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A Remark on the Topology at Infinity of a Polynomial Mapping $F:\mathbb{C}^n \to \mathbb{C}^n$ via Intersection Homology

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Abstract

In [1], Guillaume and Anna Valette associate singular varieties V_F to a polynomial mapping $F:\mathbb{C}^n \to \mathbb{C}^n$. In the case $F:\mathbb{C}^2 \to \mathbb{C}^2$, if the set $K_0(F)$ of critical values of F is empty, then F is not proper if and only if the 2-dimensional homology or intersection homology (with any perversity) of V_F is not trivial. In [2], the results of [1] are generalized in the case $F:\mathbb{C}^n \to \mathbb{C}^n$ where $n \geq 3$, with an additional condition. In this paper, we prove that for a class of non-proper *generic dominant* polynomial mappings, the results in [1] and [2] hold also for the case that the set $K_0(F)$ is not empty.

Keywords

Polynomial Mappings, Intersection Homology, Singularities

1. Introduction

In [1], Guillaume and Anna Valette provide a criteria for properness of a polynomial mapping $F:\mathbb{C}^2 \to \mathbb{C}^2$. They construct a real algebraic singular variety V_F satisfying the following property: if the set of critical values of F is empty then F is not proper if and only if the 2-dimensional homology or intersection homology (with any perversity) of V_F is not trivial ([1], Theorem 3.2). This result provides a new approach for the study of the well-known Jacobian Conjecture, which is still open until today, even in the two-dimensional case (see, for example, [3]). In [2], the result of [1] is generalized in the general case $F:\mathbb{C}^n \to \mathbb{C}^n$, where $n \geq 3$, with an additional condition ([2], Theorem 4.5). The variety V_F is a real algebraic singular variety of dimension $2n$ in some $\mathbb{R}^{2n+q}$, where $q > 0$, the singular set of which is contained in

$(K_0(F) \cup S_F) \times \{0_{\mathbb{R}}\}^q$, where $K_0(F)$ is the set of critical values and S_F is the asymptotic set of F.

This paper proves that if $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a non-proper *generic dominant* polynomial mapping, then the 2-dimensional homology and intersection homology (with any perversity) of V_F are not trivial. We prove that this result is also true for a non-proper *generic dominant* polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ $(n \geq 3)$, with the same additional condition than in [2]. To prove these results, we use the Transversality Theorem of Thom: if F is non-proper *generic* dominant polynomial mapping, we can construct an adapted $(2, \bar{p})$-allowable chain (in generic position) providing non triviality of homology and intersection homology of the variety V_F, for any perversity $\bar{p}$ (Theorems 5.1 and 5.2).

In order to compute the intersection homology of the variety V_F in the case $K_0(F) \neq \varnothing$, we have to stratify the set $K_0(F) \cup S_F$. Furthermore, the intersection homology of the variety V_F does not depend on the stratification if we use a locally topologically trivial stratification. It is well-known that a Whitney stratification is a Thom-Mather stratification and a Thom-Mather stratification is a locally topologically trivial stratification (see [4] [5] [6] [7]). In order to prove the main result, we use two facts: In [6], Thom defined a partition of the set $K_0(F)$ by "*constant rank*", which is a *local* Thom-Mather stratification; in [2], the authors provide a Whitney stratification of the asymptotic set S_F. One important point for the proof of the princial results of this paper is the following: we show that in general the set $K_0(F)$ is not closed, so we cannot define a (global) stratification of $K_0(F)$ satisfying the frontier condition. Hence, we cannot define a (global) Thom-Mather stratification of $K_0(F)$. However, we prove that the set $K_0(F) \cup S_F$ is closed and $K_0(F) \cup S_F = \overline{K_0(F)} \cup S_F$. This fact allows us to show that there exists a Thom-Mather stratification of the set $K_0(F) \cup S_F$ compatible with the partition of the set $K_0(F)$ defined by Thom in [6] and compatible with the Whitney stratification of the set S_F defined in [2] (Theorem 4.6).

This paper provides also some examples to light the results. Moreover, these examples provide also some topological properties of the well-known critical values set $K_0(F)$ associated to a complex polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$, for instance: in general, the set $K_0(F)$ is not closed; the set $K_0(F) \cup S_F$ is not smooth; $K_0(F) \cup S_F$ is not pure dimensional if F is not dominant. Via these examples, we make clear also the well-known Thom-Mather partition of $K_0(F)$ defined by Thom in [6].

2. Preliminaries

In this section we set-up our framework. All the varieties we consider in this article are semi-algebraic.

2.1. Intersection Homology

We briefly recall the definition of intersection homology. For details, we refer to the fundamental work of M. Goresky and R. MacPherson [8] (see also [4]).

Definition 2.1. Let V be a m-dimensional semi-algebraic set. A *semi-algebraic* *stratification of* V is the data of a finite semi-algebraic filtration

$$V = V_m \supset V_{m-1} \supset \cdots \supset V_0 \supset V_{-1} = \varnothing,$$

such that for every i, the set $V_i \setminus V_{i-1}$ is either an emptyset or a manifold of dimension i. A connected component of $V_i \setminus V_{i-1}$ is called *a stratum* of V.

Let S_i be a stratum of V and $\overline{S_i}$ its closure in V. If $\overline{S_i} \setminus S_i$ is the union of strata of V, for all strata S_i of V, then we say that the stratification of V satisfies the frontier condition.

Definition 2.2 (see [6] [9]). Let V be a variety in a smooth variety M. We say that a stratification of V is a *Thom-Mather stratification* if each stratum S_i is a differentiable variety of class $\mathcal{C}^\infty$ and if for each S_i, we have:

$a)$ an open neighbourhood (tubular neighbourhood) T_i of S_i in M,

$b)$ a continuous retraction π_i of T_i on S_i,

$c)$ a continuous function $\rho_i : T_i \to [0,\infty[$ which is $\mathcal{C}^\infty$ on the smooth part of $V \cap T_i$,

such that $S_i = \{x \in T_i : \rho(x) = 0\}$ and if $S_i \subset \overline{S}_j$, then

i) the restricted mapping $(\pi_i, \rho_i) : T_i \cap S_j \to S_i \times [0,\infty[$ is a smooth immersion,

ii) for $x \in T_i \cap T_j$ such that $\pi_j(x) \in T_i$, we have the following relations of commutation:

1) $\pi_i \circ \pi_j(x) = \pi_i(x)$,

2) $\rho_i \circ \pi_j(x) = \rho_i(x)$,

when the two members of these formulas are defined.

A Thom-Mather stratification satisfies the frontier conditions.

We denote by cL the open cone on the space L, the cone on the empty set being a point. Observe that if L is a stratified set then cL is stratified by the cones over the strata of L and an additional 0-dimensional stratum (the vertex of the cone).

Definition 2.3. A stratification of V is said to be *locally topologically trivial* if for every $x \in V_i \setminus V_{i-1}$, $i \geq 0$, there is an open neighborhood U_x of x in V, a stratified set L and a semi-algebraic homeomorphism

$$h : U_x \to (0;1)^i \times cL,$$

such that h maps the strata of U_x (induced stratification) onto the strata of $(0;1)^i \times cL$ (product stratification).

