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# Inequalities for Dual Orlicz Mixed Quermassintegrals

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## Abstract

In this paper, we establish the dual Orlicz-Minkowski inequality and the dual Orlicz-Brunn-Minkowski inequality for dual Orlicz mixed quermassintegrals.

## Keywords

Star Body, Orlicz Radial Sum, Dual Orlicz Mixed Volume

## 1. Introduction

Recently, Convex Geometry Analysis has made great achievement in Orlicz space (see [1]-[14]). Zhu, Zhou and Xu [12] defined the Orlicz radial sum and dual Orlicz mixed volumes. Let $\mathcal{C}^+$ be the set of convex and strictly decreasing functions $\phi:(0,+\infty)\to(0,+\infty)$ such that $\lim_{t\to\infty}\phi(t)=0$, $\lim_{t\to 0}\phi(t)=\infty$ and $\phi(0)=\infty$.

Let $K$ and $L$ be two star bodies about the origin in $\mathbb{R}^n$ and $a,b\geq 0$; the Orlicz radial sum $a\cdot K\tilde{+}_\phi b\cdot L$ was defined by [13]

$$\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)=\sup\left\{t>0:a\phi\left(\frac{\rho_K(u)}{t}\right)+b\phi\left(\frac{\rho_L(u)}{t}\right)\leq\phi(1)\right\},\forall u\in S^{n-1}. \tag{1.1}$$

The case $\phi(t)=t^{-p}\,(p\geq 1)$ of the Orlicz radial sum is the $L_p$ harmonic radial sum, which was defined by Lutwak (see [15]).

Let $f_r'$ denote the right derivative of a real-valued function $f$. For $\phi\in\mathcal{C}^+$, there is $\phi_r'(1)<0$ because $\phi$ is convex and strictly decreasing. The dual Orlicz mixed volume $\tilde{V}_\phi(K,L)$ is defined by

$$\frac{n}{\phi_r'(1)}\tilde{V}_\phi(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K\tilde{+}_\phi\,\varepsilon\cdot L)-V(K)}{\varepsilon}. \tag{1.2}$$

In this paper, we will define the dual Orlicz mixed quermassintegral

$\widetilde{W}_{\phi,i}(K,L)(i=0,\cdots,n-1)$ by

$$\frac{n-i}{\phi_r'(1)}\widetilde{W}_{\phi,i}(K,L)=\lim_{\varepsilon\to0^+}\frac{\widetilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}. \quad (1.3)$$

The main purpose of this paper is to establish the dual Orlicz-Minkowski inequality and the dual Orlicz-Brunn-Minkowski inequality for dual Orlicz mixed quermassintegrals.

**Theorem 1.1** Let $K$ and $L$ be two star bodies about the origin in $\mathbb{R}^n$ and $\phi\in\mathcal{C}^+$. If $0\le i<n-1$, then

$$\widetilde{W}_{\phi,i}(K,L)\ge\widetilde{W}_i(K)\phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right), \quad (1.4)$$

with equality if and only if $K$ and $L$ are dilates of each other.

**Theorem 1.2** Let $K$ and $L$ be two star bodies about the origin in $\mathbb{R}^n$ and $\phi\in\mathcal{C}^+$. If $0\le i<n-1$, then

$$\phi(1)\ge\phi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K\tilde{+}_\phi L)}\right)^{\frac{1}{n-i}}\right)+\phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K\tilde{+}_\phi L)}\right)^{\frac{1}{n-i}}\right), \quad (1.5)$$

with equality if and only if $K$ and $L$ are dilates of each other.

This paper is organized as follows: In Section 2 we introduce above interrelated notations and their background materials. Section 3 contains the proofs of our main results.

## 2. Notation and Background Material

The radial function $\rho_K(u):S^{n-1}\to[0,\infty)$ of a compact star-shaped about the origin $K\in\mathbb{R}^n$ is defined, for $u\in S^{n-1}$, by

$$\rho_K(u)=\max\{\lambda\ge0:\lambda u\in K\}. \quad (2.1)$$

If $\rho_K(\cdot)$ is positive and continuous, then $K$ is called a star body about the origin. The set of star bodies about the origin in $\mathbb{R}^n$ is denoted by $\mathcal{S}_0^n$. Obviously, for $K,L\in\mathcal{S}_0^n$,

$$K\subseteq L\Leftrightarrow\rho_K(u)\le\rho_L(u),\ \forall u\in S^{n-1}. \quad (2.2)$$

If $\dfrac{\rho_K(u)}{\rho_L(u)}$ is independent of $u\in S^{n-1}$, then we say star bodies $K$ and $L$ are dilates of each other.

