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Banach Limits Revisited*

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Abstract

Order unit normed linear spaces are a special type of regularly ordered normed linear spaces and therefore the first section is a short collection of the fundamental results on this type of normed linear spaces. The connection between order unit normed linear spaces and base normed linear spaces within the category of regularly ordered normed linear spaces is described in Section 2, and Section 3 at last, contains the results on Banach limits in an arbitrary order unit normed linear space. It is shown that the original results on Banach limits are valid for a greater range.

Keywords

Order Unit Normed Spaces, Base Normed Spaces, Banach Limits

1. Introduction

Most, if not all, publications where Banach limits are investigated take place in an order unit normed real linear space. Order unit normed linear spaces are a special type of regularly ordered normed linear spaces and therefore the first section is a short collection of the fundamental results on this type of normed linear spaces, for the reader's convenience. The connection between order unit normed linear spaces and base normed linear spaces within the category of regularly ordered normed linear spaces is described in Section 2, and Section 3 at last, contains the results on Banach limits in an arbitrary order unit normed linear space. It is shown that the original results on Banach limits are valid in a for greater range. For a further generalisation of vector valued Banach limits in a different direction we refer to a recent paper of R.Armario, F. Kh. Garsiya-Pacheko and F. Kh Peres-Fernandes [1].

2. Regularly Ordered Normed Linear Spaces

An ordered normed linear space E with order $" \leq "$, norm $\|\square\|_E$ and order cone

*Dedicated to Reinhard Börger, a brilliant and enthusiastic mathematician full of new ideas.

$C(E)$ is called *regularly ordered* iff the cone $C(E)$ is $\|\square\|_E$-closed and proper and $\|\square\|_E$ is a Riesz norm, *i.e.* if

(Ri 1) For $x, y \in E, -y \le x \le y$ implies $\|x\|_E \le \|y\|_E$, *i.e.* $\|\square\|_E$ is absolutely monotone, and

(Ri 2) For $x \in E$ with $\|x\|_E < 1$ there exists a $y \in E$ with $\|y\|_E < 1$ and $-y \le x \le y$ hold. (see [2] [3]).

Lemma 1. *Let, for an ordered linear space* E *with proper and* $\|\square\|_E$ *-closed cone* $C(E)$ (Ri 1) *hold. Then each of the following two conditions is equivalent to* (Ri 2)

(Ri 3) *For* $x \in E$ *and* $\varepsilon > 0$ *there exists a* $y \in E$ *such that* $-y \le x \le y$ *and* $\|y\|_E < \|x\|_E + \varepsilon$ *hold.*

(Ri 4) *For any* $x \in E$

$$\|x\|_E = \inf\left\{\|y\|_E \mid -y \le x \le y\right\},$$

holds.

Proof. The proof is straightforward. Condition (Ri 2) implies that $C(E)$ generates E If (Ri 2) holds, then for $x \in E$ and $\varepsilon > 0$ there is $y \in E$ with

$$-y \le \frac{x}{\|x\|_E + \varepsilon} \le y,$$

hence (Ri 3) is proved for $y_0 := \left(\|x\|_E + \varepsilon\right) y$.

(Ri 3) implies that $C(E)$ generates E and (Ri 1) implies

$$\|x\|_E \le \inf\left\{\|y\|_E \middle| -y \le x \le y\right\} =: \|x\|_0 .$$

Because of (Ri 3), for $x \in E$ and $\varepsilon > 0$ there is a $y \in E$ such that $-y \le x \le y$ and $\|y\|_E < \|x\|_E + \varepsilon$, for any $\varepsilon > 0$ and hence $\|x\|_0 - \varepsilon \le \|x\|_E \le \|x\|_0$ proving (Ri 4). Moreover, (Ri 4) obviously implies (Ri 2) which completes the proof. □

In [3] K. Ch. Min introduced regularly ordered normed spaces as a natural and canonical generalization of Riesz spaces. A crucial point in this generalization was the definition of the corresponding homomorphisms compatible and most closely related to the structure of these spaces, such that, in addition, the set of these special homomorphisms is again a regularly ordered normed linear space in a canonical way. This is done by

Definition 1. *If* $E_i, i = 1,2$ *are regularly ordered linear spaces a bounded linear mapping* $f : E_1 \to E_2$ *is called* positive *iff* $f\left(C\left(E_1\right)\right) \subset C\left(E_2\right)$ *holds. A bounded linear mapping is called* regular *iff it can be expressed as the difference of two positive linear mappings* [3].

The set

$$\mathrm{Reg-Ord}\left(E_1, E_2\right) := \left\{ f \mid f : E_1 \to E_2 \text{ regular linear mapping} \right\}$$

is a linear space by the obvious operations. One introduces the cone

$$C\left(\mathrm{Reg-Ord}\left(E_1, E_2\right)\right) := \left\{ f \mid f \in \mathrm{Reg-Ord}\left(E_1, E_2\right), f\left(C\left(E_1\right)\right) \subset C\left(E_2\right)\right\}$$

which is obviously proper and generates $\mathrm{Reg-Ord}\left(E_1, E_2\right)$. One often writes $x_1 \ge 0$

as abbreviation for $x_1 \in C(E_1)$ and consequently calls an $f \in \mathrm{Reg}-\mathrm{Ord}(E_1, E_2)$ positive and writes $f \geq 0$, if $f(x_1) \geq 0$, for $x_1 \geq 0$ in E_1, *i.e.* $x_1 \in C(E_1)$. The positive part of the unit ball in a regularly ordered space E with norm $\|\square\|_E$ is denoted by

$$\Delta(E) = C(E) \cap \bigcirc(E) \ , \ \bigcirc(E) = \left\{ x \middle| x \in E \ \text{ and } \ \|\square\|_E \leq 1 \right\}.$$

Lemma 2. *Let* E_i *be regularly ordered normed linear spaces with norm* $\|\square\|_i$ *and cone* $C(E_i), i = 1, 2$. *If* $g \in C(\mathrm{Reg}-\mathrm{Ord}(E_1, E_2))$ *and* $\|g\|_\infty$ *denotes the usual supremum norm, then*

$$\|g\|_\infty := \sup\left\{ \|g(x_1)\|_2 \middle| x_1 \in \Delta(E_1) \right\}$$

holds.