Theorem 2.4 (see [6] [7]). *A Thom-Mather stratification is a locally topologically trivial stratification.*

Definition 2.5 ([7]). One says that the *Whitney* (b) *condition* is realized for a stratification if for each pair of strata (S, S') and for any $y \in S$ one has: Let $\{x_n\}$ be a sequence of points in S' with limit y and let $\{y_n\}$ be a sequence of points in S tending to y, assume that the sequence of tangent spaces $\{T_{x_n} S'\}$ admits a limit T for n tending to $+\infty$ (in a suitable Grassmanian manifold) and that the sequence of directions $x_n y_n$ admits a limit λ for n tending to $+\infty$ (in the corresponding projective manifold), then $\lambda \in T$.

A stratification satisfying the Whitney (b) condition is called a *Whitney stra-*

tification.

Theorem 2.6 ([5]). *Every Whitney stratification is a Thom-Mather stratification, hence satisfies the topological triviality.*

The definition of perversities has originally been given by Goresky and MacPherson:

Definition 2.7. A *perversity* is an (m + 1)-uple of integers $\overline{p} = (p_0, p_1, p_2, p_3, \cdots, p_m)$ such that $p_0 = p_1 = p_2 = 0$ and $p_{\alpha+1} \in \{p_\alpha, p_\alpha + 1\}$, for $\alpha \geq 2$.

Traditionally we denote the zero perversity by $\overline{0} = (0, 0, \cdots, 0)$, the maximal perversity by $\overline{t} = (0, 0, 0, 1, \cdots, m-2)$, and the middle perversities by $\overline{m} = \left(0, 0, 0, 0, 1, 1, \cdots, \left[\frac{m-2}{2}\right]\right)$ (lower middle) and $\overline{n} = \left(0, 0, 0, 1, 1, 2, 2, \cdots, \left[\frac{m-1}{2}\right]\right)$ (upper middle). We say that the perversities $\overline{p}$ and $\overline{q}$ are *complementary* if $\overline{p} + \overline{q} = \overline{t}$.

Let V be a semi-algebraic variety such that V admits a locally topologically trivial stratification. We say that a semi-algebraic subset $Y \subset V$ is $(\overline{p}, i)$-*allowable* if

(2.8) $$\dim(Y \cap V_{m-\alpha}) \leq i - \alpha + p_\alpha \text{ for all } \alpha \geq 2.$$

Define $IC_i^{\overline{p}}(V)$ to be the $\mathbb{R}$-vector subspace of $C_i(V)$ consisting in the chains ξ such that $|\xi|$ is $(\overline{p}, i)$-allowable and $|\partial\xi|$ is $(\overline{p}, i-1)$-allowable.

Definition 2.9 The i^{th} *intersection homology group with perversity* $\overline{p}$, with real coefficients, denoted by $IH_i^{\overline{p}}(V)$, is the i^{th} homology group of the chain complex $IC_*^{\overline{p}}(V)$.

Notice that, the notation $IH_*^{\overline{p},c}(V)$ refers to the intersection homology with compact supports, the notation $IH_*^{\overline{p},cl}(V)$ refers to the intersection homology with closed supports. In the compact case, they coincide.

Theorem 2.10 ([8] [10]) *The intersection homology is independent on the choice of the stratification satisfying the locally topologically trivial conditions.*

The Poincaré duality holds for the intersection homology of a (singular) variety:

Theorem 2.11 (Goresky, MacPherson [8]). *For any orientable compact stratified semi-algebraic* m*-dimensional variety* V, *the generalized Poincaré duality holds*:

$$IH_k^{\overline{p}}(V) \simeq IH_{m-k}^{\overline{q}}(V),$$

where $\overline{p}$ and $\overline{q}$ are complementary perversities.

For the non-compact case, we have:

$$IH_k^{\overline{p},c}(V) \simeq IH_{m-k}^{\overline{q},cl}(V).$$

2.2. The Asymptotic Set

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. Let us denote by S_F the set of points at which F is non proper, *i.e.*,

(2.12) $$S_F := \{a \in \mathbb{C}^n \text{ such that } \exists \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^n, |x_k| \text{ tends to infinity and } F(x_k) \text{ tends to } a\}$$

where $|x_k|$ is the Euclidean norm of x_k in $\mathbb{C}^n$. The set S_F is called the asymptotic set of F.

In this paper, we will use the following important theorem:

Theorem 2.13. [11] *Let* $F:\mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial mapping. If* F *is dominant, i.e.,* $\overline{F(\mathbb{C}^n)} = \mathbb{C}^n$, *then* S_F *is either an empty set or a hypersurface.*

3. The Variety V_F

We recall in this section the construction of the variety V_F and the results obtained in [1] and [2]: Let $F:\mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. We consider F as a real mapping $F:\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. By $\text{Sing}\, F$ we mean the set of critical points of F. Thanks to the lemma 2.1 of [1], there exists a covering $\{U_1,\cdots,U_p\}$ of $M_F = \mathbb{R}^{2n} \setminus \text{Sing}(F)$ by semi-algebraic open subsets (in $\mathbb{R}^{2n}$) such that on every element of this covering, the mapping F induces a diffeomorphism onto its image. We may find some semi-algebraic closed subsets $V_i \subset U_i$ (in M_F) which cover M_F as well. By the Mostowski's Separation Lemma (see [12], p. 246), for each $i = 1,\cdots,p$, there exists a Nash function $\psi_i : M_F \to \mathbb{R}$, such that ψ_i is positive on V_i and negative on $M_F \setminus U_i$. We can choose the Nash functions ψ_i such that $\psi_i(x_k)$ tends to zero where $\{x_k\}$ is a sequence in M_F tending to infinity. We define

$$V_F := \overline{(F,\psi_1,\cdots,\psi_p)(M_F)},$$

that means, V_F is the closure of the image of M_F by $(F,\psi_1,\cdots,\psi_p)$.

The variety V_F is a real algebraic singular variety of dimension $2n$ in $\mathbb{R}^{2n+q}$, with $q > 0$, the singular set of which is contained in $(K_0(F) \cup S_F) \times \{0_{\mathbb{R}}\}^q$, where $K_0(F)$ is the set of critical values and S_F is the asymptotic set of F.

Theorem 3.1 ([2]). *Let* $F:\mathbb{C}^n \to \mathbb{C}^n$ *be a generically finite polynomial mapping with nowhere vanishing Jacobian. There exists a filtration of* V_F:

$$V_F = V_{2n} \supset V_{2n-1} \supset V_{2n-2} \supset \cdots \supset V_1 \supset V_0 \supset V_{-1} = \varnothing$$

such that:

1) for any $i < n$, $V_{2i+1} = V_{2i}$,

2) the corresponding stratification satisfies the Whitney (b) condition.

Recall the condition "F is nowhere vanishing Jacobian" means that the set of critical values $K_0(F)$ of F is an emptyset.

The following corollary comes directly from the Theorem 3.1 above.

Corollary 3.2. *Let* $F:\mathbb{C}^n \to \mathbb{C}^n$ *be a generically finite polynomial mapping. Then there exists a Whitney stratification of the asymptotic set* S_F.

