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Some Integral Inequalities of Simpson Type for Strongly Extended *s*-Convex Functions

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Abstract

The main purpose of this survey paper is to point out some very recent developments on Simpson's inequality for strongly extended *s*-convex function. Firstly, the concept of strongly extended *s*-convex function is introduced. Next a new identity is also established. Finally, by this identity and Hölder's inequality, some new Simpson type for the product of strongly extended *s*-convex function are obtained.

Keywords

Simpson Type Inequality, Integral Identity, Strongly Extended *s*-Convex Function

1. Introduction

Convex function is a kind of important function and has wide applications in pure and applied mathematics [1]. Since convex analysis appeared in 1960s, there has been tremendous interest in generalizing convex function [2]. In recent years, the generalized convex function and its application have been hot issues. The main purpose of this survey paper is to point out some very recent developments on Simpson's inequality for strongly extended s-convex function.

First, some definitions concerning various convex functions are listed.

Definition 1.1. A function $f: I \subseteq R = (-\infty, \infty) \to R$ is said to be convex if

$$f\left(\lambda x + (1-\lambda) y\right) \le \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

The *s*-convex function was defined in [3] as follows.

Definition 1.2. A function $f: I \subseteq R \to R_0 = [0, \infty)$ is said to be *s*-convex if

$$f\left(\lambda x + (1-\lambda) y\right) \le \lambda^s f(x) + (1-\lambda)^s f(y) \quad (1.1)$$

for some $s \in (0,1]$, where $x, y \in I, \lambda \in [0,1]$.

If $s = 1$, the *s*-convex function becomes a convex function on R_0.

In [4], the authors introduced the class of real functions of extended *s*-convex, defined as follows.

Definition 1.3. ([4]). A function $f : I \subseteq R \to R_0$ is said to be extended *s*-convex if

$$f\left(\lambda x + (1-\lambda) y\right) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for some $s \in [-1,1]$, where $x, y \in I, \lambda \in (0,1)$.

In [5] the concept of strongly convex functions below was innovated.

Definition 1.4. ([5]) A function $f : [a,b] \to R$ is said to be strongly convex with modulus $c > 0$, if

$$f\left(\lambda x + (1-\lambda) y\right) \le \lambda f(x) + (1-\lambda) f(y) - c\lambda(1-\lambda)(x-y)^2 \tag{1.3}$$

is valid for all $x, y \in [a,b]$, $\lambda \in [0,1]$.

In [6] the concept of strongly *s*-convex functions was introduced as follows.

Definition 1.5. A function $f : I \subseteq R \to R_0$ is said to be strongly *s*-convex with modulus $c > 0$, and some $s \in (0,1]$ if

$$f\left(\lambda x + (1-\lambda) y\right) \le \lambda^s f(x) + (1-\lambda)^s f(y) - c\lambda(1-\lambda)(x-y)^2 \tag{1.4}$$

is valid all $x, y \in [a,b]$, $\lambda \in [0,1]$.

The following inequalities of Hermite-Hadamard type were established for some of the above convex functions.

Theorem 1.1. ([7]). Let $f : I^\circ \subseteq R \to R$ be differentiable on I°, $a, b \in I^\circ$ with $a < b$.

(1) If $|f'|$ is convex function on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}. \tag{1.5}$$

(2) If $|f'|^{p/(p-1)}$ is convex function on $[a,b]$, $p > 1$, then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}.\end{aligned} \tag{1.6}$$

Theorem 1.2. ([8]). Let $f : I \subset R \to R$ be differentiable on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is *s*-convex function on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]\left(|f'(a)|^q + |f'(b)|^q\right)^{1/q}.\end{aligned} \tag{1.7}$$

Theorem 1.3. ([9]). Let $f: I \subset R_0 \to R$ be differentiable on I°, $a,b \in I^\circ$ with $a<b$, and $f' \in L[a,b]$. If $|f'|$ is *s*-convex function on $[a,b]$ for some fixed $s \in (0,1]$, then

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(b-a)\left(|f'(a)|+|f'(b)|\right). \tag{1.8}$$

In [6], Ju Hua *et al.* established the following theorem.