Proof. For $x_1 \in E_1$ with $\|x_1\|_1 < 1$ there is $y_1 \in E_1, \|y_1\| < 1$, with $-y_1 \leq x_1 \leq y_1$ which implies

$$-g(y_1) \leq g(x_1) \leq g(y_1)$$

and

$$\|g(x_1)\|_2 \leq \|g(y_1)\|_2 ,$$

hence

$$\begin{aligned} \|g\|_\infty &= \sup\left\{ \|g(x_1)\|_2 \middle| \|x_1\|_1 \leq 1 \right\} \\ &\leq \sup\left\{ \|g(y_1)\|_2 \middle| \|y_1\|_1 \leq 1, y_1 \in C(E_1) \right\} \\ &= \sup\left\{ \|g(y_1)\|_2 \middle| y_1 \in \Delta(E_1) \right\} \\ &\leq \sup\left\{ \|g(y_1)\|_2 \middle| \|y_1\|_1 \leq 1 \right\} = \|g\|_\infty . \end{aligned}$$

□

Now, we proceed to define the norm $\|\square\|^*$ in the space $\mathrm{Reg}-\mathrm{Ord}(E_1, E_2)$ by

$$\|f\|^* := \inf\left\{ \|g\|_\infty \middle| -g \leq f \leq g \right\}. \tag{*}$$

Proposition 1. *For regularly ordered normed spaces* $E_1, E_2, \|\square\|^*$ *is a Riesz norm on* $\mathrm{Reg}-\mathrm{Ord}(E_1, E_2)$ *and makes* $\mathrm{Reg}-\mathrm{Ord}(E_1, E_2)$ *a regularly ordered normed linear space. For* $f \geq 0$ $\|f\|^* = \|f\|_\infty$ *holds and in general*

$$\|f\|_\infty \leq \|f\|^* .$$

Proof. The proof that $\|\square\|^*$ is a seminorm is straightforward. In order to show that $\|\square\|_\infty \leq \|\square\|^*$ one starts with $f, g \in \mathrm{Reg}-\mathrm{Ord}(E_1, E_2)$ and $-g \leq f \leq g$. Let $\|x_1\|_1 < 1$ and $-y_1 \leq x_1 \leq y_1$ with $\|y_1\|_1 < 1, x_1, y_1 \in E_1$. Then $x_1 + y_1 \geq 0$ follows and

$$-g(x_1 + y_1) \leq f(x_1 + y_1) \leq g(x_1 + y_1). \tag{i}$$

Using $g - f \geq 0$ and $y_1 - x_1 \geq 0$ one obtains in the same way

$$-g(y_1 - x_1) \leq f(y_1 - x_1) \leq g(y_1 - x_1)$$

and, multiplying by -1

$$-g(y_1 - x_1) \le f(x_1 - y_1) \le g(y_1 - x_1). \tag{ii}$$

Adding (i) and (ii) yields

$$-g(y_1) \le f(x_1) \le g(y_1)$$

hence

$$\|f(x_1)\|_2 \le \|g(y_1)\|_2$$

and

$$\begin{aligned}\|f\|_\infty &= \sup\left\{\|f(x_1)\|_2 \,\middle|\, \|x_1\|_1 \le 1\right\} \\ &\le \sup\left\{\|g(y_1)\|_2 \,\middle|\, \|y_1\|_1 \le 1 \quad \text{and} \quad -y_1 \le x_1 \le y_1\right\} \le \|g\|_\infty .\end{aligned}$$

Now $(*)$ yields $\|f\|_\infty \le \|f\|^*$, *i.e.* $\|\square\|^*$ is a norm. If $f \ge 0$ then

$$\|f\|^* := \inf\left\{\|g\|_\infty \mid 0 \le f \le g\right\} \le \|f\|_\infty ,$$

hence $\|f\|^* = \|f\|_\infty$ and $\|\square\|^*$ is a Riesz norm because of Lemma 1, (Ri 4) and the definition of $(*)$. $C(\mathrm{Reg}-\mathrm{Ord}(E_1, E_2))$ is obviously $\|\square\|_\infty$-closed and therefore also $\|\square\|^*$-closed because of $\|f\|_\infty \le \|f\|^*$.

$\square$

In the following $\|\square\|^*$ will always denote this norm of regular linear mappings. Note that Reg-Ord is a symmetric, complete and cocomplete monoidal closed category and the inner hom-functor $\mathrm{Reg}-\mathrm{Ord}(\square,\square)$ has as an adjoint, the tensor product [3].

3. Order Unit and Base Ordered Normed Linear Spaces

The order unit normed linear spaces are a special type of regularly ordered normed linear spaces , as are the base normed linear spaces [3] [4]. For investigating a special type of mathematical objects, however, it is always best to use the type of mappings most closely related to the special structure of the objects (the Bourbaki Principle). Hence, for investigating order unit normed spaces we do not look at the full subcategory of Reg-Ord generated by the order unit normed spaces but introduce a more special type of regular linear mappings. The same method, by the way, has been successful for another type of regularly ordered spaces, namely the base normed (Banach) spaces (cp. [3] [5] [6]).

