

2) one wished that the proof in 1) would not involve any need to make a commitment as to the true nature of $\frac{1}{0}$. I have no problem defining $\frac{1}{0}$, along with L. Euler, as an expression of infinity neither negative or positive. But $\frac{1}{0}$ is mostly described as undefined and I therefore refrain from identifying it as infinity as far as the proof is concerned.

Proof of the number circle is provided in section 14 below. This proof leaves no doubt, I believe, that there is continuity at $\frac{1}{0}$. And the proof does not require any commitment as to the true nature of the expression $\frac{1}{0}$. Before proceeding to section 14, some problems with sequence (17) may be noted.

Is it possible to rearrange sequence (17) as follows?

$$\frac{1}{3}\frac{1}{2}\ 1\ \ 2\ \ 1000\ \ 1000000\ \ \frac{1}{+0}\left(=+\frac{1}{0}\right)\frac{1}{-0}\left(=-\frac{1}{0}\right)\ -1000000\ \ -1000\ \ -1\ \ -2\ \ \frac{1}{-2}\frac{1}{-3}$$

And therefore also as follows?

$$-\frac{1}{0}?-1000000-1000-1-2\frac{1}{-2}\frac{1}{-3}0\frac{1}{3}\frac{1}{2}1210001000000+\frac{1}{0}?$$

This sequence accords with the reigning concept of the number line starting at minus infinity at one end and ending at plus infinity at the other end.

All is not clear. Someone trying to use sequence (17) as proof of the number circle may be accused of circular reasoning by equating $-\frac{1}{0}$ with $+\frac{1}{0}$.

Also, there is another type of continuity that needs to be taken into consideration. What happens if, in the sequence of fractions listed above, 1 is systematically replaced by −1? The resulting sequence is as follows:

$$\frac{-1}{3}\left(=-\frac{1}{3}\right)\ \frac{-1}{2}\left(=-\frac{1}{2}\right)\ \frac{-1}{1}(=-1)\ \frac{-1}{\frac{1}{2}}(=-2)$$

$$\frac{-1}{\frac{1}{1000}}(=-1000) \quad \frac{-1}{\frac{1}{1000000}}(=-1000000)$$

$$\frac{-1}{0}(=\infty?-\infty?)$$

$$\frac{-1}{-\frac{1}{1000000}}(=+1000000) \quad \frac{-1}{-\frac{1}{1000}}(=+1000)$$

$$\frac{-1}{-\frac{1}{2}}(=+2) \quad \frac{-1}{-1}(=+1) \quad \frac{-1}{-2}\left(=\frac{1}{2}\right) \quad \frac{-1}{-3}\left(=\frac{1}{3}\right)$$

The numerator again decreases gradually across the entire sequence. In the first half of the sequence, the fractions are negative and their value keeps rising towards infinity. Then the numerator turns to 0. Immediately after, the fractions are hugely positive and keep rising towards zero and to what is in effect the beginning of the sequence. In other words, there is continuity from the end of the sequence to the beginning of the sequence.

But what happens at $\frac{-1}{0}$? This fraction borders both infinitely large positive numbers on one side and infinitely large negative numbers on the other side. At the same time, it is easy to think of $\frac{-1}{0}$ as infinitely large even if mathematicians—in the wake of nineteenth century mathematics—would label it as "undefined". L. Euler had no difficulty whatsoever in equating $\frac{-1}{0}$ with infinity.

In sum, one is faced with two types of continuity. One type of continuity passes through $\frac{1}{0}$ and the other passes through $\frac{-1}{0}$. The fraction $\frac{-1}{0}$ could also be written as $\frac{1}{-0}$. But is there such a thing as minus zero?

The two rivaling sequences presented above make it tempting to equate $\frac{1}{0}$ with $\frac{-1}{0}$. Equating the two would solve the problem. But can one do this?

And what is the relation between $-\frac{1}{0}$, $\frac{-1}{0}$, and $\frac{1}{-0}$ exactly?