Theorem 1.4. Let $f: I \subseteq R_0 \to R_0$ be differentiable mapping on I° and $a,b \in I$ with $a<b$. If $f'' \in (L[a,b])$ and $|f''|^q$ is strongly s-convex on $[a,b]$ for $q \ge 1$, $s \in (0,1]$, then

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\le \frac{6^{1/q}(b-a)^2}{324}\left\{\left[\frac{(s-3)3^{s+2}+(s+7)2^{s+2}}{3^s(s+1)(s+2)(s+3)}|f''(a)|^q\right.\right.\\ &\left.+\frac{1}{3^s(s+2)(s+3)}|f''(b)|^q-\frac{c(b-a)^2}{45}\right]^{1/q}\\ &+\left[\frac{(s-1)2^{s+2}+s+5}{3^s(s+1)(s+2)(s+3)}\left(|f''(a)|^q+|f''(b)|^q\right)-\frac{11c(b-a)^2}{270}\right]^{1/q}\\ &+\left[\frac{1}{3^s(s+2)(s+3)}|f''(a)|^q\right.\\ &\left.\left.+\frac{(s-3)3^{s+2}+(s+7)2^{s+2}}{3^s(s+1)(s+2)(s+3)}|f''(b)|^q-\frac{c(b-a)^2}{45}\right]^{1/q}\right\}.\end{aligned} \tag{1.9}$$

In this paper, the authors introduce the concept of strongly extended *s*-convex function and establish a new identity. By this identity and Hölder's inequality, some new Simpson type for the product of strongly extended *s*-convex function and discussed and some results are obtained.

2. Definition and Integral Identities

Now the concept of strongly extended *s*-convex function is introduced.

Definition 2.1. A function $f:[a,b] \to R$ is said to be strongly extended s-convex with modulus $c>0$, if

$$f(tx+(1-t)y) \le t^s f(x)+(1-t)^s f(y)-ct(1-t)(x-y)^2 \tag{2.1}$$

is valid for all $x,y \in [a,b]$ and $t \in (0,1)$, some $s \in [-1,1]$.

For establishing new integral inequalities of Simpson type involving the strongly extended s-convex function, the following identity is needed:

Lemma 2.1. Let $f: I \subseteq R \to R$ be differentiable on I° and where $a,b \in I$ with

$a<b$. If $f'\in L([a,b])$, then the following identity holds:

$$\frac{1}{4}\left[f(a)+f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$
$$=\frac{b-a}{9}\left[\int_0^1\left(t-\frac{3}{4}\right)f'\left((1-t)a+t\frac{2a+b}{3}\right)\mathrm{d}t\right.$$
$$+\int_0^1\left(t-\frac{2}{4}\right)f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)\mathrm{d}t \tag{2.2}$$
$$\left.+\int_0^1\left(t-\frac{1}{4}\right)f'\left((1-t)\frac{a+2b}{3}+tb\right)\mathrm{d}t\right].$$

Proof. By straightforward computation, the result is followed. The proof is completed.

Lemma 2.2. ([4]). Let $f:I\subseteq R\to R$ be differentiable on I° and $a,b\in I$ with $a<b$. If $f'\in L([a,b])$, then

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$
$$=(b-a)\left(\int_0^{\frac{1}{2}}tf'(ta+(1-t)b)\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)f'(ta+(1-t)b)\mathrm{d}t\right). \tag{2.3}$$

3. Some Integral Inequalities of Simpson Type

Theorem 3.1. Let $f:I\subseteq R\to R_0$ be differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in L([a,b])$ and $|f'|^q$ is strongly extended s-convex on $[a,b]$ for $q\ge 1$, $s\in(-1,1]$, then

$$\left|\frac{1}{4}\left[f(a)+f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le\frac{b-a}{9\times16^{1-1/q}}\left\{5^{1-1/q}\left[\frac{3\times2^{2s+1}s+2^{2s+2}+1}{2^{2s+3}(s+1)(s+2)}|f'(a)|^q\right.\right.$$
$$\left.+\frac{3^{s+2}-2^{2s+2}+2^{2s+1}s}{2^{2s+3}(s+1)(s+2)}\left|f'\left(\frac{2a+b}{3}\right)\right|^q-\frac{71c(b-a)^2}{13824}\right]^{1/q}$$
$$+4^{1-1/q}\left[\frac{2^{s+1}+1}{2^{s+1}(s+1)(s+2)}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)-\frac{c(b-a)^2}{288}\right]^{1/q} \tag{3.1}$$
$$+5^{1-1/q}\left[\frac{3^{s+2}-2^{2s+2}+2^{2s+1}s}{2^{2s+3}(s+1)(s+2)}\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right.$$
$$\left.\left.+\frac{3\times2^{2s+1}s+2^{2s+2}+1}{2^{2s+3}(s+1)(s+2)}|f'(b)|^q-\frac{71c(b-a)^2}{13824}\right]^{1/q}\right\}.$$