Definition 2. *For two order unit normed linear spaces* E_i *with order unit* $e_i, i = 1, 2,$ *define*

$$Bs_0(E_1, E_2) := \left\{ f \mid f \in \mathrm{Reg}-\mathrm{Ord}(E_1, E_2),\ f \ge 0 \quad \text{and} \quad f(e_1) = e_2 \right\}$$

and

$$C_0(E_1, E_2) := R_+ Bs_0(E_1, E_2).$$

Proposition 2. *Let* E_1, E_2 *be order unit spaces with order units* e_1, e_2. *Then*

i) $Bs_0(E_1, E_2)$ is a $\|\square\|_\infty$-closed convex base of $C_0(E_1, E_2)$ and $Bs_0(E_1, E_2) \subset \bigcirc_\infty(E_1, E_2)$, the $\|\square\|_\infty$-closed unit ball of $\mathrm{Reg}-\mathrm{Ord}(E_1, E_2)$ in the supremum norm $\|\square\|_\infty$

ii) $C_0(E_1,E_2)$ is a $\|\square\|_\infty$-closed proper subcone of $C(\mathrm{Reg}-\mathrm{Ord}(E_1,E_2))$.

Proof. (1) Let $f_n \in Bs_0(E_1,E_2), n \in N$, and $f \in \mathrm{Reg}-\mathrm{Ord}(E_1,E_2)$ with $\lim\limits_{n\to\infty}\|f_n - f\|_\infty = 0$. Then, for $x_1 \in E_1$, $f(x_1) = \lim\limits_{n\to\infty} f_n(x_1)$ follows which implies $f \ge 0$ and $f(e_1) = e_2$, *i.e.* $f \in Bs_0(E_1,E_2)$ showing that $Bs_0(E_1,E_2)$ is $\|\square\|_\infty$-closed. Now, $f \in Bs_0(E_1,E_2)$ and $-e_1 \le x_1 \le e_1$ imply $-e_2 \le f(x_1) \le e_2$ *i.e.* $\|f\|_\infty \le 1$ and even $\|f\|_\infty = 1$ because $f(e_1) = e_2$, Hence $Bs_0(E_1,E_2) \subset \bigcirc_\infty(E_1,E_2)$ follows, even $Bs_0(E_1,E_2) \subset \partial(\bigcirc_\infty(E_1,E_2))$.

Let $\sum\limits_{i=1}^{n}\alpha_i f_i, \alpha_i \ge 0, 1 \le i \le n, \sum\limits_{i=1}^{n}\alpha_i = 1$, be a convex combination of $f_i \in Bs_0(E_1,E_2)$. Then

$$\sum_{i=1}^{n}\alpha_i f_i \ge 0 \quad \text{and} \quad \left(\sum_{i=1}^{n}\alpha_i f_i\right)(e_1) = \sum_{i=1}^{n}\alpha_i e_2 = e_2,$$

follows, *i.e.*

$$\sum_{i=1}^{n}\alpha_i f_i \in Bs_0(E_1,E_2),$$

which proves that $Bs_0(E_1,E_2)$ is convex.

Now $\alpha f = \beta g$ with $\alpha,\beta > 0$ and $f, g \in Bs_0(E_1,E_2)$ implies $\alpha e_2 = \alpha f(e_1) = \beta g(e_1) = \beta e_2$, and $\alpha = \beta$, *i.e.* $Bs_0(E_1,E_2)$ is a $\|\square\|_\infty$-closed base of $C(E_1,E_2)$ and $0 \notin Bs_0(E_1,E_2)$.

(*ii*) This follows from (i) (see [7], 3.9 p. 128). □

Corollary 1. *For order unit normed linear spaces* $E_i, i = 1,2,$

$$\mathrm{Ord}-\mathrm{Unit}(E_1,E_2) = C(E_1,E_2) - C(E_1,E_2)$$

is a base-normed ordered linear space with base $Bs_0(E_1,E_2)$ *and base norm denoted by* $\|\square\|_0$. $C(E_1,E_2)$ *and* $Bs_0(E_1,E_2)$ *are closed in the base norm* $\|\square\|_0$.

Proof. That $\mathrm{Ord}-\mathrm{Unit}(E_1,E_2)$ is a base normed space follows from Proposition 2 and the definition. That base and cone are base normed closed follows from the fact that they are $\|\square\|_0$.-closed (see Proposition 2) and because the $\|\square\|_0$.-topology is weaker than the $\|\square\|_0$.-topology (see Proposition 2 and [7], 3.8.3, p. 121).

□

Remark 1. *If* $\mathrm{Reg}-\mathrm{Ord}(E_1,E_2)$ *is a Banach space, with the norm* $\|\square\|_0$. *because* $E_i, i = 1,2,$ *are Banach spaces, then* $Bs_0(E_1,E_2)$ *is superconvex* (see [3] [6]) *and* $\mathrm{Reg}-\mathrm{Ord}(E_1,E_2)$ *is a base normed Banach space* (see [3] [4] [7]).

Definition 3. *The order unit normed linear spaces together with the linear mappings* $f: E_1 \to E_2$ *with* $f \in \mathrm{Ord}-\mathrm{Unit}(E_1,E_2)$ *constitute the category* Ord-Unit *of order-unit normed linear spaces which is a not full subcategory of* Reg-Ord.

There is an equally important subcategory of Reg-Ord, the category of based normed linear spaces.

Definition 4. *A base normed ordered linear space* "*base normed linear space*" *for short, is a regular ordered linear space* E *with proper closed cone* $C(E)$ *and norm* $\|\square\|_E$ *which is induced by a base* $Bs(E)$ *of* $C(E)$ (see [4] [7]). *If* $E_i, i = 1,2$ *are* *base normed linear spaces, put*

$$Bs(E_1,E_2):=\{f \mid f:E_1\to E_2 \text{ linear and } f(Bs(E_1))\subset Bs(E_2)\}.$$

The elements of $Bs(E_1,E_2)$ *are monotone mappings,* $Bs(E_1,E_2)$ *is a base set in* $\mathrm{Reg-Ord}(E_1,E_2)$ *and it is* $\|\square\|_\infty$ *-closed. Let* $C(E_1,E_2)$ *denote the proper closed cone generated by* $\mathrm{Bs}(E_1,E_2)$.

$$\mathrm{BN-Ord}(E_1,E_2):=C(E_1,E_2)-C(E_1,E_2)$$

is a base normed space of special mappings from E_1 *to* E_2. *The base normed linear spaces and these linear mappings form the not full subcategory* **BN-Ord** of **Reg-Ord** (*see* [6] [8] [9]), *which is therefore a closed category.*

What remains in this connection is to investigate special morphisms particularly adapted to these subcategories between spaces belonging to two different of these subcategories **Ord-Unit** and **BN-Ord**. We start this with investigating the intersection of these subcategories.

