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On the Representations of Γ_1-Nonderanged Permutation Group $\mathcal{G}_p$

Ibrahim A. Aminu, Ejima Ojonugwa, Kazeem O. Aremu

Usmanu Danfodiyo University, Sokoto, Nigeria
Email: unusoj1@yahoo.com

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Abstract

Representation theory is concerned with the ways of explaining or visualizing a group as a group of matrices. In this paper, we extend the permutation pattern of $\omega_i = \left((1)(1+i)_{mp}(1+2i)_{mp}\cdots(1+(p-1)i)_{mp}\right)$ to a two-line notation. We consider the representations of this Γ_1 non-deranged permutation group $\mathcal{G}_p$ ($p \geq 5$ and p a prime). Also we reveal some interesting properties and results of the character $\chi(\omega_i)$ of $\mathcal{G}_p$ where $\omega_i \in \mathcal{G}_p$.

Keywords

Representation, Non-Deranged, Permutation Group, Γ_1-Permutation Group, $\mathbb{F}\mathcal{G}_p$-Module, Character

1. Introduction

The beauty of Group as a topic is the various properties that can arise from its studies. Its interesting nature has encouraged various studies in this field over the years. For instance, for every n a positive integer, the set of all permutations of $\{1,2,\cdots,n\}$, under the product operation of composition is a group. This group is known as a symmetric group (Permutation group) of degree n. According to [1], the study of the symmetric group by Georg Frobenius in 1903, opened the door to the various works that was further developed by many mathematicians, including Percy MacMahon, W. V. D. Hodge, G. de B. Robinson, Gian-Carlo Rota, Alain Lascoux, Marcel-Paul Schtzenberger and Richard P.

Conscious efforts by different researchers over the years led to the discovery of other form of permutation patterns, groups and their subsequent representations; [2] shows how functions acting on a finite set can be conveniently expressed using matrices, whereby the composition of functions corresponds to multiplication of the

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matrices. Essentially, they considered the induced action on the vector space with the elements of the set acting as a basis. This action extends to tensor powers of the vector space and can be extended also to symmetric powers, antisymmetric powers, etc., that yielded representations of the multiplicative semi-group of functions and representations of permutation groups.

To be precise, [3] described a representation as a homomorphism from G into a group of invertible matrices. [4] described a representation as an (linear) action of a group or Lie algebra on a vector space. (Say, for every $g \in G$ there is an associated operator $\pi(g)$, which acts on the vector space V.) In fact, π is a representation of G acting on the space V. Most of the informations contained in the representation of a group can be distilled into one simple statistic, the trace of the corresponding matrices; [5].

Over the years, deranged permutation, a permutation with no fix points has been studied with various results established. One of the many works in this field is the group theoretical interpretation of Bara'at Model by [6] to establish a deranged permutation pattern. The theoretic and topological properties have also been studied and established by [7]. More recently is the use of Catalan numbers by [8] to develop the scheme for prime numbers $P \geq 5$ and $\Omega \subseteq N$ which generate the cycles of permutation patterns using $\omega_i = \left((1)(1+i)_{mp}(1+2i)_{mp}\cdots(1+(p-1)i)_{mp}\right)$ to determine the arrangements.

This permutation pattern was further worked upon by [9] to establish a permutation group. This was achieved by embedding an identity element $\{1\}$ in the collection of $\omega_i = \left((1)(1+i)_{mp}(1+2i)_{mp}\cdots(1+(p-1)i)_{mp}\right)$. Furthermore on the discovery of the special permutation group G_p, several other works have been done to show some interesting results and properties of $\mathcal{G}_p$. Some of these works include [10] a paper that studied the Algebraic properties of the (132)—avoiding class of this permutation pattern and its applications to graph. The comparison of the group permutation pattern and generalized permutation patterns using Wilf-equivalence has also been studied by [11].

Besides, as established by [12], that not every transitive group contains a derangement. Hence we will in this paper, take a lead from the representations of symmetry groups as shown by [13] [14] and [15] to show the representations of Γ_1 non-deranged permutation group; this will be achieved by extending the work of [8], to a two-line notation; we will also introduce another identity element for this Γ_1 non-deranged permutation group while we study some other results as it relates to representations of groups.