In the end, there is a lot of conflict surrounding the sequences described above. Perhaps, someone will be able to derive from these sequences a kind of proof of L. Euler's belief. I personally am not at this time. I see much that is extremely suggestive. But I do not discern anything that constitutes positive proof.

J. Wallis already noted the remarkable properties of a sequence like (17). He may have been the first to note that huge negative numbers can follow huge positive numbers. His views are discussed in section 17 below. But J. Wallis did not prove the number circle. In fact, he did not conceive of it. He describes negative numbers that following huge positive numbers as "greater than infinity". By the same token, he should have come to the conclusion that there are also positive numbers that are smaller than infinity.

I do not agree with J. F. Scott, the principal student of J. Wallis's mathematical work, when he writes that "[h]is error lay in the fact that he assumed $\frac{1}{a}$ to increase continually as a by units diminished, and that this increase persisted when $a = 0$" [6]. Rather, I believe J. Wallis noticed something remarkable. But he could not explain it. J. F. Scott writes as follows:

Thus Wallis arrived at a position which he did not understand, namely, the transition from a positive to a negative quantity by way of infinity [6].

And also as follows:

But here a formidable obstacle presented itself, which not even the genius of Wallis was able to overcome [15].

L. Euler provided an explanation. It is time to prove the explanation.

14. First Proof of the Number Circle or Cycle: Inductive, by Means of Two Dimensional Infinite Series

What is it that needs to be proven in order to prove the veracity of the number cycle? The QED needs to be defined precisely because the danger of producing a circular proof is great. It is clear that there is continuity from negative to positive numbers and *vice versa* at zero (0), as follows:

$$\cdots-5\cdots-2\cdots-1\cdots-\frac{1}{2}\cdots-\frac{1}{5}\cdots-\frac{1}{1000000}\cdots-0\cdots\frac{1}{1000000}\cdots\frac{1}{5}\cdots\frac{1}{2}\cdots1\cdots2\cdots5\cdots.$$

The sequence could also be presented as follows:

$$\cdots5\cdots2\cdots1\cdots\frac{1}{2}\cdots\frac{1}{5}\cdots\frac{1}{1000000}\cdots0\cdots-\frac{1}{1000000}\cdots-\frac{1}{5}\cdots-\frac{1}{2}\cdots-1\cdots-2\cdots-5\cdots.$$

At 0, the transitions from negative to positive numbers and from positive to negative numbers happen at the smallest possible negative numbers and the smallest possible positive numbers.

What needs to be proven is that there is likewise continuity from negative to positive numbers and *vice versa* across infinity, infinity being neither negative nor positive:

$$\begin{array}{l}\cdots1\cdots2\cdots5\cdots1000000\cdots1000000000\cdots\infty\\ \cdots-1000000000\cdots-1000000\cdots-5\cdots-2\cdots-1\cdots.\end{array} \tag{18}$$

Sequence (18) could also be represented as follows:

$$\begin{array}{l}\cdots-1\cdots-2\cdots-5\cdots-1000000\cdots-1000000000\cdots\infty\\ \cdots1000000000\cdots1000000\cdots5\cdots2\cdots1\cdots.\end{array}$$

In other words, all that separates the largest possible negative numbers from the largest possible positive numbers is infinity. In every other regard, there is continuity.

To be successful, the anatomy of the desired proof is best performed in two steps, as far as I can see. Why?

It is clear that the largest possible negative numbers and the largest possible positive numbers are not continuous in that the largest possible negative number is immediately adjacent to the largest possible negative number or the largest possible negative number immediately adjacent to the largest possible positive number. They do not touch, as it were. Rather, they are adjacent in that they share a *common boundary*. To use a metaphor, they are like two gardens that can be viewed as continuous because all that separates them is a fence.

It is therefore necessary to prove two matters distinctly and separately from one another:

1) that there is continuity from the largest possible positive numbers to the largest possible negative numbers, and *vice versa*, involving a common boundary;

2) that the identity of the common boundary is infinity, which is neither negative nor positive.