Proposition 3. *Let* $(E,C(E),\|\square\|_E)$ *be a regular ordered normed linear space. Then* $\|\square\|_E$ *is a base and order unit norm iff* $(E,C(E),\|\square\|_E)$ *is isomorphic to* $(R,[0,\infty],|\square|)$ *by a regular positive isomorphism.*

Proof. If $e\in E$ is the order unit and if we omit the index E at the norm, then trivially $\|e\|=1$ and $e>0$ hold. Let $b\in Bs(E)$ and assume $b\neq e$. As $Bs(E)\subset\bigcirc(E)=[-e,e]$ (see [7]), $0<b<e$ holds and $d:=e-b>0$ follows or $e=b+d$, which implies $1=\|e\|=\|b+d\|=\|b\|+\|d\|$, because $\|\square\|$ is additive on $C(E)$. This implies $\|d\|=0$ and hence $e=b$ which gives a contradiction. Therefore $Bs(E)=\{e\}$ and the assertion follows as $C(E)=R_+Bs(E)=R_+e$ and

$$E=C(E)-C(E)=Re.$$

Hence, the isomorphism is $i:E\to R$ defined by $i(\alpha e)=\alpha, \alpha\in R.$ □

It should be noted that this isomorphism is an isomorphism in the category Ord-Unit of order unit normed spaces and also in BN-Ord. So, loosely speaking,

$$\mathrm{Ord-Unit}\cap\mathrm{BN-Ord}=\{R\}.$$

Now the "*general connection*" between Ord-Unit and BN-Ord is investigated via the morphisms:

Proposition 4. *If* E_1 *is a base normed and* E_2 *an order unit normed linear space, then* $\mathrm{Reg-Ord}(E_1,E_2)$ *is an order unit normed linear space.*

Proof. Define $\varepsilon:C(E_1)\to C(E_2)$ by $\varepsilon(Bs(E_1))=\{e_2\}$ and extend ε positive linearly by $\varepsilon(\alpha x_1)=\alpha e_2$, for $\alpha\geq 0, x_1\in Bs(E_1)$, to $\varepsilon:C(E_1)\to C(E_2)$ which can be uniquely extended to $\varepsilon:E_1\to E_2$, a monotone, linear mapping in $\mathrm{Reg-Ord}$ in the usual way. Obviously, $\|\varepsilon\|_\infty=\|\varepsilon\|$, with $\|\square\|$ the Reg-Ord norm, as ε is a positive mapping. Take a $g\in\mathrm{Reg-Ord}(E_1,E_2)$ with $\|g\|\leq 1$, *i.e.* $g(\bigcirc(E_1))\subset\bigcirc(E_2)=[-e_2,e_2]$ and hence $-e_2\leq g(b_1)\leq e_2$ for $b_1\in Bs(E_1)$ or $-\varepsilon(b_1)\leq g(b_1)\leq\varepsilon(b_1)$ whence $-\varepsilon(c_1)\leq g(c_1)\leq\varepsilon(c_1)$ for $c_1\in C(E_1)$. For arbitrary $x_1\in E_1, x_1=c_1-d_1, c_1,d_1\in C(E_1)$ $-\varepsilon(c_1)\leq g(c_1)\leq\varepsilon(c_1)$ and

$-\varepsilon(d_1) \le -g(d_1) \le \varepsilon(d_1)$ follows implying $-\varepsilon(x_1) \le g(x_1) \le \varepsilon(x_1)$ for $x_1 \in E_1$ or $-\varepsilon \le g \le \varepsilon$. This means, for arbitrary $g \ne 0$, that $-\|g\|\varepsilon \le g \le \|g\|\varepsilon$. This shows that ε is an order unit in $\text{Reg}-\text{Ord}(E_1, E_2)$. Denoting the order unit norm by $\|\square\|_0$ $\|g\|_0 \le \|g\|$ follows. □

This is a slightly different version of the proof of Theorem 1 in Ellis [7].

Surprisingly a corresponding result also holds if $E_1 \in$ Ord-Uni and E_2 BN-Ord

Proposition 5. *If* E_1 *is an order unit and* E_2 *is a base normed ordered linear space, then* Reg-Ord (E_1, E_2). *is a base normed ordered linear space.*

Proof. Define

$$Bs_0(E_1, E_2) := \left\{ f \mid f \in C\left(\text{Reg}-\text{Ord}(E_1, E_2)\right) \text{ and } f(e_1) \in Bs(E_2) \right\}$$

where e_1 denotes the order unit of E_1 One shows first that $Bs_0(E_1, E_2)$ is a base set. For this, let $g \in C\left(\text{Reg}-\text{Ord}(E_1, E_2)\right)$ $g \ne 0$, *i.e.* $g > 0$ and $g(e_1) > 0$, implying for

$$f := \frac{1}{\|g(e_1)\|_2} g$$

that is $f > 0$, and $\|f(e_1)\|_2 = 1$, hence $f(e_1) \in Bs(E_2)$. this implies that $Bs_0(E_1, E_2) \ne \varnothing$.

For $f, g \in Bs_0(E_1, E_2)$ and $0 \le \alpha \le 1$, obviously $\alpha f + (1-\alpha) g \in Bs_0(E_1, E_2)$ and $Bs_0(E_1, E_2)$ is convex. Besides, the above proof shows, that any $g \in C\left(\text{Reg}-\text{Ord}(E_1, E_2)\right), g > 0,$ can be written as $g = \alpha f$ with $\alpha \ge 0,\ f \in Bs_0(E_1, E_2)$.