Theorem 3.3 ([1]). *Let* $F:\mathbb{C}^2 \to \mathbb{C}^2$ *be a polynomial mapping with nowhere vanishing Jacobian. The following conditions are equivalent*:

1) F is non proper,

2) $H_2(V_F) \neq 0$,

3) $IH_2^{\bar{p}}(V_F) \neq 0$ for any perversity $\bar{p}$,

4) $IH_2^{\bar{p}}(V_F) \neq 0$ for some perversity $\bar{p}$.

Form here, we denote by $\hat{F}_i$ the homogeneous component of F_i of highest degree, or *the leading form* of F_i.

Theorem 3.4 [2] *Let* $F:\mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial mapping with nowhere vanishing*

Jacobian. If $\mathrm{Rank}_{\mathbb{C}}\left(D\hat{F}_i\right)_{i=1,\cdots,n} \geq n-1$, *where* $\hat{F}_i$ *is the leading form of* F_i, *then the following conditions are equivalent*:

1) F is non proper,
2) $H_2\left(V_F\right) \neq 0$,
3) $IH_2^{\bar{p}}\left(V_F\right) \neq 0$ for any (or some) perversity $\bar{p}$
4) $IH_{2n-2}^{\bar{p}}\left(V_F\right) \neq 0$, for any (or some) perversity $\bar{p}$.

Notice that with the notations $H_*\left(V\right)$ (*resp.* $IH_*^{\bar{p}}\left(V\right)$), we mean the homology (*resp.*, the intersection homology) with both compact supports and closed supports.

Remark 3.5. There exist may-be a lots of varieites V_F associated to the same polynomial mapping $F:\mathbb{C}^n \to \mathbb{C}^n$, but for any variety V_F, its properties in the Theorems 3.3 and 3.4 do not change.

The purpose of this paper is to prove that if $F:\mathbb{C}^n \to \mathbb{C}^n$ $\left(n \geq 2\right)$ is a non-proper *generic dominant* polynomial mapping, then the 2-dimensional homology and intersection homology (with any perversity) of V_F are not trivial. In order to compute the intersection homology of the variety V_F in the case $K_0\left(F\right) \neq \varnothing$, we have to stratify the set $K_0\left(F\right) \cup S_F$. Furthermore, the intersection homology of the variety V_F does not depend on the stratification of V_F if we use a locally topologically trivial stratification. By theorem 2.4, a Thom-Mather stratification is a locally topologically trivial stratification. In the following section, we provide an explicit Thom-Mather stratification of the set $K_0\left(F\right) \cup S_F$.

4. A Thom-Mather Stratification of the Set $K_0\left(F\right) \cup S_F$

We begin this section by giving an example to show that in general the set $K_0\left(F\right)$ of a polynomial mapping $F:\mathbb{C}^n \to \mathbb{C}^n$ is neither closed, nor smooth, nor pure dimensional. Recall that a set X is pure dimensional of dimension m if any point of this set admits a m-dimensional neighbourhood in X.

Example 4.1. Let us consider the polynomial mapping $F:\mathbb{C}^3_{(x_1,x_2,x_3)} \to \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ such that

$$F\left(x_1,x_2,x_3\right) = \left(x_1^3 - x_1x_2x_3, x_2x_3, x_3x_1\right).$$

Then, the jacobian determinant $\left|J_F\left(x\right)\right|$ of F is given by $x_1x_3\left(3x_1^2 - x_2x_3\right)$. If $\left|J_F\left(x\right)\right| = 0$ then $x_1 = 0$ or $x_3 = 0$ or $3x_1^2 = x_2x_3$. So we have the following cases:

+ if $x_1 = 0$ then $F\left(0,x_2,x_3\right) = \left(0,x_2x_3,0\right)$ and the axis $0\alpha_2$ is contained in $K_0\left(F\right)$,
+ if $x_3 = 0$ then $F\left(x_1,x_2,0\right) = \left(x_1^3,0,0\right)$ and the axis $0\alpha_1$ is contained in $K_0\left(F\right)$,
+ if $3x_1^2 = x_2x_3$ then $F\left(x_1,x_2,x_3\right) = \left(-2x_1^3, 3x_1^2, x_3x_1\right) = \left(\alpha_1,\alpha_2,\alpha_3\right)$. We observe that: if $x_1 = 0$ then $\alpha_1 = \alpha_2 = \alpha_3 = 0$; If $x_1 \neq 0$ then $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Moreover, since $3x_1^2 = x_2x_3$ and $x_1 \neq 0$, then $x_3 \neq 0$, this implies $\alpha_3 \neq 0$. Furthermore, we have $27\alpha_1^2 = 4\alpha_2^3$. Let

$$\left(\mathscr{S}\right) = \left\{\left(\alpha_1,\alpha_2,\alpha_3\right) \in \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)} : 27\alpha_1^2 = 4\alpha_2^3, \alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_3 \neq 0\right\},$$

then $\left(\mathscr{S}\right) \cup \left\{0\right\}$ is contained in $K_0\left(F\right)$.

So, we have $K_0(F) = (\mathscr{S}) \cup 0\alpha_1 \cup 0\alpha_2$ (see Figure 1).

Notice that $K_0(F)$ does not contain neither $0\alpha_3 \setminus \{0\}$, nor the curve $(\mathscr{C})$ of equation $27\alpha_1^2 = 4\alpha_2^3$ in the plane $(0\alpha_1\alpha_2)$. However $\{0\} \subseteq K_0(F)$ and this is the singular point of $K_0(F)$. So, the set $K_0(F)$ is neither closed, nor smooth, nor pure dimensional.

From the example 4.1, in general the set $K_0(F)$ is not closed, so we cannot stratify $K_0(F)$ in such a way that the stratification satisfies the frontier condition. The following proposition allows us to provide a stratification satisfying the frontier condition of the set $K_0(F) \cup S_F$.

Proposition 4.2. *The set* $K_0(F) \cup S_F$ *is closed. Moreover, we have*

$$K_0(F) \cup S_F = \overline{K_0(F)} \cup S_F.$$

To prove this proposition, we need the three following lemmas.

Lemma 4.3. *For a polynomial mapping* $F: \mathbb{C}^n \to \mathbb{C}^n$, *the set of the solutions of* $|J_F(x)| = 0$ *is closed, where* $|J_F(x)|$ *is the jacobian determinant of* F *at* x.

Chứng minh. Considering a sequence $\{x_k\}$ contained in the set $\{x \in \mathbb{C}^n : |J_F(x)| = 0\}$ such that x_k tends to x_0. Since F is a polynomial mapping, then $|J_F(x)|$ is also a polynomial mapping and $|J_F(x)|$ is continuous. Hence $|J_F(x_k)|$ tends to $|J_F(x_0)|$. Since $|J_F(x_k)| = 0$ for all x_k, we have $|J_F(x_0)| = 0$. So x_0 belongs to the set $\{x : |J_F(x)| = 0\}$. We conclude that the set of the solutions of $|J_F(x)| = 0$ is closed. □

Lemma 4.4. *The set* $\overline{K_0(F)} \setminus K_0(F)$ *is contained in the set* S_F.