Towards delivering proof of 1) and 2), it is necessary to perform two conversions.

The first conversion is as follows. The numbers in sequence (18) can be converted into the form $\frac{1}{1-a}$ as follows:

$$\cdots\frac{1}{1-2}\cdots\frac{1}{1-\frac{3}{2}}\cdots\frac{1}{1-\frac{6}{5}}\cdots\frac{1}{1-\frac{1000001}{1000000}}\cdots\frac{1}{1-\frac{1000000001}{1000000000}}\cdots$$

$$\cdots\frac{1}{1-?}\cdots$$

$$\cdots\frac{1}{1-\frac{999999999}{1000000000}}\cdots\frac{1}{1-\frac{999999}{1000000}}\cdots\frac{1}{1-\frac{4}{5}}\cdots\frac{1}{1-\frac{1}{2}}\cdots\frac{1}{1-0}\cdots. \quad (19)$$

I have left ∞ out of consideration for the time being in converting sequence (18) into sequence (19). It is part of what needs to be proven. Still, it is possible to determine the value of *a*, 1), as $\frac{1000000001}{1000000000}$ becomes ever smaller in a rightward direction on an infinite trajectory toward $\frac{10000.0000\cdots1}{1000000000\cdots}$ by adding an equal number of zeros at… in both numerator and denominator and, 2), as $\frac{999999999}{1000000000}$ becomes ever larger in a leftward direction on an infinite trajectory toward $\frac{999999999\cdots}{1000000000\cdots}$ by adding as many 9 s to the numerator of $\frac{999999999\cdots}{1000000000\cdots}$ as 0s to the denominator. Totally independently from any consideration of ∞, it seems more than obvious inductively that *a* becomes 1 in the dimension of infinity. Accordingly, the expression $\frac{1}{1-?}$ at this point is as follows:

$$\frac{1}{1-1}.$$

Importantly, with the addition of $\frac{1}{1-1}$, sequence (19) seems to become fully continuous. But the problem is the interpretation of $\frac{1}{1-1}$. It is not clear what to make of it. An obvious equivalent is as follows:

$$\frac{1}{0}.$$

For reasons laid out in section 13 above, I try to avoid this expression if I can. There has been much controversy about it and it is common for mathematicians to leave it undefined.

In agreement with L. Euler and J. Wallis, I have no doubt that

$$\frac{1}{0}=\infty .$$

But verifying this Equation is in fact part of what needs to be proven and assuming it to be true at this point would therefore constitute circular reasoning.

I therefore leave the expression

$$\frac{1}{1-1}$$

as it is.

The need is for an unambiguous proof of continuity that does involve $\frac{1}{1-1}$ but makes it unnecessary to deal with $\frac{1}{0}$.

Towards that purpose, a second conversion may be applied converting the fractions in sequence (19), including $\frac{1}{1-1}$, into infinite series. Again, hardly any mathematical technique has proven as successful as infinite series. And it will appear that the technique will be indispensable in trying to understand the geography of infinity itself better. It takes infinity to explain infinity, as it were.

The technique of converting fractions into infinite series has been laid out at great length above. Accordingly, there is no doubt that each of the infinite series of fractions in Equation (19) can be converted into an infinite series as follows, as infinity is expanded into a second dimension:

$$-1=\frac{1}{1-2}=1+2+4+8+16+\cdots;$$

$$-2=\frac{1}{1-\frac{3}{2}}=1+\frac{3}{2}+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\left(\frac{3}{2}\right)^4+\cdots=1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\cdots;$$

$$-5=\frac{1}{1-\frac{6}{5}}=1+\frac{6}{5}+\left(\frac{6}{5}\right)^2+\left(\frac{6}{5}\right)^3+\left(\frac{6}{5}\right)^4+\cdots=1+\frac{6}{5}+\frac{36}{25}+\frac{216}{125}+\frac{1296}{625}+\cdots;$$