Obviously $0 \notin Bs_0(E_1, E_2)$ and if $\alpha f = \beta g$ with $\alpha, \beta \ge 0$ and $f, g \in Bs(E_1, E_2)$ implying

$$\alpha f(e_1) = \beta g(e_1)$$

from which $\alpha = \beta$ follows because of $f(e_1), g(e_1) \in Bs(E_1, E_2)$ and finally $f = g$. □

It is interesting that by defining the subspaces $\text{Ord}-\text{Unit}(E_1, E_2)$ and $\text{BN}-\text{Ord}(E_1, E_2)$ of $\text{Reg}-\text{Ord}(E_1, E_2)$ for order unit or base normed spaces E_1, E_2, respectively, one gets a number of results which for the bigger space $\text{Reg}-\text{Ord}(E_1, E_2)$ have either not yet been proved or were more difficult to prove because the assumptions for $\text{Reg}-\text{Ord}(E_1, E_2)$ are weaker (see [10] [11]). The Propositions 4 and 5 are an exception because here the general space $\text{Reg}-\text{Ord}(E_1, E_2)$ has the special structure of an order unit or base normed spaces, respectively.

There are different ways to generalize the structure of R in many fields of mathematics. In analysis one is primarily interested in aspects of order, norm and convergence. Now, essentially, R with 1, the usual order and the absolute value (considered as a norm) forms the intersection $\text{Ord}-\text{Unit} \cap \text{BN}-\text{Ord} = \{R\}$, which both generalize R in different (dual) directions. The above results seem to indicate that the order unit spaces are at least as important as generalizations of R as the base normed spaces while in many publications the latter type seems to play the dominant role. Propositions 4 and 5 are particularly interesting because the hom-spaces have a special structure if the arguments do not belong to the same of the two subcategories $\mathrm{Reg}-\mathrm{Ord}$ and $\mathrm{BN}-\mathrm{Ord}$

4. Banach Limits

For the introduction of Banach Limits we first prove, following a proof method of W. Roth in [12], Theorem 2.1, a special variant of the Hahn-Banach Theorem for order unit normed linear spaces.

Theorem 6. (**Hahn-Banach Theorem for Order Unit Spaces**) *Let E be an order unit normed space with order unit e, ordering cone $C(E)$ and norm $\|\square\|_E$ and let the following conditions be satisfied*:

i) $p: E \to R$ is a sublinear monotone function with $p(e)=1$,

ii) $S: E \to E$ is a surjective positive linear mapping.

iii) For any $x \in E$, the set mapping T_x is a right inverse of $S: E \to E$ with $T_x(S(x)) = x$.

T_x is monotone and $-T_x(y) = T_{-x}(-y)$, $T_x(y+z) = T_x(y) + T_0(z)$, $x, y, z \in E$.

iv) G is a muliplicative group G of positive automorphisms of E.

v) For any $x, y \in E$ and for every $\gamma \in G$, $p(S(x)) = p(x)$, $p(T_x(y)) = p(y)$ and $p(\gamma(x)) = p(x)$ hold.

Then there exists a positive linear functional $\mu: E \to R$ with:

a) $\mu(x) \le p(x)$ and $\mu(e) = 1$,

b) $\mu(S(x)) = \mu(x)$,

c) $\mu(T_z(x)) = \mu(x)$,

d) $\mu(\gamma(x)) = \mu(x)$,

for $x, z \in E$ and $\gamma \in G$.

Proof. Define

$$M_p := \{ s \mid s: E \to R, \text{ sublinear and monotone with } -p(-x) \le s(x) \le p(x),$$
$$s(S(x)) = s(T_z(x)) = s(\gamma(x)) = s(x), \text{ for all } x, z \in E \text{ and } \gamma \in G \}.$$

Obviously $p \in M_p$ holds, hence $M_p \neq \varnothing$. A partial order "$\le$" is defined in M_p by putting, for $s_1, s_2 \in M_p$,

$$s_1 \le s_2 \text{ if and only if } s_1(x) \le s_2(x) \quad \text{for all } x \in E.$$

Let $O \subset M_p$ be a non-empty, with respect to "$\le$" totally ordered subset and define

$$s_0(x) = \inf \{ s(x) \mid s \in O \}, \ x \in E.$$

As $-p(-x) \le s(x) \le p(x)$ for all $s \in O$ s_0 is well defined and finite and

$$-p(-x) \le s_0(x) \le p(x)$$

holds.

If $x \le y$ then $s(x) \le s(y)$ for all $s \in O$ and hence for all $x \le y$, $s_0(x) \le s_0(y)$ follows, *i.e.* s_0 is monotone.

Let $x, y \in E$ then for all $s \in O$ $s_0(x+y) \le s(x+y) \le s(x) + s(y)$ and hence

$$s_0(x+y) \le s_0(x) + s_0(y).$$

As obviously $s_0(\alpha x) = \alpha s_0(x)$ for $\alpha \ge 0$ it follows that s_0 is sublinear. Also $s_0(x)$ trivially satisfies the conditions a)-d) as well as $s_0(e) = 1$. Consequently, $s_0 \in M_p$ and s_0 is a lower bound of O in M_p. Zorn's Lemma now implies the existence of (at least) one minimal element in M_p with respect to $\le$ which will be denoted by μ.