Proof. Let $a \in \overline{K_0(F)} \setminus K_0(F)$. There exists a sequence $\{a_k\} \subset K_0(F)$ such that a_k tends to a. Then there exists a sequence $\{x_k\}$ contained in the set $\{x : |J_F(x)| = 0\}$ such that $F(x_k) = a_k$, for all k, where $|J_F(x)|$ is the determinant of the Jacobian

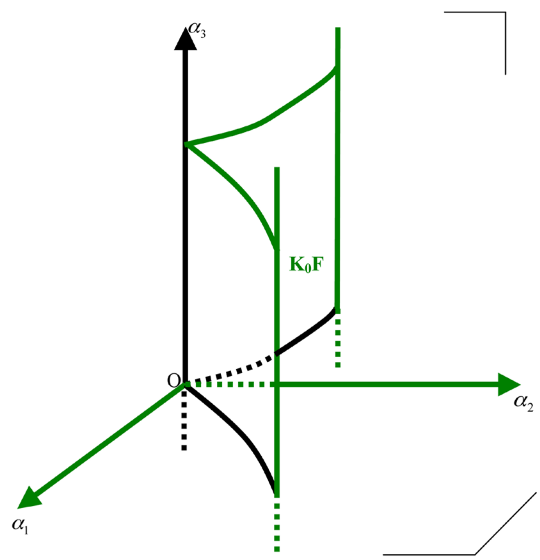

Figure 1. The set $K_0(F)$ of the polynomial mapping $F = (x_1^3 - x_1x_2x_3, x_2x_3, x_3x_1)$.

matrix of F. Assume that the sequence $\{x_k\}$ tends to x_0 and x_0 is finite. Since the set $\{x \in \mathbb{C}^n : |J_F(x)| = 0\}$ is closed, then x_0 belongs to the set $\{x \in \mathbb{C}^n : |J_F(x)| = 0\}$. Moreover, since F is a polynomial mapping, then $F(x_k)$ tends to $F(x_0)$. Hence a_k tends to $F(x_0)$ and $a = F(x_0)$. Since x_0 is finite, then $a \in K_0(F)$, which provides the contradiction. Then x_k tends to infinity and a belongs to S_F. □

Considering now the graph of F in $\mathbb{C}^n \times \mathbb{C}^n$, that means

$$\operatorname{graph} F = \{(a, F(a)) : a \in \mathbb{C}^n\} \subset \mathbb{C}^n \times \mathbb{C}^n.$$

Let $\overline{\operatorname{graph} F}$ be the projective closure of $\operatorname{graph} F$ in $\mathbb{CP}^n \times \mathbb{C}^n$. We have the following lemma:

Lemma 4.5. *The asymptotic set* S_F *of a polynomial mapping* $F : \mathbb{C}^n \to \mathbb{C}^n$ *is the image of the set* $\overline{\operatorname{graph} F} \setminus \operatorname{graph} F$ *by the canonical projection* $\pi_2 : \mathbb{CP}^n \times \mathbb{C}^n \to \mathbb{C}^n$.

This lemma is well-known. In fact, this is the first observation of Jelonek [11] when he studied the geometry of the asymtotic set S_F. We can find this fact, for example, in the introduction of [1]. We provide here a demonstration of this observation.

Proof. Firstly, we show the inclusion $S_F \subset \pi_2(\overline{\operatorname{graph} F} \setminus \operatorname{graph} F)$. Let $a' \in S_F$, there exists a sequence $\{\xi_k\} \subset \mathbb{C}^n$ such that ξ_k tends to infinity and $F(\xi_k)$ tends to a'. The limit of the sequence $\{(\xi_k, F(\xi_k))\}$ is $a^* = (\infty, a')$, where $a^* \in \overline{\operatorname{graph} F} \setminus \operatorname{graph} F \subset (\mathbb{CP}^n \times \mathbb{C}^n)$ and $a' = \pi_2(a^*) \in \pi_2(\overline{\operatorname{graph} F} \setminus \operatorname{graph} F)$.

Now we show the inclusion $\pi_2(\overline{\operatorname{graph} F} \setminus \operatorname{graph} F) \subset S_F$. Let $a' \in \pi_2(\overline{\operatorname{graph} F} \setminus \operatorname{graph} F)$, then there exists $a^* = (a, a') \in \overline{\operatorname{graph} F} \setminus \operatorname{graph} F$ such that $a^* \in \overline{\operatorname{graph} F}$ but $a^* \notin \operatorname{graph} F$. Then we have $a' \neq F(a)$. Moreover, there exists a sequence $\{(\xi_k, F(\xi_k))\} \subset \operatorname{graph} F$ such that $(\xi_k, F(\xi_k))$ tends to (a, a'). Hence the sequence ξ_k tends to a and $F(\xi_k)$ tends to a'. Since F is a polynomial mapping, then $F(\xi_k)$ tends to $F(a)$. But $a' \neq F(a)$, then $a = \infty$, and ξ_k tends to infinity. Thus we have $a' \in S_F$. □

We prove now the proposition 4.2.

Proof. By the lemma 4.5, the set S_F is the image of the set $(\overline{\operatorname{graph} F} \setminus \operatorname{graph} F)$ by the canonical projection $\pi_2 : \mathbb{PC}^n \times \mathbb{C}^n \to \mathbb{C}^n$. Then the set S_F is closed. Moreover, we have

$$\overline{K_0(F)} \cup S_F = K_0(F) \cup (\overline{K_0(F)} \setminus K_0(F)) \cup S_F.$$

By the lemma 4.4, we have $\overline{K_0(F)} \setminus K_0(F) \subset S_F$, then $K_0(F) \cup S_F = \overline{K_0(F)} \cup S_F$. Consequently, the set $K_0(F) \cup S_F$ is closed. □

Theorem 4.6. *Let* $F : \mathbb{C}^n \to \mathbb{C}^n$ *be a generically finite polynomial mapping. Let* $(\mathcal{K})$ *be the partition of* $K_0(F)$ *defined by Thom in* [6] *and let* $(\mathcal{S})$ *be the stratification of* S_F *defined in* [2] (*see Theorem* 3.1 *and Corollary* 3.2). *Then there exists a Thom-Mather stratification of the set* $K_0(F) \cup S_F$ *compatible with* $(\mathcal{S})$ *and* $(\mathcal{K})$.

Proof. By the Proposition 4.2, we have $K_0(F) \cup S_F = \overline{K_0(F)} \cup S_F$. So, in order to define a Thom-Mather stratification of $K_0(F) \cup S_F$, we have to define a Thom-Mather stratification of the set $\overline{K_0(F)} \cap S_F$.

Considering the partition $(\mathcal{K})$ of $K_0(F)$ defined by Thom [6] and the stratification $(\mathcal{S})$ of S_F defined in [2]. Notice that:

+ $(\mathcal{K})$ is a local Thom-Mather partition ([6], Theorem 4.B.1).
+ Since F is a generically finite polynomial mapping, then by the Theorem 4.1 in [2] (see Theorem 3.1), $(\mathcal{S})$ is a Whitney stratification. Hence $(\mathcal{S})$ is a Thom-Mather stratification (Theorem 2.6).