Proof. Using Lemma 2.1 and by Hölder's inequality, the followings can be obtained:

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{b-a}{9}\left[\int_0^1\left|t-\frac{3}{4}\right|\left|f'\left((1-t)a+t\frac{2a+b}{3}\right)\right|\mathrm{d}t\right. \\
&\quad \left.+\int_0^1\left|t-\frac{2}{4}\right|\left|f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)\right|\mathrm{d}t+\int_0^1\left|t-\frac{1}{4}\right|\left|f'\left((1-t)\frac{a+2b}{3}+tb\right)\right|\mathrm{d}t\right] \\
&\le \frac{b-a}{9}\left[\left(\int_0^1\left|t-\frac{3}{4}\right|\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|t-\frac{3}{4}\right|\left|f'\left((1-t)a+t\frac{2a+b}{3}\right)\right|^q\mathrm{d}t\right)^{1/q}\right. \\
&\quad +\left(\int_0^1\left|t-\frac{2}{4}\right|\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|t-\frac{2}{4}\right|\left|f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)\right|^q\mathrm{d}t\right)^{1/q} \\
&\quad \left.+\left(\int_0^1\left|t-\frac{1}{4}\right|\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|t-\frac{1}{4}\right|\left|f'\left((1-t)\frac{a+2b}{3}+tb\right)\right|^q\mathrm{d}t\right)^{1/q}\right],
\end{aligned} \tag{3.2}$$

where,

$$\begin{aligned}
&\int_0^1\left|t-\frac{3}{4}\right|\mathrm{d}t=\frac{5}{16},\ \int_0^1 t(1-t)\left|t-\frac{3}{4}\right|\mathrm{d}t=\frac{71}{1536},\ \int_0^1\left|t-\frac{2}{4}\right|\mathrm{d}t=\frac{1}{4}, \\
&\int_0^1 t(1-t)\left|t-\frac{2}{4}\right|\mathrm{d}t=\frac{1}{32},\ \int_0^1\left|t-\frac{1}{4}\right|\mathrm{d}t=\frac{5}{16},\ \int_0^1 t(1-t)\left|t-\frac{1}{4}\right|\mathrm{d}t=\frac{71}{1536}.
\end{aligned} \tag{3.3}$$

Again $|f'|^q$ is strongly extended s-convex on $[a,b]$, so

$$\begin{aligned}
&\int_0^1\left|t-\frac{3}{4}\right|\left|f'\left((1-t)a+t\frac{2a+b}{3}\right)\right|^q\mathrm{d}t \\
&\le \int_0^1\left|t-\frac{3}{4}\right|\left((1-t)^s|f'(a)|^q+t^s\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)\mathrm{d}t \\
&\quad -\frac{c(b-a)^2}{9}\int_0^1 t(1-t)\left|t-\frac{3}{4}\right|\mathrm{d}t \\
&=\frac{3\times 2^{2s+1}s+2^{2s+2}+1}{2^{2s+3}(s+1)(s+2)}|f'(a)|^q \\
&\quad +\frac{3^{s+2}-2^{2s+2}+2^{2s+1}s}{2^{2s+3}(s+1)(s+2)}\left|f'\left(\frac{2a+b}{3}\right)\right|^q-\frac{71c(b-a)^2}{13824}.
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\int_0^1\left|t-\frac{2}{4}\right|\left|f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)\right|^q\mathrm{d}t \\
&\le \int_0^1\left|t-\frac{2}{4}\right|\left((1-t)^s\left|f'\left(\frac{2a+b}{3}\right)\right|^q+t^s\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)\mathrm{d}t \\
&\quad -\frac{c(b-a)^2}{9}\int_0^1 t(1-t)\left|t-\frac{2}{4}\right|\mathrm{d}t \\
&=\frac{2^{s+1}+1}{2^{s+1}(s+1)(s+2)}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)-\frac{c(b-a)^2}{288}.
\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\int_0^1\left|t-\frac{1}{4}\right|\left|f'\left((1-t)\frac{a+2b}{3}+tb\right)\right|^q\mathrm{d}t\\&\le\int_0^1\left|t-\frac{1}{4}\right|\left((1-t)^s\left|f'\left(\frac{a+2b}{3}\right)\right|^q+t^s\left|f''(b)\right|^q\right)\mathrm{d}t\\&\quad-\frac{c(b-a)^2}{9}\int_0^1 t(1-t)\left|t-\frac{1}{4}\right|\mathrm{d}t\\&=\frac{3^{s+2}-2^{2s+2}+2^{2s+1}s}{2^{2s+3}(s+1)(s+2)}\left|f'\left(\frac{a+2b}{3}\right)\right|^q\\&\quad+\frac{3\times 2^{2s+1}s+2^{2s+2}+1}{2^{2s+3}(s+1)(s+2)}\left|f'(b)\right|^q-\frac{71c(b-a)^2}{13824}.\end{aligned}\tag{3.6}$$