Define for $x_0 \in E$:

$$\alpha_0(x_0) = \sup\left\{-p(-x_1) - \mu(x_2) \middle| x_1, x_2 \in E \text{ and } x_1 \le x_0 + x_2\right\}.$$

As, for $x \in E$, $-p(-x) \le \mu(x) \le p(x)$

$$-p(-x_1) \le \mu(x_1) \le \mu(x_0 + x_2) \le \mu(x_0) + \mu(x_2),$$

$$-p(x_1) - \mu(x_2) \le \mu(x_0)$$

and $\alpha_0(x_0) \le \mu(x_0) \le p(x_0)$ follows. (1)

Taking $x_1 = x_0$ and $x_2 = 0$ in the defining equation of α_0 yields

$$-p(-x_0) \le \alpha_0(x_0) \le p(x_0) \tag{2}$$

implying

$$\alpha_0(e) = 1. \tag{2a}$$

Now, the remaining equations in the assertion will be proved for $\alpha_0(x)$. Take the inequality $x_1 \le S(x_0) + x_2$ from the defining equation of $\alpha_0(S(x_0))$, then

$$T_{x_0}(x_1) \le T_{x_0}(S(x_0) + x_2) = x_0 + T_0(x_2)$$

contributing

$$-p(-T_{x_0}(x_1)) - \mu(T_0(x_2)) = -p(T_{-x_0}(-x_1)) - \mu(x_2) = -p(-x_1) - \mu(x_2)$$

to $\alpha_0(x_0)$. Conversely, $x_1 \le x_0 + x_2$ leads to

$$S(x_1) \le S(x_0 + x_2) = S(x_0) + S(x_2)$$

contributing

$$-p(-S(x_1)) - \mu(S(x_2)) = -p(-x_1) - \mu(x_2)$$

to the definition of $\alpha_0(S(x_0))$ and one gets

$$\alpha_0(S(x_0)) = \alpha_0(x_0). \tag{3a}$$

To show the invariance of $\alpha_0(x_0)$ under $T_z, z \in E$, start with $x_1 \le x_0 + x_2$ from $\alpha_0(x_0)$. Then

$$T_z(x_1) \le T_z(x_0 + x_2) = T_z(x_0) + T_0(x_2)$$

contributing

$$-p(-T_z(x_1)) - \mu(T_0(x_2)) = -p(-x_1) - \mu(x_2)$$

to $\alpha_0(T_z(x_0))$.

An inequality $x_1 \le T_z(x_0)+x_2$ of $\alpha_0(T_z(x_0))$. leads to

$$S(x_1) \le S(T_z(x_0)+x_2) = x_0 + S(x_2)$$

and

$$-p(-S(x_1)) - \mu(S(x_2)) = -p(-x_1) - \mu(x_2)$$

as contribution to $\alpha_0(x_0)$. Hence

$$\alpha_0(T_z(x_0)) = \alpha_0(x_0), z \in E. \tag{3b}$$

Verbatim, this proof carries over to the equation

$$\alpha_0(\gamma(x_0)) = \alpha_0(x_0), z \in E, \gamma \in G. \tag{3c}$$

A new function is now introduced by

$$\tilde{\mu}(x) := \inf\left\{\mu(y) + \lambda\alpha_0(x_0) \middle| \lambda \ge 0 \quad \textit{and} \quad x \le y + \lambda x_0, x_0, y \in E\right\}. \tag{4}$$

If $\lambda = 0$ then $x \le y + \lambda x_0$ is $x \le y$ and therefore

$$-p(-x) \le \mu(x) \le \mu(y) = \mu(y) + 0\alpha_0(x_0).$$

For $\lambda > 0$, $x \le y + \lambda x_0$ implies $\frac{x}{\lambda} \le x_0 + \frac{y}{\lambda}$ and yields

$-p\left(-\frac{x}{\lambda}\right) - \mu\left(\frac{y}{\lambda}\right) \le \alpha_0(x_0)$ because of the definition of $\alpha_0(x_0)$. Hence

$$-p(-x) \le \mu(y) + \lambda\alpha_0(x_0) \text{ for } \lambda > 0$$

and

$$-p(-x) \le \tilde{\mu}(x) \tag{5}$$

follows which implies in, particular, $\tilde{\mu}(x) \ge 0$ for $x \ge 0$, as $-p(-x)$ is positive.

Taking $y = x$ and $\lambda = 0$ one has

$$-p(-x) \le \tilde{\mu}(x) \le \mu(x) \le p(x) \tag{6}$$

in particular $-p(-x) \le \tilde{\mu}(x) \le p(x)$. If $x_1 \le x_2$ and $x_2 \le y_2 + \lambda x_0$ in (4) then

$$\tilde{\mu}(x_1) \le \tilde{\mu}(x_2) \tag{7}$$

follows *i.e.* monotonicity.

Consider now, for $x_i \in E, i = 1, 2,$

$$\tilde{\mu}(x_i) = \inf\left\{\mu(y_i) + \lambda_i\alpha_0(x_0) \middle| y_i \in E, \lambda_i \ge 0 \textit{ and } x_i \le y_i + \lambda_i x_0\right\},$$

then

$$\tilde{\mu}(x_1 + x_2) \le \mu(y_1 + y_2) + (\lambda_1 + \lambda_2)\alpha_0(x_0) \le \mu(y_1) + \lambda_1\alpha_0(x_0) + \mu(y_2) + \lambda_2\alpha_0(x_0)$$

i.e.

$$\tilde{\mu}(x_1 + x_2) \le \tilde{\mu}(x_1) + \tilde{\mu}(x_2).$$

Now for $\sigma > 0$,

$$\tilde{\mu}(\sigma x) := \inf\left\{\mu(y_\sigma) + \lambda_\sigma\alpha_0(x_0) \middle| y_\sigma \in E, \lambda_\sigma \ge 0 \text{ and } \sigma x \le y_\sigma + \lambda_\sigma x_0\right\}$$

and

$$\sigma\tilde{\mu}(x) := \inf\left\{\mu(\sigma y) + \sigma\lambda\alpha_0(x_0) \middle| y \in E, \lambda \geq 0 \text{ and } \sigma x \leq \sigma y + \lambda\sigma x_0\right\}.$$

The mapping

$$(y,\lambda) \mapsto (\sigma y, \sigma\lambda)), \sigma > 0,$$

is, for fixed $\sigma > 0$, bijective, therefore

$$\tilde{\mu}(\sigma x) = \sigma\tilde{\mu}(x), \sigma \geq 0, \tag{8}$$

holds because for $\sigma = 0$, one has $\tilde{\mu}(0x) = \tilde{\mu}(0) = 0$. So $\tilde{\mu}$ is sublinear.