We define now a partition of of $\overline{K_0(F)} \cap S_F$, denoted by $(\bar{\mathcal{K}}) \cap (\mathcal{S})$, as follows:

$$(\bar{\mathcal{K}}) \cap (\mathcal{S}) := \{\bar{K} \cap S : K \in (\mathcal{K}), S \in (\mathcal{S})\}.$$

Since $(\mathcal{K})$ is a local Thom-Mather partition, then $(\bar{\mathcal{K}})$ is a Thom-Mather stratification. Since a Thom-Mather stratification is a particular case of a Whitney stratification (Theorem 2.6), then we can use the result in [13], we have $(\bar{\mathcal{K}}) \cap (\mathcal{S})$ is a Thom-Mather stratification (see Tranversal intersection of stratifications in [13], p. 4).

Finally, we define a stratification of $K_0(F) \cup S_F$, denoted by $(\mathcal{K}) \cup (\mathcal{S})$, as follows:

$$\begin{aligned}(\mathcal{K}) \cup (\mathcal{S}) := &\{K \setminus (\bar{\mathcal{K}} \cap S) : K \in (\mathcal{K}), S \in (\mathcal{S})\} \\ &\cup \{S \setminus (\bar{\mathcal{K}} \cap S) : K \in (\mathcal{K}), S \in (\mathcal{S})\} \cup ((\bar{\mathcal{K}}) \cap (\mathcal{S}))\end{aligned}.$$

By the Proposition 4.2, since $K_0(F) \cup S_F$ is closed, then the obtained partition is a Thom-Mather stratification. It is clear that this stratification is compatible with $(\mathcal{S})$ and $(\mathcal{K})$ defined by [6] and [2], respectively. □

Remark 4.7. Another way to define a Thom-Mather stratification of the asymptotic set S_F is to use "*la méthode des façons*" in [14]. In fact, the stratification of the asymptotic set S_F defined by "*la méthode des façons*" is a Thom-Mather stratification (see [15]).

The following example is for making clear the idea "define a partition of the set $K_0(F)$ by *constant rank*" defined by Thom in [6].

Example 4.8. Let us consider the example 4.1: let $F : \mathbb{C}^3_{(x_1,x_2,x_3)} \to \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ be the polynomial mapping such that $F(x_1,x_2,x_3) = (x_1^3 - x_1x_2x_3, x_2x_3, x_3x_1)$.

We provide a partition of the set $K_0(F)$ by "*constant rank*" defined by Thom in [6] of this example, consisted in the five following steps.

1) Step 1: Subdividing the singular set $\operatorname{Sing} F$ of F into subvarieties V^i, where $V^i = \{(x_1,x_2,x_3) \in \mathbb{C}^3 : \operatorname{Rank} J_F(x_1,x_2,x_3) = i\}$. From the example 4.1, we have:

$$V^0 = \{(0,0,0)\},$$

$$V^1 = \{(0,x_2,0) : x_2 \neq 0\},$$

$$V^2 = \{(0,x_2,x_3) : x_3 \neq 0\} \cup \{(x_1,x_2,0) : x_1 \neq 0\} \cup \{(x_1,x_2,x_3) : x_3 \neq 0, x_2x_3 = 3x_1^2\}.$$

2) Step 2: Subdividing the sets V^i in step 1 into smooth varieties. Since V^2 is not smooth, so we need to subdivide V_2 into $V_1^2 := \{(x_1,x_2,x_3) : 3x_1^2 = x_2x_3, x_3 \neq 0\}$, $V_2^2 := \{(0,x_2,x_3) : x_3 \neq 0\}$ and $V_3^2 := \{(x_1,x_2,0) : x_1 \neq 0\}$.

3) Step 3: Making a partition of the set $\operatorname{Sing} F$ from the subsets V_j^i in the steps 1 and 2. Since $V_1^2 \cap V_2^2 = 0x_3 \setminus \{0\}$, so let us consider:

$$V_1'^2 := V_1^2 \setminus 0x_3, \quad V_2'^2 := V_2^2 \setminus 0x_3,$$
$$V_3'^2 := V_3^2 \setminus 0x_3, \quad V_1'^1 := V^1 \setminus \{0\} = 0x_2 \setminus \{0\},$$
$$V_2'^1 = 0x_3 \setminus \{0\}, \quad V'^0 := \{0\}.$$

We get a partition of $\text{Sing}\, F$.

4) Step 4: Computing $\text{Rank}\, J_{F|_{TV_i'^j}}$. We have

$$\text{Rank}\, J_{F|_{TV_1'^2}} = 2,$$
$$\text{Rank}\, J_{F|_{TV_2'^2}} = \text{Rank}\, J_{F|_{TV_3'^2}} = 1,$$
$$\text{Rank}\, J_{F|_{TV_1'^1}} = \text{Rank}\, J_{F|_{TV_2'^1}} = \text{Rank}\, J_{F|_{TV'^0}} = 0.$$

5) Step 5: Computing $W_j^{i,k} := F\left(\left\{x \in V_j'^i : \text{Rank}\, J_{F|_{T_x V_j'^i}} = k\right\}\right)$. We have

$$W^{2,2} = (\mathscr{S}), \quad W_2^{2,1} = 0\alpha_2, \quad W_3^{2,1} = 0\alpha_1, \quad W_1^{1,0} = W_2^{1,0} = W^{0,0} = \{0\}.$$

Recall that $(\mathscr{S}) = \left\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)} : 27\alpha_1^2 = 4\alpha_2^3, \alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_3 \neq 0\right\}$.

Each $W_j^{i,k}$ is a k-dimensional smooth variety of $K_0(F)$. So we get a partition of $K_0(F)$ by smooth varieties (see Figure 2).

Remark 4.9. If $K_0(F) \setminus S_F$ is smooth, then we can define easily a stratification of the set $K_0(F) \cup S_F$. But in general, $K_0(F) \setminus S_F$ is not smooth. We can check this fact in the following example:

$$F : \mathbb{C}^3 \to \mathbb{C}^3, \qquad F = \left(x_1^3 - x_1x_2x_3, x_2x_3, x_3\right).$$

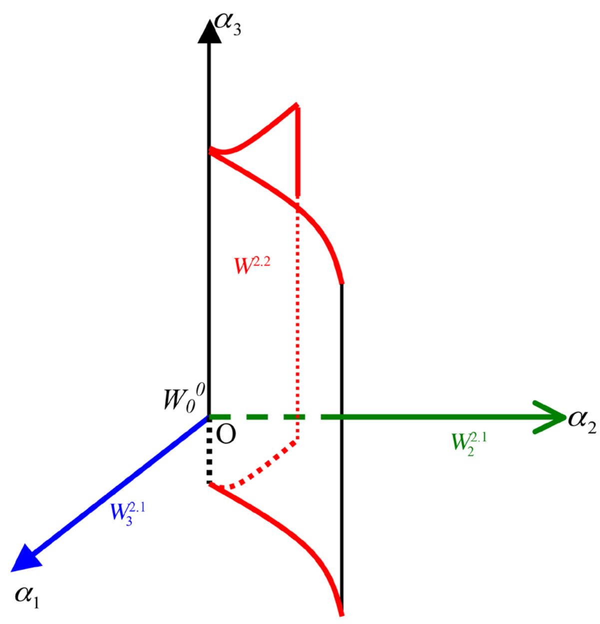

Figure 2. The partition of $K_0(F)$ defined by Thom of the polynomial mapping $F = \left(x_1^3 - x_1x_2x_3, x_2x_3, x_3x_1\right)$.