If $K_i\in\mathcal{S}_0^n(i=1,2,\cdots,m)$ and $\lambda_i(i=1,2,\cdots,m)$ are nonnegative real numbers, then the volume of $\lambda_1K_1\tilde{+}\cdots\tilde{+}\lambda_mK_m$ is a homogeneous polynomial of degree $n$ in $\lambda_i$ given by

$$V(\lambda_1K_1\tilde{+}\cdots\tilde{+}\lambda_mK_m)=\sum_{i_1,\cdots,i_n}\tilde{V}(K_{i_1},\cdots,K_{i_n})\lambda_{i_1}\cdots\lambda_{i_n},$$

where the sum is taken over all $n$-tuples $(i_1,\cdots,i_n)$ of positive integers not exceeding $m$.

The coefficient $\tilde{V}\left(K_{i_1},\cdots,K_{i_n}\right)$ depends only on the bodies $K_{i_1},\cdots,K_{i_n}$, and is uniquely determined by the above identity, it is called the dual mixed volume of $K_{i_1},\cdots,K_{i_n}$. More explicitly, the dual mixed volume $\tilde{V}\left(K_{i_1},\cdots,K_{i_n}\right)$ has the following integral representation [16]:

$$\tilde{V}\left(K_{i_1},\cdots,K_{i_n}\right)=\frac{1}{n}\int_{S^{n-1}}\rho_{K_{i_1}}(u)\cdots\rho_{K_{i_n}}(u)\,\mathrm{d}S(u), \tag{2.3}$$

where $S$ is the Lebesgue measure on $S^{n-1}$.

The coefficients $\tilde{V}\left(K_{i_1},\cdots,K_{i_n}\right)$ are nonnegative, symmetric and monotone (with respect to set inclusion). They are also multilinear with respect to the radial sum and $\tilde{V}(K,\cdots,K)=V(K)$. Let $K_1=\cdots=K_{n-i}=K$ and $K_{n-i+1}=\cdots=K_n=L$, then the dual mixed volume $\tilde{V}(K_1,\cdots,K_n)$ is usually written as $\tilde{V}_i(K,L)$. If $L=B$, then $\tilde{V}_i(K,B)$ is the dual quermassintegral $\tilde{W}_i(K)$. For $0\le i\le n-1$, the dual mixed quermassintegral $\tilde{W}_i(K,L)$ denotes the dual mixed volume $\tilde{V}\Big(\underbrace{K,\cdots,K}_{n-i-1},\underbrace{B,\cdots,B}_{i},L\Big)$. For $L=K$, then $\tilde{W}_i(K,L)=\tilde{W}_i(K)$.

The dual mixed quermassintegral $\tilde{W}_i(K,L)$ has the following integral representation:

$$\tilde{W}_i(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i-1}(u)\rho_L(u)\,\mathrm{d}S(u), \tag{2.4}$$

where $S$ is the Lebesgue measure on $S^{n-1}$.

By using the Minkowski's integral inequality, we can obtain the dual Minkowski inequality for dual mixed quermassintegrals: If $K,L\in\mathcal{S}_0^n$, and $0\le i<n-1$, then

$$\tilde{W}_i(K,L)^{n-i}\le\tilde{W}_i(K)^{n-i-1}\tilde{W}_i(L), \tag{2.5}$$

equality holds if and only if $K$ and $L$ are dilates of each other.

Suppose that $\mu$ is a probability measure on a space $X$ and $g:X\to I\subset\mathbb{R}$ is a $\mu$-intergrable function, where $I$ is a possibly infinite interval. Jessen's inequality states that if $\phi:X\to I\subset\mathbb{R}$ is a convex function, then

$$\int_X\phi\left(g(x)\right)\mathrm{d}\mu(x)\ge\phi\left(\int_X g(x)\,\mathrm{d}\mu(x)\right). \tag{2.6}$$

If $\phi$ is strictly convex, equality holds if and only if $g(x)$ is a constant for $\mu$-almost all $x\in X$ (see [17]).

## 3. Main Results

Let $K,L\in\mathcal{S}_0^n$ and $\phi\in\mathcal{C}^+$. For $i=0,\cdots,n-1$, the dual Orlicz mixed quermassintegral $\tilde{W}_{\phi,i}(K,L)$ is defined by

$$\tilde{W}_{\phi,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)\,\mathrm{d}S(u). \tag{3.1}$$

For $L=K$, then $\tilde{W}_{\phi,i}(K,K)=\phi(1)\tilde{W}_i(K)$. The case $i=0$ of the dual Orlicz mixed quermassintegral $\tilde{W}_{\phi,i}(K,L)$ is the dual Orlicz mixed volume $\tilde{V}_\phi(K,L)$, which was defined by Zhu, Zhou and Xu [12].