$$-1000000=\frac{1}{1-\frac{1000001}{1000000}}$$
$$=1+\frac{1000001}{1000000}+\left(\frac{1000001}{1000000}\right)^2+\left(\frac{1000001}{1000000}\right)^3+\left(\frac{1000001}{1000000}\right)^4+\cdots;$$

$$-1000000000=\frac{1}{1-\frac{1000000001}{1000000000}}$$
$$=1+\frac{1000000001}{1000000000}+\left(\frac{1000000001}{1000000000}\right)^2+\left(\frac{1000000001}{1000000000}\right)^3+\left(\frac{1000000001}{1000000000}\right)^4+\cdots;$$

$$\frac{1}{1-1}==1+1+(1)^2+(1)^3+(1)^4+\cdots=1+1+1+1+1+\cdots;$$

$$1000000000=\frac{1}{1-\frac{999999999}{1000000000}}$$
$$=1+\frac{999999999}{1000000000}+\left(\frac{999999999}{1000000000}\right)^2+\left(\frac{999999999}{1000000000}\right)^3+\left(\frac{999999999}{1000000000}\right)^4+\cdots;$$

$$1000000=\frac{1}{1-\frac{999999}{1000000}}=1+\frac{999999}{1000000}+\left(\frac{999999}{1000000}\right)^2+\left(\frac{999999}{1000000}\right)^3+\left(\frac{999999}{1000000}\right)^4+\cdots;$$

$$5=\frac{1}{1-\frac{4}{5}}=1+\frac{4}{5}+\left(\frac{4}{5}\right)^2+\left(\frac{4}{5}\right)^3+\left(\frac{4}{5}\right)^4+\cdots=1+\frac{4}{5}+\frac{16}{25}+\frac{64}{125}+\frac{256}{625}+\cdots;$$

$$2=\frac{1}{1-\frac{1}{2}}=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\cdots=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots;$$

$$1=\frac{1}{1-0}=1+0+(0)^2+(0)^3+(0)^4+\cdots=1+0+0+0+0+\cdots.$$

It seems to me that this two dimensional representation of infinity provides the proof that is sought.

The expression $\frac{1}{1-1}$ has vanished and is now just one infinite series among an infinite series of infinite series. Each infinite series is equivalent to an integer because the series was converted from an integer through the mediation of a fraction.

A crucial property of the infinite series is that *all the numbers are positive*, as one can see from the following presentation:

$$1+2+4+8+16+\cdots;$$

$$1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\cdots;$$

$$1+\frac{6}{5}+\frac{36}{25}+\frac{216}{125}+\frac{1296}{625}+\cdots;$$

$$1+\frac{1000001}{1000000}+\left(\frac{1000001}{1000000}\right)^2+\left(\frac{1000001}{1000000}\right)^3+\left(\frac{1000001}{1000000}\right)^4+\cdots;$$

$$1+\frac{1000000001}{1000000000}+\left(\frac{1000000001}{1000000000}\right)^2+\left(\frac{1000000001}{1000000000}\right)^3+\left(\frac{1000000001}{1000000000}\right)^4+\cdots;$$

$$1+1+1+1+1+\cdots;$$

$$1+\frac{9999999999}{1000000000}+\left(\frac{999999999}{1000000000}\right)^2+\left(\frac{999999999}{1000000000}\right)^3+\left(\frac{999999999}{1000000000}\right)^4+\cdots;$$

$$1+\frac{999999}{1000000}+\left(\frac{999999}{1000000}\right)^2+\left(\frac{999999}{1000000}\right)^3+\left(\frac{999999}{1000000}\right)^4+\cdots;$$

$$1+\frac{4}{5}+\frac{16}{25}+\frac{64}{125}+\frac{256}{625}+\cdots;$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots;$$

$$1+0+0+0+0+\cdots.$$

It is therefore abundantly obvious that the infinite series descends continuously through positive numbers. Consider, for example, just the second term, as follows:

$$1+2+\cdots;$$

$$1+\frac{3}{2}+\cdots;$$

$$1+\frac{6}{5}+\cdots;$$

$$1+\frac{1000001}{1000000}+\cdots;$$

$$1+\frac{1000000001}{1000000000}+\cdots;$$

$$1+1+\cdots;$$

$$1+\frac{999999999}{1000000000}+\cdots;$$

$$1+\frac{999999}{1000000}+\cdots;$$

$$1+\frac{4}{5}+\cdots;$$

$$1+\frac{1}{2}+\cdots;$$

$$1+0+\cdots.$$

The second term descends continuously from 2 to 0. There is no disruption of the continuity whatsoever at any point.

This undeniable continuity makes it possible to prove Part One of the sought proof. It is that infinitely large positive numbers and infinitely large negative numbers are continuous in that they share a single common boundary, as follows:

$$-1 = 1 + 2 + \cdots;$$

$$-2 = 1 + \frac{3}{2} + \cdots;$$

$$-5 = 1 + \frac{6}{5} + \cdots;$$

$$-1000000 = 1 + \frac{1000001}{1000000} + \cdots;$$

$$-1000000000 = 1 + \frac{1000000001}{1000000000} + \cdots;$$

$$? = 1 + 1 + \cdots;$$

$$1000000000 = 1 + \frac{999999999}{1000000000} + \cdots;$$

$$1000000 = 1 + \frac{999999}{1000000} + \cdots;$$

$$5 = 1 + \frac{4}{5} + \cdots;$$

$$2 = 1 + \frac{1}{2} + \cdots;$$

$$1 = 1 + 0 + \cdots.$$

This infinite series of infinite series on the right-hand, 1), guarantees the veracity of continuity and, 2), also reveals the identity of the common boundary in terms of an infinite series. The common boundary is as follows:

$$1 + 1 + 1 + 1 + 1 + \cdots.$$

QED, as far as Part One is concerned.

The second part of the proof concerns the identity of the common boundary

$$1 + 1 + 1 + 1 + 1 + \cdots.$$

It is tempting to equate this infinite series with ∞ because the infinite series was derived from the expression $\frac{1}{1-1}$ and this expression can be rewritten as $\frac{1}{0}$, which J. Wallis and L. Euler equated with ∞. The problem, however, as was already pointed out, is that equating $\frac{1}{0}$ with ∞ is what needs to be proven.

In fact, two universally held tenets currently completely dominate mathematics, the following:

1) there is such a thing as negative infinity $(-\infty)$ and there is such a thing as positive infinity $(+\infty)$;

2) the expression $\frac{1}{0}$ is undefined (I do not know who first promoted this

concept, presumably in the nineteenth century).

By sharp contrast, the design of the present paper is to demonstrate once and for all the following opposite tenets:

1) infinity is neither negative nor positive;

2) $\frac{1}{0} = \infty$.

I have the impression that this set of two tenets has been either positively proven or positively disproven. Therefore, the question arises: What does the evidence tell us?

Consider again an excerpt of the infinite series of infinite series already mentioned above:

$$-1000000 = 1 + \frac{1000001}{1000000} + \cdots;$$

$$-1000000000 = 1 + \frac{1000000001}{1000000000} + \cdots;$$

$$? = 1 + 1 + \cdots;$$

$$1000000000 = 1 + \frac{999999999}{1000000000} + \cdots;$$

$$1000000 = 1 + \frac{999999}{1000000} + \cdots.$$

The infinite sums on the right-hand side leave no doubt about the identity of the boundary between the largest possible positive numbers and the largest possible negative numbers. It is, as was noted above,

$$1 + 1 + 1 + 1 + 1 + \cdots.$$

But what about expressing this boundary in terms other than by an infinite series? Let us consider the numbers on the left-hand side of the above sequence.