Remark 4.10. In all examples in this paper and in [16], the set $K_0(F)\cup S_F$ is pure dimensional if F is dominant. So we can suggest the following conjecture:

Conjecture 4.11. *If* $F:\mathbb{C}^n\to\mathbb{C}^n$ *is a dominant polynomial mapping then the set* $K_0(F)\cup S_F$ *is pure dimensional.*

Notice that the above conjecture is not true in the case F is not dominant, as shown in the following example:

$$F:\mathbb{C}^3\to\mathbb{C}^3,\qquad F=\left(x_1^2-x_2x_3,x_2-x_3,x_1-x_3\right).$$

5. The Homology and Intersection Homology of the Variety V_F

In this section, we prove the principal results of the paper, which are the two following theorems.

Theorem 5.1. *Let* $F:\mathbb{C}^2\to\mathbb{C}^2$ *be a non-proper generic dominant polynomial mapping. Then for any variety* V_F *associated to* F, *we have*

1) $H_2(V_F)\neq 0$,
2) $IH_2^{\bar{p}}(V_F)\neq 0$ for any perversity $\bar{p}$,
3) $IH_2^{\bar{p}}(V_F)\neq 0$ for some perversity $\bar{p}$.

Theorem 5.2. *Let* $F:\mathbb{C}^n\to\mathbb{C}^n$ $(n\geq 3)$ *be a non-proper generic dominant polynomial mapping. If* $\mathrm{Rank}_{\mathbb{C}}\left(D\hat{F}_i\right)_{i=1,\cdots,n}\geq n-1$, *where* $\hat{F}_i$ *is the leading form of* F_i, *then for any variety* V_F *associated to* F, *we have*

1) $H_2(V_F)\neq 0$,
2) $IH_2^{\bar{p}}(V_F)\neq 0$ for any (or some) perversity $\bar{p}$,
3) $IH_{2n-2}^{\bar{p}}(V_F)\neq 0$, for any (or some) perversity $\bar{p}$.

Before proving these theorems, we recall some necessary definitions and lemmas.

Definition 5.3. *A semi-algebraic family of sets* (*parametrized by* $\mathbb{R}$) is a semi-algebraic set $A\subset\mathbb{R}^n\times\mathbb{R}$, the last variable being considered as parameter.

Remark 5.4. A semi-algebraic set $A\subset\mathbb{R}^n\times\mathbb{R}$ will be considered as a family parametrized by $t\in\mathbb{R}$. We write A_t, for "the fiber of A at t", *i.e.*:

$$A_t:=\left\{x\in\mathbb{R}^n:(x,t)\in A\right\}.$$

Lemma 5.5 ([1] lemma 3.1). Let β be a j-cycle and let $A\subset\mathbb{R}^n\times\mathbb{R}$ be a compact semi-algebraic family of sets with $|\beta|\subset A_t$ for any t. Assume that $|\beta|$ bounds a $(j+1)$-chain in each A_t, $t>0$ small enough. Then β bounds a chain in A_0.

Definition 5.6 ([1]). Given a subset $X\subset\mathbb{R}^n$, we define the "*tangent cone at infinity*", called "*contour apparent à l'infini*" in [16] by:

$$C_\infty(X):=\left\{\lambda\in\mathbb{S}^{n-1}(0,1)\text{ such that }\exists\eta:(t_0,t_0+\varepsilon]\to X\text{ semi-algebraic},\right.$$
$$\left.\lim_{t\to t_0}\eta(t)=\infty,\lim_{t\to t_0}\frac{\eta(t)}{|\eta(t)|}=\lambda\right\}$$

Lemma 5.7 ([2] lemma 4.10). Let $F=(F_1,\cdots,F_m):\mathbb{R}^n\to\mathbb{R}^m$ be a polynomial mapping and V be the zero locus of $\hat{F}:=\left(\hat{F}_1,\cdots,\hat{F}_m\right)$, where $\hat{F}_i$ is the leading form of F_i. If X is a subset of $\mathbb{R}^n$ such that $F(X)$ is bounded, then $C_\infty(X)$ is a subset of $\mathbb{S}^{n-1}(0,1)\cap V$, where $V=\hat{F}^{-1}(0)$.

Proof. (*Proof of the Theorem* 5.1).

The proof of this theorem consists into three steps:

+ In the first step, we use the Transversality Theorem of Thom (see [17], p. 34): if F is non-proper *generic* dominant polynomial mapping, we can construct an adapted $(2,\bar{p})$-allowable chain in generic position providing non triviality of homology and intersection homology of the variety V_F, for any perversity $\bar{p}$.
+ In the second step, we use the same idea than in [1] to prove that the chain that we create in the first step cannot bound a 3-chain in V_F.
+ In the third step, we provide an explicit stratification of the singular set of V_F, so that the properties of the homology and the intersection homology of the set V_F in the theorem do not change for all the varieties V_F associated to F.

a) Step 1: Let $F:\mathbb{C}^2\to\mathbb{C}^2$ be a generic polynomial mapping, then $\dim_{\mathbb{R}} V_F=4$ ([1], proposition 2.3). Assume that $S_F\neq\emptyset$. We claim that $S_F\cap K_0(F)\neq\emptyset$. In fact, since F is dominant, then by the Theorem 2.11, we have $\dim_{\mathbb{R}} S_F=1$. Moreover, since F is generic then $\dim_{\mathbb{C}} K_0(F)=1$. Thanks again to the genericity of F, we have $S_F\cap K_0(F)\neq\emptyset$. Let $x_0\in S_F\setminus K_0(F)$, then there exists a complex Puiseux arc γ in $\mathbb{R}^4$, where

$$\gamma: D(0,\eta)\to\mathbb{R}^4,\quad \gamma=uz^{\alpha}+\cdots,$$

(with α is a negative integer, u is an unit vector of $\mathbb{R}^4$ and $D(0,\eta)$ a small 2-dimensional disc centered in 0 and radius η) tending to infinity in such a way that $F(\gamma)$ converges to x_0. Then, the mapping $h_F\circ\gamma$, where $h_F=(F,\psi_1,\cdots,\psi_p)$ (see the construction of the variety V_F, Section 3) provides a singular 2-simplex in V_F that we will denote by c. We prove now the simplex c is $(\bar{p},2)$-allowable for any perversity $\bar{p}$. In fact, since $\dim_{\mathbb{C}} S_F=1$, the condition (see 2.8)

$$0=\dim_{\mathbb{R}}\{x_0\}=\dim_{\mathbb{R}}\left(\left(S_F\times\{0_{\mathbb{R}^p}\}\right)\cap|c|\right)\leq 2-\alpha+p_{\alpha},$$

where $\alpha=\operatorname{codim}_{\mathbb{R}} S_F=2$ holds for any perversity $\bar{p}$ since $p_2=0$.