2. Notation

In an attempt to simplify this paper, basic concepts and notation as related to the work are defined below.

Definition 2.1:

Γ_1-non deranged permutation group $\mathcal{G}_p$ is a special permutation group with a fixed element on the first column from the left.

Definition 2.2:

A permutation of a set X is a bijective function $\rho: X \to X$. It is a quantity or function that carries n indices or variables (where each can run from $1,\cdots,N$). For instance, Let Ω be a non-empty ordered set such that $\Omega \subset \mathbb{N}$. Let $\mathcal{G}_p = \{\omega_i : 1 \leq i \leq p-1\}$ be a subgroup of symmetry group S_n, such that every ω_i is generated by arbitrary set Ω for any prime $p \geq 5$ using the following

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & \cdots & (1+(p-1)i)_{mp} \end{pmatrix}. \tag{1}$$

Lemma 2.3:

The order of the group $\mathcal{G}_p$ $p \geq 5$ p, a prime is $p-1$ ($|\mathcal{G}_p| = p-1$).

Proof. We recall that Langrange's theorem says that order of the group is divisible by the order of the subgroup. If $\mathcal{G}_p \subseteq S_p$ then

$$\frac{|S_p|}{|\mathcal{G}_p|} = \frac{p!}{p-1} = q$$

where q is a positive integer. We claim that $p-1$ is a factor of $p!$ hence $p-1$ is the order of $\mathcal{G}_p$. □

Example 2.3.1:

For $p = 5$ Equation (1) will generate a Γ_1 permutation group $\mathcal{G}_5$

$$\omega_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = e\,(\text{the identity permutation}),\ \omega_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix},$$

$$\omega_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix},\ \omega_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$$

and written G_5 in cycle form,

$$\omega_1 = (e),\ \omega_2 = (2354),\ \omega_3 = (3245),\ \omega_4 = (43)(25).$$

Definition 2.4:

A representation of G over $\mathbb{F}$ is a homomorphism ρ from G to $GL(n,\mathbb{F})$, for some n. The degree of the ρ is the integer n. Thus if ρ is a function from G to $GL(n,\mathbb{F})$, then ρ is a representation if and only if $(gh)\rho = (g\rho)(h\rho)$ for all $g,h \in G$: Since a representation is a homomorphism, it follows that for every representation $\rho: G \to GL(n,\mathbb{F})$, we have

1) $1\rho = I_n$
2) $g^{-1}\rho = (g\rho)^{-1}$

for all $g \in G$, where I_n denotes the $n \times n$ identity matrix.

Definition 2.5:

Let G be a subgroup of S_n, so that G is a group of permutations of $\{1,\cdots,n\}$. Let V be a n-dimensional vector space over $\mathbb{F}$, with bases $v_1 \cdots v_n$ for each i with $1 \le i \le n$ and each permutation $g \in G'$ define $v_i g = v_{ig}$ for all i, and all $g \in G$ is called the permutation module for G over F. We call $v_1,\cdots,v_n$ the natural basis of V.

Definition 2.6:

Two-line notation is a notation used to describe a permutation on a (usually finite) set. For a finite set suppose S is a finite set and ρ is a permutation. The two-line notation for ρ is a description of ρ in two aligned rows. The top row lists the elements of S, and the bottom row lists, under each element of S, its image under ρ.

If $S = \{a_1, a_2, \cdots, a_n\}$, the two-line notation for ρ is:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ \rho(a_1) & \rho(a_2) & \rho(a_3) & \cdots & \rho(a_n) \end{pmatrix}$$

Definition 2.7:

Consider a finite set S and an ordering of the elements of S, with the elements (in order), given as $a_1, a_2, \cdots, a_n$. For a permutation ρ of S, the one-line notation for ρ is the string $\rho(a_i)$. The one-line notation for a permutation is a compressed form for the two-line notation where the first line is omitted because it is implicitly understood.

3. Representation of $\mathcal{G}_p$

In considering Γ_1-non deranged permutation group $\mathcal{G}_p$, and its representation. Let $\mathcal{G}_p$ be a group and let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $GL(p,\mathbb{F})$ denotes the group of invertible $p \times p$ matrices with entries in $\mathbb{F}$.