It is clear from the left-hand side of the above sequence that the expression $1 + 1 + 1 + 1 + 1 + \cdots$ is obtained after increasing positive numbers in all infinity. Everyone now assumes that, if one increases a quantity forever, one finds an infinitely large quantity or positive infinity. At the same time, it is clear from the left above sequence that the expression $1 + 1 + 1 + 1 + 1 + \cdots$ is obtained after decreasing positive numbers in all infinity. Everyone now assumes that, if one decreases a quantity forever, one finds an infinitely large quantity or positive infinity.

A problem arises at this time. There is only one expression $1 + 1 + 1 + 1 + 1 + \cdots$. The above sequence makes clear that the expression is encountered when making positive numbers infinitely larger, that is, at infinity. But identifying the infinity in question as positive contradicts the fact that the same expression is also encountered in the same sequence when making negative numbers infinitely smaller, that is, at infinity. Then again, identifying infinity instead as negative contradicts the notion that the same expression is also encountered in the same sequence when making positive numbers infinitely larger.

Defining infinity as either positive or negative is absurd in the mathematical sense. The simple conclusion is that infinity is neither positive nor negative. But it is still infinity. QED, as far as Part Two of the proof is concerned.

This proof may be in need of some extra confirmation. Such confirmation is derived from the following consideration.

In the expression $1+1+1+1+1+\cdots$, one has the impression of rising towards positive infinity. And yet the facts evidence that this expression is located where both negative numbers descend to infinity and positive numbers rise to infinity at the same time.

The attention turns to the expression $-1-1-1-1-1-\cdots$. One has the impression that this expression rises towards negative infinity. And yet, the facts evidence that it too is located where both negative numbers descend to infinity and positive numbers rise to infinity at the same time. It has already been established that infinity, which is neither positive nor negative, is the boundary between the largest possible positive numbers and the largest possible negative numbers.

Consider again an excerpt of the infinite series of infinite series already mentioned above:

$$-1000000 = 1 + \frac{1000001}{1000000} + \cdots;$$

$$-1000000000 = 1 + \frac{1000000001}{1000000000} + \cdots;$$

$$= 1 + 1 + \cdots;$$

$$1000000000 = 1 + \frac{999999999}{1000000000} + \cdots;$$

$$1000000 = 1 + \frac{999999}{1000000} + \cdots.$$

It is quite legitimate to multiply all the terms in these equations with -1. The result is as follows:

$$+1000000 = -1 - \frac{1000001}{1000000} - \cdots;$$

$$+1000000000 = -1 - \frac{1000000001}{1000000000} - \cdots;$$

$$? = -1 - 1 - \cdots;$$

$$-1000000000 = -1 - \frac{999999999}{1000000000} - \cdots;$$

$$-1000000 = -1 - \frac{999999}{1000000} - \cdots.$$

Evidently, the expression $-1-1-1-1-1-\cdots$ exhibits the exact same properties as the expression $+1+1+1+1+1+\cdots$. It ends where both negative numbers decrease to infinity and positive numbers rise to infinity. It therefore reaches an infinity that cannot possibly be defined as either negative or positive if one is to avoid absurdity in the mathematical sense.

Therefore, not only $+1+1+1+1+1+\cdots$, but also $-1-1-1-1-1-\cdots$, needs to be defined as infinity being neither positive nor negative.

The inevitable conclusion is that

$$(+)1+1+1+1+1+\cdots=-1-1-1-1-1-\cdots.$$

Then again, there may be those that view this Equation as an argument for refusing to accept the veracity of the number circle. It should be noted at this point that I feel that what precedes provides sufficient mathematical proof of the number circle. I have therefore no doubt that the above Equation is true. But realizing that not everyone may be easily convinced, I will present a completely independent algebraic proof that the Equation is indeed true. This proof is presented in section 17 below.

15. +2 Both Larger and Smaller than −2

If the number circle is true, as I firmly believe that it is (as a consequence of mathematical proof presented above and below), then one of its most remarkable implications is that every number is at the same time both smaller and larger than any other number.