Notice that $V_F\setminus S_F$ is not smooth in general. In fact, $\operatorname{Sing}(V_F\setminus S_F)\subset K_0(F)$. Let us consider a stratum V_i of the stratification of $K_0(F)\cup S_F$ defined in the Theorem 4.6 and denote $\beta=\operatorname{codim}_{\mathbb{R}} V_i$. Assume that $\beta\geq 2$, we can choose the Puiseux arc γ such that c lies in the regular part of $V_F\setminus\left(S_F\times\{0_{\mathbb{R}^q}\}\right)$, thanks to the genericity of F. In fact, this comes from the generic position of transversality. So c is $(\bar{p},2)$-allowable. Hence we only need to consider the cases $\beta=0$ and $\beta=1$. Then:

1) If c intersects V_i: since $x_0\in S_F\setminus K_0(F)$, then we have $0\leq\dim_{\mathbb{R}}(V_i\cap|c|)\leq 1$. Considering the condition

$$\dim_{\mathbb{R}}(V_i\cap|c|)\leq 2-\beta+p_{\beta}. \tag{5.8}$$

We see that $2-\beta+p_{\beta}\geq 1$, for $\beta=0$ and $\beta=1$. So the condition (5.8) holds.

2) If c does not meet V_i, then the condition

$$-\infty=\dim_{\mathbb{R}}\emptyset=\dim_{\mathbb{R}}(V_i\cap|c|)\leq 2-\beta+p_{\beta}$$

holds always.

In conclusion, the simplex c is $(\overline{p},2)$-allowable for any perversity $\overline{p}$.

We can always choose the Puiseux arc such that the support of ∂c lies in the regular part of $V_F \setminus \left(S_F \times \{0_{\mathbb{R}^p}\}\right)$ and ∂c bounds a 2-dimensional singular chain e of $\operatorname{Reg}\left(V_F \setminus \left(S_F \times \{0_{\mathbb{R}^p}\}\right)\right)$. So $\sigma = c - e$ is a $(\overline{p},2)$-allowable cycle of V_F.

b) Step 2: We claim that σ cannot bound a 3-chain in V_F. Assume otherwise, *i.e.* assume that there is a 3-chain in V_F, satisfying $\partial\tau = \sigma$. Let

$$A := h_F^{-1}\left(|\sigma| \cap \left(V_F \setminus \left(S_F \times \{0_{\mathbb{R}^p}\}\right)\right)\right),$$

$$B := h_F^{-1}\left(|\tau| \cap \left(V_F \setminus \left(S_F \times \{0_{\mathbb{R}^p}\}\right)\right)\right).$$

By definition 5.6, the sets $C_\infty(A)$ and $C_\infty(B)$ are subsets of $\mathbb{S}^3(0,1)$. Observe that, in a neighborhood of infinity, A coincides with the support of the Puiseux arc γ. The set $C_\infty(A)$ is equal to $\mathbb{S}^1 \cdot a$ (denoting the orbit of $a \in \mathbb{C}^2$ under the action of $\mathbb{S}^1$ on $\mathbb{C}^2$, $(\mathrm{e}^{i\eta}, z) \mapsto \mathrm{e}^{i\eta} z$). Let V be the zero locus of the leading forms $\hat{F} := (\hat{F}_1, \hat{F}_2)$. Since $F(A)$ and $F(B)$ are bounded, then by lemma 5.7, the sets $C_\infty(A)$ and $C_\infty(B)$ are subsets of $V \cap \mathbb{S}^3(0,1)$.

For R large enough, the sphere $\mathbb{S}^3(0,R)$ with center 0 and radius R in $\mathbb{R}^4$ is transverse to A and B (at regular points). Let

$$\sigma_R := \mathbb{S}^3(0,R) \cap A, \qquad \tau_R := \mathbb{S}^3(0,R) \cap B.$$

Then σ_R is a chain bounding the chain τ_R. Considering a semi-algebraic strong deformation retraction $\Phi : W \times [0;1] \to \mathbb{S}^1 \cdot a$, where W is a neighborhood of $\mathbb{S}^1 \cdot a$ in $\mathbb{S}^3(0,1)$ onto $\mathbb{S}^1 \cdot a$. Considering R as a parameter, we have the following semi-algebraic families of chains:

1) $\tilde{\sigma}_R := \dfrac{\sigma_R}{R}$, for R large enough, then $\tilde{\sigma}_R$ is contained in W,
2) $\sigma'_R = \Phi_1(\tilde{\sigma}_R)$, where $\Phi_1(x) := \Phi(x,1), \quad x \in W$,
3) $\theta_R = \Phi(\tilde{\sigma}_R)$, we have $\partial\theta_R = \sigma'_R - \tilde{\sigma}_R$,
4) $\theta'_R = \tau_R + \theta_R$, we have $\partial\theta'_R = \sigma'_R$.

As, near infinity, σ_R coincides with the intersection of the support of the arc γ with $\mathbb{S}^3(0,R)$, for R large enough the class of σ'_R in $\mathbb{S}^1 \cdot a$ is nonzero.

Let $r = 1/R$, consider r as a parameter, and let $\{\tilde{\sigma}_r\}$, $\{\sigma'_r\}$, $\{\theta_r\}$ as well as $\{\theta'_r\}$ the corresponding semi-algebraic families of chains.

Let us denote by $E_r \subset \mathbb{R}^4 \times \mathbb{R}$ the closure of $|\theta_r|$, and set $E_0 := \left(\mathbb{R}^4 \times \{0\}\right) \cap E$. Since the strong deformation retraction Φ is the identity on $C_\infty(A) \times [0,1]$, we see that

$$E_0 \subset \Phi\left(C_\infty(A) \times [0,1]\right) = \mathbb{S}^1 \cdot a \subset V \cap \mathbb{S}^3(0,1).$$

Let us denote by $E'_r \subset \mathbb{R}^4 \times \mathbb{R}$ the closure of $|\theta'_r|$, and set $E'_0 := \left(\mathbb{R}^4 \times \{0\}\right) \cap E'$. Since A bounds B, then $C_\infty(A)$ is contained in $C_\infty(B)$. We have

$$E'_0 \subset E_0 \cup C_\infty(B) \subset V \cap \mathbb{S}^3(0,1).$$

The class of σ'_r in $\mathbb{S}^1 \cdot a$ is, up to a product with a nonzero constant, equal to the generator of $\mathbb{S}^1 \cdot a$. Therefore, since σ'_r bounds the chain θ'_r, the cycle $\mathbb{S}^1 \cdot a$ must bound a chain in $|\theta'_r|$ as well. By Lemma 5.5, this implies that $\mathbb{S}^1 \cdot a$ bounds a chain in E'_0 which is included in $V \cap \mathbb{S}^3(0,1)$.