**Corollary 3.1** The dual Orlicz mixed quermassintegral $\tilde{W}_{\phi,i}(K,\cdot)$ is monotone with respect to set inclusion.

**Proof.** Let $L_1, L_2 \in \mathcal{S}_0^n$ and $L_1 \subseteq L_2$. By (3.1), (2.2) and the fact that $\phi$ is strictly decreasing on $(0,\infty)$, we have

$$\begin{aligned}\tilde{W}_{\phi,i}(K,L_1) &= \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho_{L_1}(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)\,\mathrm{d}S(u)\\ &\geq \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho_{L_2}(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)\,\mathrm{d}S(u)\\ &= \tilde{W}_{\phi,i}(K,L_2).\end{aligned}$$

**Lemma 3.1** [12] Let $K, L \in \mathcal{S}_0^n$ and $u \in S^{n-1}$. If $\phi \in \mathcal{C}^+$, then

$$a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) = \phi(1)$$

if and only if

$$\rho_{a\cdot K \tilde{+}_\phi b\cdot L}(u) = t.$$

**Lemma 3.2** [12] Let $K, L \in \mathcal{S}_0^n$ and $\phi \in \mathcal{C}^+$. Then

$$\lim_{\varepsilon\to 0^+} \frac{\rho_{K\tilde{+}_\phi \varepsilon\cdot L}(u) - \rho_K(u)}{\varepsilon} = \frac{\rho_K(u)}{\phi_r'(1)}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right), \tag{3.2}$$

uniformly for all $u \in S^{n-1}$.

**Theorem 3.1** Let $K, L \in \mathcal{S}_0^n$ and $\phi \in \mathcal{C}^+$. For $i = 0, \cdots, n-1$, then

$$\lim_{\varepsilon\to 0^+} \frac{\tilde{W}_i(K \tilde{+}_\phi \varepsilon\cdot L) - \tilde{W}_i(K)}{\varepsilon} = \frac{n-i}{n\phi_r'(1)}\int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)\,\mathrm{d}S(u).$$

**Proof.** Suppose $\varepsilon > 0, K, L \in \mathcal{S}_0^n$, and $u \in S^{n-1}$. Note that $K \tilde{+}_\phi \varepsilon \cdot L \to K$ as $\varepsilon \to 0^+$ (see [12]). By Lemma 3.2, it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{\rho^{n-i}_{K\tilde{+}_\phi \varepsilon\cdot L}(u) - \rho_K^{n-i}(u)}{\varepsilon} &= (n-i)\rho^{n-i-1}_{K\tilde{+}_\phi \varepsilon\cdot L}(u)\Big|_{\varepsilon=0} \cdot \lim_{\varepsilon\to 0^+} \frac{\rho_{K\tilde{+}_\phi \varepsilon\cdot L}(u) - \rho_K(u)}{\varepsilon}\\ &= \frac{(n-i)\rho_K^{n-i}(u)}{\phi_r'(1)}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),\end{aligned}$$

uniformly on $S^{n-1}$.

Hence

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{\tilde{W}_i(K \tilde{+}_\phi \varepsilon\cdot L) - \tilde{W}_i(K)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\left(\frac{1}{n}\int_{S^{n-1}} \frac{\rho^{n-i}_{K\tilde{+}_\phi \varepsilon\cdot L}(u) - \rho_K^{n-i}(u)}{\varepsilon}\,\mathrm{d}S(u)\right)\\ &= \frac{1}{n}\int_{S^{n-1}} \lim_{\varepsilon\to 0^+} \frac{\rho^{n-i}_{K\tilde{+}_\phi \varepsilon\cdot L}(u) - \rho_K^{n-i}(u)}{\varepsilon}\,\mathrm{d}S(u)\\ &= \frac{n-i}{n\phi_r'(1)}\int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)\,\mathrm{d}S(u).\end{aligned}$$

We complete the proof of Theorem 3.1. □

From (3.1) and Theorem 3.1, we have

$$\frac{n-i}{\phi_r'(1)}\tilde{W}_{\phi,i}(K,L)=\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i\left(K\tilde{+}_\phi\,\varepsilon\cdot L\right)-\tilde{W}_i(K)}{\varepsilon}. \tag{3.3}$$

For $K\in\mathcal{S}_0^n$, since $\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\,\mathrm{d}S(u)=\tilde{W}_i(K)$, then $\frac{\rho_K^{n-i}(\cdot)\,\mathrm{d}S(\cdot)}{n\tilde{W}_i(K)}$ is a probability measure on $S^{n-1}$.