On a circle, there are two ways of reaching one point from any other point, clockwise and counterclockwise. The same must therefore be true of the number circle.

It will be useful to illustrate this fundamental point with a simple example.

Let us take two points on the number circle, $+2$ and -2. Anyone who accepts the veracity of the number circle, as I do, needs to be able to specify in strict mathematical terms:

1) how $+2$ is larger than -2 and

2) how $+2$ is smaller than -2.

What does it mean for one number to be both larger and smaller than another number? This is the challenge.

Evidently, if $+2$ is larger than -2, then it should be possible to *subtract* something from $+2$ and obtain -2. In this case, the solution is more than obvious. If one subtracts 4 from $+2$, one obtains -2. In subtracting 4 from $+2$, one passes through zero. In sum,

$$2-4=-2.$$

But what does one need to add to $+2$ to obtain -2, in order to demonstrate that $+2$ can also be thought of as being smaller than -2? The answer is: one adds

1) a certain number

2) and an infinite series that is shortened at the beginning.

This number and this infinite series are obtained as follows.

-2 can be written as follows:

$$\frac{1}{1-\frac{3}{2}}=\frac{1}{\frac{2}{2}-\frac{3}{2}}=\frac{1}{-\frac{1}{2}}=-2.$$

The fraction

$$\frac{1}{1-\frac{3}{2}}$$

can be converted into the following infinite series according to the techniques laid out above:

$$1+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\cdots.$$

This infinite series can be rewritten as follows:

$$1+1+\frac{5}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\frac{729}{64}\cdots.$$

It follows that, if one adds

$$\frac{5}{4}$$

as well as an infinite series from which the term is removed, namely

$$\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\frac{2187}{128}\cdots$$

to $+2$, one obtains -2.

Or the following Equation applies:

$$(+)2+\frac{5}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\cdots=-2.$$

In adding the expression

$$\frac{5}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\frac{2187}{128}\cdots$$

to $+2$, one obtains -2.

In adding said expression to $+2$ to obtain -2, one passes through infinity. In adding -4 to $+2$, one passes through zero. In adding

$$\frac{5}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\frac{2187}{128}\cdots,$$

one passes through infinity. Therefore, it is clear that there are two ways of moving on the number circle, clockwise and counterclockwise as it were.

But moving through infinity is quite different from moving through 0. There are in fact an infinite number of ways of moving from one number to another number through infinity, and all are equally mathematically exact according to proven mathematical techniques.

For example, it is a fact that

$$-2=2+4+8+16+32+\cdots.$$

Therefore, the difference between $+2$ and -2 can also be defined by the following infinite series:

$$4+8+16+32+64+\cdots.$$

The remarkable thing about infinity is that all processes involving infinity happen in an infinite number of ways and each of these ways is fully mathematically precise.

It is time to map the geography of infinity. Some of this will be proposed in what follows. But a fuller account of the geography of infinity will need to be postponed to future papers.

16. Second Proof of the Number Circle or Cycle: Demonstration that $1+1+1+1+1+\cdots=-1-1-1-1-1-\cdots$

It was demonstrated above that both

$$1+1+1+1+1+\cdots$$

and

$$-1-1-1-1-1-\cdots$$

are identical in both bordering at the same time the largest possible negative numbers and the largest possible positive numbers. It was concluded above that both are equal to infinity, which is neither negative nor positive. I believe this proof to be sufficient.

But it is possible to deliver a different kind of proof that the two are indeed equal to one another. The first of the two has been obtained by resolving the fraction

$$\frac{1}{1-1}$$

into an infinite series according to accepted techniques. The essence of the technique is to divide 1 by only part of the fraction, namely 1.