Substituting the above (3.3)-(3.6) into the inequality (3.2) results in the inequality (3.1).

Theorem 3.1 is proved.

Corollary 3.2. Under conditions of Theorem 3.1, if $q=1$, then

$$\begin{aligned}&\left|\frac{1}{4}\left[f(a)+f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\&\le\frac{b-a}{9}\left\{\frac{3\times 2^{2s+1}s+2^{2s+2}+1}{2^{2s+3}(s+1)(s+2)}\left|f'(a)\right|\right.\\&\quad+\frac{3^{s+2}+2^{2s+3}-2^{2s+2}+2^{s+2}+2^{2s+1}s}{2^{2s+3}(s+1)(s+2)}\left|f'\left(\frac{2a+b}{3}\right)\right|\\&\quad+\frac{3^{s+2}+2^{2s+3}-2^{2s+2}+2^{s+2}+2^{2s+1}s}{2^{2s+3}(s+1)(s+2)}\left|f'\left(\frac{a+2b}{3}\right)\right|\\&\quad\left.+\frac{3\times 2^{2s+1}s+2^{2s+2}+1}{2^{2s+3}(s+1)(s+2)}\left|f'(b)\right|-\frac{95c(b-a)^2}{6912}\right\}.\end{aligned}$$

Theorem 3.3. Let $f:I\subseteq R\to R_0$ be differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in L([a,b])$ and $|f'|$ is strongly extended s-convex on $[a,b]$ for $q\ge 1$, $s=-1$, then

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\&\le\frac{b-a}{8^{1-1/q}}\left\{\left[\frac{\left|f'(a)\right|^q+(2\ln 2-1)\left|f'(b)\right|^q}{2}-\frac{5c(b-a)^2}{192}\right]^{1/q}\right.\\&\quad\left.+\left[\frac{(2\ln 2-1)\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}-\frac{5c(b-a)^2}{192}\right]^{1/q}\right\}.\end{aligned}\tag{3.7}$$

Proof. Since $|f'|^q$ is strongly extended *s*-convex on $[a,b]$, using Lemma 2.2 and by Hölder's inequality, the followings can be obtained:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$

$$\le (b-a)\left[\int_0^{1/2} t\left|f'(ta+(1-t)b)\right|\mathrm{d}t+\int_{1/2}^1 (1-t)\left|f'(ta+(1-t)b)\right|\mathrm{d}t\right]$$

$$\le (b-a)\left[\left(\int_0^{1/2} t\mathrm{d}t\right)^{1-1/q}\left(\int_0^{1/2} t\left|f'(ta+(1-t)b)\right|^q\mathrm{d}t\right)^{1/q}\right.$$

$$\left.+\left(\int_{1/2}^1 (1-t)\mathrm{d}t\right)^{1-1/q}\left(\int_{1/2}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q\mathrm{d}t\right)^{1/q}\right]$$

$$\le \frac{b-a}{8^{1-1/q}}\left\{\left[\int_0^{1/2} t\left(t^{-1}\left|f'(a)\right|^q+(1-t)^{-1}\left|f'(b)\right|^q-ct(1-t)(b-a)^2\right)\mathrm{d}t\right]^{1/q}\right.$$

$$\left.+\left[\int_{1/2}^1 (1-t)\left(t^{-1}\left|f'(a)\right|^q+(1-t)^{-1}\left|f'(b)\right|^q-ct(1-t)(b-a)^2\right)\mathrm{d}t\right]^{1/q}\right\}$$

$$=\frac{b-a}{8^{1-1/q}}\left\{\left[\frac{\left|f'(a)\right|^q+(2\ln 2-1)\left|f'(b)\right|^q}{2}-\frac{5c(b-a)^2}{192}\right]^{1/q}\right.$$

$$\left.+\left[\frac{(2\ln 2-1)\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}-\frac{5c(b-a)^2}{192}\right]^{1/q}\right\}.$$

Theorem 3.3 is proved.