**Proof of Theorem 1.1**

By (3.1), (2.6), (2.5) and the fact that $\phi$ is decreasing on $(0,\infty)$, we obtain

$$\begin{aligned}\frac{\tilde{W}_{\phi,i}(K,L)}{\tilde{W}_i(K)}&=\frac{1}{n\tilde{W}_i(K)}\int_{S^{n-1}}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)\,\mathrm{d}S(u)\\&\ge\phi\left(\frac{1}{n\tilde{W}_i(K)}\int_{S^{n-1}}\frac{\rho_L(u)}{\rho_K(u)}\rho_K^{n-i}(u)\,\mathrm{d}S(u)\right)\\&=\phi\left(\frac{\tilde{W}_i(K,L)}{\tilde{W}_i(K)}\right)\\&\ge\phi\left(\frac{\tilde{W}_i(K)^{\frac{n-i-1}{n-i}}\tilde{W}_i(L)^{\frac{1}{n-i}}}{\tilde{W}_i(K)}\right)\\&=\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right)\end{aligned}$$

This gives the desired inequality. Since $\phi$ is strictly decreasing, from the equality condition of the dual Minkowski inequality (2.5), we have that $K$ and $L$ are dilates of each other.

Conversely, when $L=\lambda K$, by (3.1), we have

$$\tilde{W}_{\phi,i}(K,L)=\tilde{W}_i(K)\phi(\lambda)=\tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right).\ \square$$

The following uniqueness is a direct consequence of the dual Orlicz-Minkowski inequality (1.4).

**Corollary 3.2** Suppose $\phi\in\mathcal{C}^+$, and $\mathcal{M}\subset\mathcal{S}_0^n$ such that $K,L\in\mathcal{M}$. For $0\le i<n-1$, if

$$\tilde{W}_{\phi,i}(M,K)=\tilde{W}_{\phi,i}(M,L),\ \text{for all } M\in\mathcal{M}, \tag{3.4}$$

or

$$\frac{\tilde{W}_{\phi,i}(K,M)}{\tilde{W}_i(K)}=\frac{\tilde{W}_{\phi,i}(L,M)}{\tilde{W}_i(L)},\ \text{for all } M\in\mathcal{M}, \tag{3.5}$$

then $K=L$.

**Proof.** Suppose (3.4) holds. If we take *K* for *M*, then from (3.1), we obtain

$$\phi(1)\tilde{W}_i(K) = \tilde{W}_{\phi,i}(K,K) = \tilde{W}_{\phi,i}(K,L).$$

Hence, from the dual Orlicz-Minkowski inequality (1.4), we have

$$\phi(1) \ge \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),$$

with equality if and only if *K* and *L* are dilates of each other. Since $\phi$ is strictly decreasing on $(0,\infty)$, we have

$$\tilde{W}_i(L) \ge \tilde{W}_i(K),$$

with equality if and only if *K* and *L* are dilates of each other. If we take *L* for *M*, we similarly have $\tilde{W}_i(L) \le \tilde{W}_i(K)$. Hence, $\tilde{W}_i(K) = \tilde{W}_i(L)$ and from the equality condition we can conclude that *K* and *L* are dilates of each other. However, since they have the same volume they must be equal.

Next, suppose (3.5) holds. If we take *K* for *M*, then from (3.1), we obtain

$$\phi(1) = \frac{\tilde{W}_{\phi,i}(K,K)}{W_i(K)} = \frac{\tilde{W}_{\phi,i}(L,K)}{\tilde{W}_i(L)}.$$

Then, from the dual Orlicz-Minkowski inequality (1.4), we have

$$\phi(1) \ge \phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right),$$

with equality if and only if *K* and *L* are dilates of each other. Since $\phi$ is strictly decreasing on $(0,\infty)$, we have

$$\tilde{W}_i(K) \ge \tilde{W}_i(L),$$

with equality if and only if *K* and *L* are dilates of each other. If we take *L* for *M*, we similarly have $\tilde{W}_i(K) \le \tilde{W}_i(L)$. Hence, $\tilde{W}_i(K) = \tilde{W}_i(L)$ and from the equality condition we can conclude that *K* and *L* are dilates of each other. However, since they have the same volume they must be equal.

From the dual Orlicz-Minkowski inequality, we will prove the following dual Orlicz-Brunn-Minkowski inequality which is more general than Theorem 1.2.