We now show that $\tilde{\mu}$ also satisfies the equations of the assertions of the theorem. Take $x \leq y + \lambda x_0$ from the defining set of $\tilde{\mu}(x)$. Then

$$S(x) \leq S(y + \lambda x_0) = S(y) + \lambda S(x_0) \text{ follows,}$$

contributing

$$\mu(S(y)) + \lambda\alpha_0(S(x_0)) = \mu(y) + \lambda\alpha_0(x_0)$$

to $\tilde{\mu}(S(x))$. Conversely, $S(x) \leq y + \lambda x_0$ contributing $\mu(y) + \lambda\alpha_0(x_0)$ to $\tilde{\mu}(S(x))$ yields by applying T_x

$$x \leq T_x(y + \lambda x_0) = T_x(y) + T_0(\lambda x_0).$$

Hence

$$\mu(T_x(y)) + 1 \cdot \alpha_0(T_0(\lambda x_0)) = \mu(y) + 1 \cdot \alpha_0(\lambda x_0) = \mu(y) + \lambda\alpha(x_0).$$

This implies

$$\tilde{\mu}(S(x)) = \tilde{\mu}(x). \tag{9}$$

The proof of the remaining two equations of the assertions follows almost verbatim this pattern of proof and one gets:

$$\tilde{\mu}(T_z(x)) = \tilde{\mu}(x), \ x, z \in E$$

$$\tilde{\mu}(\gamma(x)) = \tilde{\mu}(x), \ x \in E, \gamma \in G$$

and (6) implies $\tilde{\mu}(e) = 1$. Hence $\tilde{\mu} \in M_p$ is proved which implies

$$\tilde{\mu} = \mu \tag{10}$$

because of (6) and the minimality of $\mu \in M_p$.

Now, looking again at the definition (4) of $\tilde{\mu}$ and putting $y = 0, \lambda = 1$ and $x = x_0$ one gets

$$\tilde{\mu}(x_0) \leq \alpha_0(x_0) \tag{11}$$

which together with (1) yields $\tilde{\mu}(x_0) \leq \alpha_0(x_0) \leq \mu(x_0)$ and in combination with (1) and (10) gives

$$\alpha_0(x) = \mu(x), \quad x \in E. \tag{12}$$

Now, for $x_0, y_0 \in E$,

$$\alpha_0(x_0) = \sup\{-p(-x_1) - \mu(x_2) | x_1, x_2 \in E \quad \text{and} \quad x_1 \le x_0 + x_2\}.$$

and

$$\alpha_0(y_0) = \sup\{-p(-y_1) - \mu(y_2) | y_1, y_2 \in E \quad \text{and} \quad y_1 \le y_0 + y_2\}.$$

and since $-p(-x)$ and $-\mu(x)$ are superlinear, one has

$$-p(-x_1) - \mu(x_2) - p(-y_1) - \mu(y_2) \le -p(-x_1 - y_1) - \mu(x_2 + y_2) \le \alpha_0(x_0 + y_0)$$

which implies $\alpha_0(x_0) + \alpha_0(y_0) \le \alpha_0(x_0 + y_0)$ that is $\alpha_0(x)$ is superadditive and because of (12) positively homogeneous, *i.e.* superlinear. This implies that $\mu(x)$ is linear because of (12) and satisfies all the equations in the assertion, which completes the proof. □

Banach limits are almost always defined as continuous extensions of a continuous linear functional in an order unit normed space. Hence, for the introduction of Banach limits we need Theorem 6 in a continuous form. Surprisingly Theorem 6 already contains all the necessary continuity conditions as the following Corollary shows:

Corollary 2. *Let the assertions* (i)-(v) *of Theorem* 6 *be satisfied and put* $U_p := \{x | p(x) = -p(-x)\}$ *and* $\lambda_0(x) := p(x)$ *for* $x \in U_p$. *Then*

i) $e \in U_p$ and U_p is an isometrical order unit normed subspace of E which is closed.

ii) $p : E \to R$ is continuous and $\lambda_0 : U_p \to R$ is a positive, continuous linear functional with $\lambda_0(e) = 1$.

iii) Any $\mu : E \to R$ satisfying Theorem 6 is a positive, continuous linear extension of λ_0.

Proof. i): Obviously, $e \in U_p$ and, hence, $xU_p \ne \{0\}$. For $x \in U_p \subset E, -\|x\|_E e \le x \le \|x\|_E e$ holds and this is an inequality in E and U_p which proves i).

ii): $p(x)$ and $-p(-x)$ are both monotone, $p(x)$ sublinear and $-p(-x) \le p(x)$ superlinear. As for $x \in U_p, p(x) = -p(-x)$ holds, $p(x)$ is linear and positive on U_p. If $-e \le x \le e$ then $-1 \le p(x) \le 1$, and $|p(x)| \le 1$ s. th. norm of p is $\|p\| = \sup\{|p(x)| | -e \le x \le e\} = 1$ and p is in 0 continuous hence also for any $x \in E$ and $|p(x)| \le \|x\|_E, x \in E$. Hence, we get for any μ in Theorem 6

$$-\|x\|_E \le -p(-x) \le \mu(x) \le p(x) \le \|x\|_E$$

implying the continuity of μ and $\|\mu\| \le 1$, even $\|\mu\| = 1$ because $\mu(e) = 1$. In particular, this holds also for λ_0. □

It is remarkable that with respect to the continuity properties, the continuity of $S, T_x, x \in E_i$, and $\gamma \in G$, do not play any role.