It is altogether acceptable to restyle the above expression as

$$\frac{1}{-1+1}.$$

This expression invites the attempt to divide 1 by the other part of the denominator, namely -1, according to the same proved and time-honored technique described above. I refrain from developing the division here in detail. I hope that the technique has been described in sufficient detail so that it is abundantly clear that the result of the division is no doubt

$$-1-1-1-1-1-\cdots.$$

If one does not wish to do this longhand, one shorthand definition is to use the definition already obtained above that the fraction

$$\frac{1}{-a+1}$$

can be turned into the following infinite fraction:

$$-\frac{1}{a}-\frac{1}{a^2}-\frac{1}{a^3}-\frac{1}{a^4}-\frac{1}{a^5}-\cdots.$$

It follows that the expression

$$\frac{1}{-1+1}$$

is equivalent to

$$-\frac{1}{1}-\frac{1}{1^2}-\frac{1}{1^3}-\frac{1}{1^4}-\frac{1}{1^5}-\cdots,$$

or

$$-1-1-1-1-1-\cdots.$$

In sum, there is mathematically no doubt whatsoever that

$$-1-1-1-1-1-\cdots=1+1+1+1+1+1+\cdots.$$

This second proof only confirms what was already conclusively proven in the first proof: Positive infinity is the same as negative infinity and there is therefore only one infinity neither negative nor positive.

17. Towards a Complete Mapping of the Geography of Infinity in Number Theory in the Footsteps of L. Euler: Preview of Forthcoming Topics

Much has been left incomplete in what precedes. Some topics are as follows.

First, there is the matter of J. Wallis's "numbers greater than infinity" and how closely it approaches the truth of the matter.

Second, there is still the task of making the Equation

$$1+2+3+4+5+\cdots=-\frac{1}{12}$$

part of a comprehensive map of infinity in the realm of numbers. To this comprehensive map belongs the following Equation, which will be proven algebraically and inductively in a forthcoming article:

$$0=1+\infty+(\infty)^2+(\infty)^3+(\infty)^4+\cdots.$$

I also note in the margin that

$$1+\infty+\infty+\infty+\infty+\cdots=\infty$$

and that

$$\infty+\infty+\infty+\cdots=\infty\ .$$

Or also that

$$2\times\infty=3\times\infty=4\times\infty=n\times\infty=\infty.$$

Another Equation is as follows:

$$\frac{1}{5}=1-4+16-64+256-1024+\cdots$$

This type of Equation has been more or less universally condemned since the nineteenth century. It is said to diverge. Clearly, it moves ever farther away from $\frac{1}{5}$. So how does it finally end up at $\frac{1}{5}$? How does it have a rational number as its result, as so-called convergent numbers do?

Other Equations, already mentioned above, that need to be brought into perfect harmony with the Equation

$$1+2+3+4+5+\cdots=-\frac{1}{12}$$

are as follows:

$$\frac{1}{2}=1-1+1-1+1-\cdots;$$

$$\infty=1+1+1+1+1+1+\cdots=-1-1-1-1-1-\cdots.$$

All these Equations need to become gears in a fully comprehensive and fully coherent geography of infinite in numerical terms.

In conclusion, at stake is the geography of infinity in numerical terms. And that seems like a task that is too large for the present paper. Such a geography will be presented in subsequent papers.

In that regard, the need is to follow in the footsteps of L. Euler, the most prolific mathematician of all time. The most remarkable thing is that his approach to the nature of infinity has been completely abandoned after him. The present effort therefore presents very much a return to L. Euler, after what looks a little like a total abandonment.

In terms of style when it comes to writing mathematics, there is much inspiration to be derived from what the distinguished mathematician Georg Pólya wrote about L. Euler, as follows [16]:

Euler seems to me almost unique in one respect: he takes pains to present the relevant inductive evidence carefully, in detail, in good order. He presents it convincingly but honestly, as a genuine scientist should do. His presentation is "the candid exposition of the ideas that led him to those discoveries" and has a distinctive charm. Naturally enough, as any other author, he tries to impress his readers, but, as a really good author, he tries to impress his readers only by such things as have genuinely impressed himself.

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