The set V is a projective variety which is an union of cones in $\mathbb{R}^4$. Since $\dim_{\mathbb{C}} V \leq 1$, so $\dim_{\mathbb{R}} V \leq 2$ and $\dim_{\mathbb{R}} V \cap \mathbb{S}^3(0,1) \leq 1$. The cycle $\mathbb{S}^1 \cdot a$ thus bounds a chain in $E'_0 \subseteq V \cap \mathbb{S}^3(0,1)$, which is a finite union of circles, that provides a contradiction.

c) Step 3: We prove at first the afirmation: If F is dominant, then F is generically finite. Recall that F is generically finite if there exists a subset $U \subset \mathbb{C}^n$ in the target space such that U is dense in $\mathbb{C}^n$ and for any $a \in U$, the cardinality of $F^{-1}(a)$ is finite. To prove that F is generically finite, we do two steps:

+ Prove that $\overline{F(\mathbb{C}^n)} = F(\mathbb{C}^n) \cup S_F$. In fact, by the definition of S_F (see (2.12)), it is clear that $F(\mathbb{C}^n) \cup S_F \subset \overline{F(\mathbb{C}^n)}$. Take now $a \in \overline{F(\mathbb{C}^n)}$, then there exists a sequence $\{\xi_k\} \subset \mathbb{C}^n$ such that $F(\xi_k)$ tends to a. If ξ_k tends to infinity, then a belongs to S_F. If ξ_k does not tend to infinity, assume that ξ_k tends to $\xi \in \mathbb{C}^n$. Since F is a polynomial mapping and hence is continuous, then $F(\xi_k)$ tends to $F(\xi)$. Moreover $\mathbb{C}^n$ is a Hausdorff space, then $F(\xi) = a$. This implies that $a \in F(\mathbb{C}^n)$. Consequently, we have $\overline{F(\mathbb{C}^n)} \subset F(\mathbb{C}^n) \cup S_F$. We conclude that $\overline{F(\mathbb{C}^n)} \subset F(\mathbb{C}^n) \cup S_F$.
+ Indicate that there exists a dense subset U in the target space $\mathbb{C}^n$ in the target space such that for any $a \in U$, the cardinality of $F^{-1}(a)$ is finite. In fact, let

$$U = \mathbb{C}^n \setminus S_F.$$

Since F is dominant, then by the Theorem 2.13, the dimension of S_F is $n-1$. Hence U is dense in the the target space $\mathbb{C}^n$. With each $a \in U$, since $a \notin S_F$, and since F is a polynomial mapping, then the cardinality of $F^{-1}(a)$ is finite (see, for example, the Proposition 6 of [11]). Then F is generically finite.

Since F is generically finite, then by the Theorem 4.6, there exists an explicit Thom-Mather stratification of the set $K_0(F) \cup S_F$, which is compatible with the Thom-Mather partition of $K_0(F)$ defined by [6] and is compatible with the Whitney stratification of S_F defined in [2]. In other words, there exists an explicit Thom-Mather stratification of the variety V_F, since $K_0(F) \cup S_F$ is the singular part of the set V_F. We use this stratification to calculate the intersection homology of the variety V_F. Since the obtained stratification is a Thom-Mather stratification, then it is a locally topologically trivial stratification (Theorem 2.6). Hence the intersection homology of the variety V_F does not depend on the stratification of V_F (Theorem 2.9). Consequently, the properties of the homology and the intersection homology of the variety V_F in the theorem do not depend on the choice of the varieties associated to the polynomial mapping F. □

We prove now the Theorem 5.2.

Proof. (*Proof of the Theorem* 5.2).

Assume that $F : \mathbb{C}^n \to \mathbb{C}^n$ $(n \geq 3)$ is a non-proper generic dominant polynomial mapping. Similarly to the previous proof, we have:

- Since F is dominant, then by the Theorem 2.13, we have $\dim_{\mathbb{C}} S_F = n-1$. Moreover, since F is generic then $\dim_{\mathbb{C}} K_0(F) = n-1$. Thanks again to the genericity of F, we have $S_F \cap K_0(F) \neq \varnothing$. Let $x_0 \in S_F \setminus K_0(F)$, then there exists a complex Puiseux arc γ in $\mathbb{R}^{2n}$, where

$$\gamma : D(0,\eta) \to \mathbb{R}^{2n}, \quad \gamma = uz^{\alpha} + \cdots,$$

(with α is a negative integer and u is an unit vector of $\mathbb{R}^{2n}$) tending to infinity such a way that $F(\gamma)$ converges to x_0. Since $x_0 \in S_F \setminus K_0(F)$ and F is generic, then we can choose the arc Puiseux γ in generic position, that means the simplex c is $(\bar{p},2)$-allowable for any perversity $\bar{p}$.

- Now, with the same notations than the above proof, we have: Since $\operatorname{rank}_{\mathbb{C}}\left(D\hat{F}_i\right)_{i=1,\cdots,n} \geq n-1$ then $\operatorname{corank}_{\mathbb{C}}\left(D\hat{F}_i\right)_{i=1,\cdots,n} \leq 1$. Moreover since $\dim_{\mathbb{C}} V = \operatorname{corank}_{\mathbb{C}}\left(D\hat{F}_i\right)_{i=1,\cdots,n}$ then $\dim_{\mathbb{R}} V \leq 2$ and $\dim_{\mathbb{R}} V \cap \mathbb{S}^{2n-1}(0,1) \leq 1$. The cycle $\mathbb{S}^1 \cdot a$ bounds a chain in $E_0' \subseteq V \cap \mathbb{S}^{2n-1}(0,1)$, which is a finite union of circles, that provides a contradiction.

Hence, we get the facts (1) and (2) of the theorem. Moreover, from the Goresky-MacPherson Poincaré Duality Theorem (Theorem 2.11), we have

$$IH_2^{\bar{p},c}(V_F) = IH_{2n-4}^{\bar{q},cl}(V_F)$$

where $\bar{p}$ and $\bar{q}$ are complementary perversities. Since the chain σ that we create in the proof of the Theorem 5.1 can be either a chain with compact supports or a chain with closed supports, so we get the fact (3) of the theorem. □

Remark 5.9. The properties of the homology and intersection homology in the Theorem 5.1 and 5.2 hold for both compact supports and closed supports.

Remark 5.10. From the proofs of the Theorems 5.1 and 5.2, we see that the properties of the intersection homology in these theorems do not hold if F is not dominant. The reason is that the Theorem 2.11 is not true if F is not dominant and then the condition (5.8) does not hold. However, the properties of the homology hold even if F is not dominant. So we have the two following corollaries.

Corollary 5.11. Let $F:\mathbb{C}^2 \to \mathbb{C}^2$ be a non-proper generic polynomial mapping, then $H_2(V_F) \neq 0$.

Corollary 5.12. Let $F:\mathbb{C}^n \to \mathbb{C}^n$ $(n \geq 3)$ be a non-proper generic polynomial mapping. If $\operatorname{Rank}_{\mathbb{C}}\left(D\hat{F}_i\right)_{i=1,\cdots,n} \geq n-1$, where $\hat{F}_i$ is the leading form of F_i, then $H_2(V_F) \neq 0$.

Remark 5.13. In the previous papers [1] and [2], the condition "F is nowhere vanishing Jacobian" (see Theorems 3.3 and 3.4) implies F is dominant. Hence, the condition "F is dominant" in the Theorems 5.1 and 5.2 guarantees the condition of dimension of the set S_F (see Theorem 2.13). Moreover, we need this condition in this paper also to be free ourself from the condition $K_0(F) = \varnothing$, since the condition of dimension of S_F when F is dominant also guarantees the (generic) tranversal position of the $(2,\bar{p})$-allowable chain which provides non triviality of homology and intersection homology of the variety V_F when $K_0(F) \neq \varnothing$ in Theorems 5.1 and 5.2.

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