Definition 5. *With the notations of Corollary* 2 *any such* μ *is called a* Banach limit *of* λ_0.

One defines

$$\text{Ban} - \text{Lim}(E, e, p, S, T_\square, G) := \{\mu | \mu \text{ Banach limit of } \lambda_0\}.$$

$\mathrm{Ban}-\mathrm{Lim}(E,e,p,S,T_{\square},G)$, of course, depends on the parameters $S,T_{\square},G$, but in order to make the notation for the following not too cumbersome we will mostly omit them and write simply $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$.

Proposition 7. *For* $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$. *the following statements hold*:

i) $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$. is a convex subset of $Bs(E')$ of the base normed Banach space E', the dual space of E.

ii) $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$. is weakly-*-closed, weakly closed and also $\|\square\|'$-closed, where $\|\square\|'_E$ denotes the usual dual norm of $\|\square\|'_E$ of E'.

iii) $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$. is a superconvex base set contained in $Bs(E')$.

Proof. i): Let $(\alpha_1,\cdots,\alpha_n)\in\Omega:=\left\{(\alpha_1,\cdots,\alpha_n)\middle|\alpha_i\ge 0,\sum_{i=1}^{n}\alpha_i=1,n\in N,\right\}$ be the set of all abstract convex combinations, then, for $\lambda_i\in\mathrm{Ban}-\mathrm{Lim}(E,e,p)$, $1\le i\le n$, obviously $\sum_{i=1}^{n}\alpha_i\lambda_i\in\mathrm{Ban}-\mathrm{Lim}(E,e,p)$, as $\sum_{i=1}^{n}\alpha_i\lambda_i(x)\ge 0$ for $x\ge 0$, $\sum_{i=1}^{n}\alpha_i\lambda_i\le p$, $\sum_{i=1}^{n}\alpha_i\lambda_i(e)=\sum_{i=1}^{n}\alpha_i=1$ and obviously all other equations in Theorem 6 are satisfied, too.

ii): One first proves that $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$. is $\|\square\|'$-closed, because from this , the other two assertions of ii) then follow at once. If $\lambda_n\in\mathrm{Ban}-\mathrm{Lim}(E,e,p)$, $n\in N$, and there is a $\lambda_*\in E'$ with $\|\lambda_n-\lambda_*\|'\to 0$ for $n\to\infty$, then $|\lambda_n(x)-\lambda_*(x)|\to 0$, *i.e.* $\lim_{n\to\infty}\lambda_n(x)=\lambda_*(x)$ follows for $x\in E$, which implies $\lambda_*(x)\ge 0$, for $x\ge 0,\lambda_*(e)=1,\lambda_*(x)\le p(x),x\in X$, and $\lambda_*(x)=\lambda_0(x)$ for $x\in U_p$. Analogously, one shows the other equations in Theorem 6 for λ_* because they hold for the $\lambda_n(x),x\in E,n\in N$.

iii): Obviously, $\mathrm{Ban}-\mathrm{Lim}(E,e,p)\subset Bs(E')$ holds, hence $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$. is bounded and because of ii) $\|\square\|'$-closed. Then, it is a general result that $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$. is superconvex (see [13], Theorem 2.5).

□

Because $\mathrm{Ban}-\mathrm{Lim}(E,e,p)$. is as a subset of $Bs(E')$ trivially a base set

$$C(\mathrm{Ban}-\mathrm{Lim}(E,e,p)):=R_+\ \mathrm{Ban}-\mathrm{Lim}(E,e,p)$$

is a proper cone and

$$\mathfrak{B}an-\mathfrak{L}im(E,e,p):=C(Ban-Lim(E,e,p))-C((Ban-Lim(E,e,p))$$

is a base normed ordered linear space. To simplify notation, we will write (E,p) instead of (E,e,p) in the following.

Theorem 8. *If the norm induced by* $\mathrm{Ban}-\mathrm{Lim}(E,p)$ *in* $\mathfrak{B}\mathrm{an}-\mathfrak{L}\mathrm{im}(E,p)$ *is denoted by* $\|\square\|_{\mathrm{Ban}}$ *and the topology induced by the weak-*-topology* $\sigma(E',E)$ *in* $\mathfrak{B}\mathrm{an}-\mathfrak{L}\mathrm{im}(E,p)$ *by* τ_B^*, *then*

$$\left(\mathfrak{B}\mathrm{an}-\mathfrak{L}\mathrm{im}(E,p),\|\square\|_{\mathrm{Ban}},C(\mathrm{Ban}-\mathrm{Lim}(E,p)),\tau_B^*\right)$$

is a compact, base normed Saks space (see [13], Theorem 3.1) and an isometrical subspace of

$$\left(E', \|\square\|', C(E), \sigma(E', E)\right).$$

Proof. As $\mathrm{Ban}-\mathrm{Lim}(E,p)$ is a weakly-*-closed base set and a subset of $Bs(E')$ which is weakly-*-compact because of Alaoglu-Bourbaki it is also weakly-*-compact and the space $\mathfrak{B}\mathrm{an}-\mathfrak{L}\mathrm{im}(E,p)$ generated by the closed cone $C(\mathrm{Ban}-\mathrm{Lim}(E,p))$ (see [7], Theorem 3.8.3, [13], Theorem 3.2, [14]) is a compact, base normed Saks space. The last assertion is obvious. □

The result of Theorem 8 is essentially the definition of a functor from any category with objects satisfying the assertions of Theorem 6 to the category of compact, base normed Saks spaces ([13], Theorem 3.1). This functor will be investigated by the authors in a forthcoming paper.

5. Summary

The main result of the paper offers a Hahn-Banach theorem for order unit normed spaces (Theorem 6) from which novel conclusions on Banach limits are drawn. The result of Theorem 8 gives rise to the definition of a functor which goes from any category with objects satisfying the assertions of Theorem 6 into the category of compact, base normed Saks spaces.

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