**Theorem 3.2** Let $K, L \in \mathcal{S}_0^n$, $a, b > 0$ and $\phi \in \mathcal{C}^+$. If $0 \le i < n-1$, then

$$\phi(1) \ge a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right),$$

with equality if and only if *K* and *L* are dilates of each other.

**Proof.** Let $K_\phi = a \cdot K \tilde{+}_\phi b \cdot L$. From (2.3), Lemma 3.1 and (1.4), it follows that

$$\begin{aligned}\phi(1) &= \frac{1}{n\tilde{W}_i(K_\phi)}\int_{S^{n-1}}\phi(1)\rho_{K_\phi}^{n-i}(u)\,\mathrm{d}S(u)\\ &= \frac{1}{n\tilde{W}_i(K_\phi)}\int_{S^{n-1}}\left[a\phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right)+b\phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right)\right]\rho_{K_\varphi}^{n-i}(u)\,\mathrm{d}S(u)\\ &= \frac{a}{n\tilde{W}_i(K_\phi)}\int_{S^{n-1}}\phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right)\rho_{K_\phi}^{n}(u)\,\mathrm{d}S(u)+\frac{b}{n\tilde{W}_i(K_\phi)}\int_{S^{n-1}}\phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right)\rho_{K_\phi}^{n-i}(u)\,\mathrm{d}S(u)\\ &= \frac{a}{\tilde{W}_i(K_\phi)}\tilde{W}_{\phi,i}(K_\phi,K)+\frac{b}{\tilde{W}_i(K_\phi)}\tilde{W}_{\phi,i}(K_\phi,L)\\ &\ge a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\phi)}\right)^{\frac{1}{n-i}}\right)+b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\phi)}\right)^{\frac{1}{n-i}}\right).\end{aligned}$$

By the equality condition of the dual Orlicz-Minkowski inequality (1.4), equality in (3.6) holds if and only if $K$ and $L$ are dilates of each other.

Indeed, we also can prove the dual Orilcz-Minkowski inequality by the dual Orilcz-Brunn-Minkowski inequality.

**Proof.** For $\varepsilon \ge 0$, let $K_\varepsilon = K \tilde{+}_\phi \varepsilon \cdot L$. Note that $K_\varepsilon \to K$ as $\varepsilon \to 0^+$. By the dual Orlicz-Brunn-Minkowski inequality, the following function

$$G(\varepsilon)=\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)+\varepsilon\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)-\phi(1)$$

is non-positive. Obviously, $G(0)=0$. Thus

$$\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon}\le 0. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)+\varepsilon\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)-\phi(1)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)-\phi(1)}{\varepsilon}+\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right)\\ &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)-\phi(1)}{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}-1}\cdot\lim_{\varepsilon\to 0^+}\frac{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}-1}{\varepsilon}+\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right).\end{aligned} \tag{3.8}$$

Let $s=\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}$ and note that $s\to 1^+$ as $\varepsilon\to 0^+$. Consequently,

$$\lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)-\phi(1)}{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}-1}=\lim_{s\to 1^+}\frac{\phi(s)-\phi(1)}{s-1}=\phi_r'(1). \tag{3.9}$$

By (3.3), we have

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}-1\right)}{\varepsilon}\\&=-\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K_\varepsilon)^{\frac{1}{n-i}}-\tilde{W}_i(K)^{\frac{1}{n-i}}}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\tilde{W}_i(K_\varepsilon)^{-\frac{1}{n-i}}\\&=-\frac{1}{n-i}\tilde{W}_i(K)^{\frac{1}{n-i}-1}\cdot\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K_\varepsilon)-\tilde{W}_i(K)}{\varepsilon}\cdot\tilde{W}_i(K)^{-\frac{1}{n-i}}\\&=-\frac{1}{\phi_r'(1)}\frac{\tilde{W}_{\phi,i}(K,L)}{\tilde{W}_i(K)}.\end{aligned} \tag{3.10}$$

From (3.8), (3.9), and (3,10), it follows that

$$\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon}=-\frac{\tilde{W}_{\phi,i}(K,L)}{\tilde{W}_i(K)}+\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{3.11}$$

Combing (3.7) and (3.11), we have

$$-\frac{\tilde{W}_{\phi,i}(K,L)}{\tilde{W}_i(K)}+\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right)\le 0. \tag{3.12}$$

Therefore, the equality in (3.12) holds if and only if $G(\varepsilon)=G(0)=0$, this implies that $K$ and $L$ are dilates of each other.

**Remark 3.1** The case $i=0$ of Theorem 1.1 and Theorem 1.2 were established by Zhu, Zhou and Xu [12]. The dual forms of Theorem 1.1 and Theorem 1.2 were established by Xiong and Zou [11].

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