Theorem 3.4. Let $f: I\subseteq R\to R_0$ be differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in L([a,b])$ and $|f'|^q$ is strongly extended s-convex on $[a,b]$ for $q>1$, $s\in(-1,1]$, then

$$\left|\frac{1}{4}\left[f(a)+f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$

$$\le \frac{b-a}{9}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\left(\frac{1+3^{(2q-1)/(q-1)}}{4^{(2q-1)/(q-1)}}\right)^{1-1/q}\right.$$

$$\times\left(\frac{1}{s+1}\left|f'(a)\right|^q+\frac{1}{s+1}\left|f'\left(\frac{2a+b}{3}\right)\right|^q-\frac{c(b-a)^2}{54}\right)^{1/q} \tag{3.8}$$

$$+\frac{1}{2}\left(\frac{1}{s+1}\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\frac{1}{s+1}\left|f'\left(\frac{a+2b}{3}\right)\right|^q-\frac{c(b-a)^2}{54}\right)^{1/q}$$

$$\left.+\left(\frac{1+3^{(2q-1)/(q-1)}}{4^{(2q-1)/(q-1)}}\right)^{1-1/q}\left[\frac{1}{s+1}\left|f'\left(\frac{a+2b}{3}\right)\right|^q+\frac{1}{s+1}\left|f'(b)\right|^q-\frac{c(b-a)^2}{54}\right]^{1/q}\right\}.$$

Proof. By the Lemma 2.1 and using Hölder's inequality, the followings can be obtained:

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{b-a}{9}\left[\left(\int_0^1\left|t-\frac{3}{4}\right|^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|f'\left((1-t)a+t\frac{2a+b}{3}\right)\right|^q\mathrm{d}t\right)^{1/q}\right. \\
&+\left(\int_0^1\left|t-\frac{2}{4}\right|^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)\right|^q\mathrm{d}t\right)^{1/q} \\
&\left.+\left(\int_0^1\left|t-\frac{1}{4}\right|^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|f'\left((1-t)\frac{a+2b}{3}+tb\right)\right|^q\mathrm{d}t\right)^{1/q}\right],
\end{aligned} \tag{3.9}$$

where,

$$\begin{aligned}
&\int_0^1\left|t-\frac{3}{4}\right|^{q/(q-1)}\mathrm{d}t=\int_0^1\left|t-\frac{1}{4}\right|^{q/(q-1)}\mathrm{d}t=\frac{\left(1+3^{(2q-1)/(q-1)}\right)(q-1)}{4^{(2q-1)/(q-1)}(2q-1)}, \\
&\int_0^1\left|t-\frac{2}{4}\right|^{q/(q-1)}\mathrm{d}t=\frac{q-1}{2^{q/(q-1)}(2q-1)}.
\end{aligned} \tag{3.10}$$

Since $|f'|^q$ is strongly extended s-convex on $[a,b]$, so

$$\begin{aligned}
&\int_0^1\left|f'\left((1-t)a+t\frac{2a+b}{3}\right)\right|^q\mathrm{d}t \\
&\le\frac{1}{s+1}|f'(a)|^q+\frac{1}{s+1}\left|f'\left(\frac{2a+b}{3}\right)\right|^q-\frac{c(b-a)^2}{54},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\int_0^1\left|f'\left((1-t)\frac{2a+b}{3}+t\frac{a+2b}{3}\right)\right|^q\mathrm{d}t \\
&\le\frac{1}{s+1}\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\frac{1}{s+1}\left|f'\left(\frac{a+2b}{3}\right)\right|^q-\frac{c(b-a)^2}{54},
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\int_0^1\left|f'\left((1-t)\frac{a+2b}{3}+tb\right)\right|^q\mathrm{d}t \\
&\le\frac{1}{s+1}\left|f'\left(\frac{a+2b}{3}\right)\right|^q+\frac{1}{s+1}|f'(b)|^q-\frac{c(b-a)^2}{54}.
\end{aligned} \tag{3.13}$$

Substituting (3.10)-(3.13) into the inequality (3.9) yields (3.8). Theorem 3.4 is proved.

4. Conclusion

In this paper, the authors introduce the concept of strongly extended s-convex function and establish a new identity. Then by this identity and Hölder's inequality, some new Simpson type for the product of strongly extended s-convex function are obtained.

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