A representation of $\mathcal{G}_p$ over $\mathbb{F}$ is a homomorphism ρ from $\mathcal{G}_p$ to $GL(p,\mathbb{F})$ for some p. The degree of ρ is the prime p. Thus if ρ is a function from $\mathcal{G}_p$ to $GL(p,\mathbb{F})$, then ρ is representation if and only if $(\omega_i\omega_j)\rho = (\omega_i\rho)(\omega_j\rho)$ for all $\omega_i, \omega_j \in \mathcal{G}_p$. Since a representation is a homomorphism, it follows that for every representation $\rho: \mathcal{G}_p \to GL(p,\mathbb{F})$, we have $1\rho = I_p$ and $\omega_i^{-1}\rho = (\omega_i\rho)^{-1}$ for all $\omega_i \in \mathcal{G}_p$ where I_p denotes the $p \times p$ identity matrix

3.1. $\mathcal{G}_p$ as $\mathbb{F}\mathcal{G}_p$-Module

We need to introduce the concept of an $\mathbb{F}\mathcal{G}_p$ module, and show that there is a close connection between the $\mathbb{F}\mathcal{G}_p$ module and the representation of this Γ_1-non deranged permutation group $\mathcal{G}_p$ over $\mathbb{F}$.

Let $\mathcal{G}_p$ be a group and let $\mathbb{F}$ be $\mathbb{R}$. suppose that $\rho: G \to GL(p,\mathbb{F})$ is representation of $\mathcal{G}_p$ write $V = \mathbb{F}^p$, the vector space of all row vectors $(\lambda_1 \cdots \lambda_p)$ with $\lambda_i \in F$ for all $v \in V$ and $\omega_i \in \mathcal{G}_p$, the matrix product $v(\omega_i\rho)$ of the row vector V with the $p \times p$ matrix $\omega_i\rho$ is a row vector in V. (since the product of a

$1\times p$ matrix with an $p\times p$ matrix is again $1\times p$ matrix).

3.2. Proposition

Let V be a vector space over $\mathbb{F}$ and let $\mathcal{G}_p$ be a group then V is an $\mathbb{F}\mathcal{G}_p$-module if a multiplication $v\omega_i$ ($v\in V$, $\omega_i\in\mathcal{G}_p$) is defined satisfying the following conditions for all $u,v\in V,\lambda\in F$ and $\omega_i,\omega_j\in\mathcal{G}_p$.

1) $v\omega_i\in V$

2) $v\left(\omega_i\omega_j\right)=\left(v\omega_i\right)\omega_j$

3) $v1=v$

4) $\left(\lambda v\right)\omega_i=\lambda\left(v\omega_i\right)$

5) $\left(u+v\right)\omega_i=u\omega_i+v\omega_i$

Proof:

1) Let $v\in V,\omega_i\in\mathcal{G}_p$ such that $\omega_i=\left(\alpha_1\cdots\alpha_p\right)$. Then

$$v\omega_i=\left(v\alpha_1\cdots v\alpha_p\right)$$

which implies that $v\alpha_i=q_i$; $1\le i\le p$ and $q\in V$. Therefore $v\omega_i=q\in V$

2) Let $v\in V$ and $\omega_i,\omega_j\in\mathcal{G}_p$ and are given as $\omega_i=\left(\alpha_1\cdots\alpha_p\right)$ and $\omega_j=\left(\rho_1\cdots\rho_p\right)$, then

$$\begin{aligned}v\left(\omega_i\omega_j\right)&=v\left[\left(\alpha_1\cdots\alpha_p\right)\left(\rho_1\cdots\rho_p\right)\right]\\&=v\left(\alpha_1\rho_1\alpha_2\rho_2\cdots\alpha_p\rho_p\right)\quad\text{where } \alpha_i\rho_i=\delta_i\in\mathbb{G}_p\\&=\left(v\alpha_1\rho_1v\alpha_2\rho_2\cdots v\alpha_p\rho_p\right)\\&=\left(v\alpha_1\right)\rho_1\left(v\alpha_2\right)\rho_2\cdots\left(v\alpha_p\right)\rho_p\\&=\left(v\omega_i\right)\omega_j\end{aligned}$$

3) Let $v\in V$ and $e\in\mathcal{G}_p$

$$\begin{aligned}ve&=v\left(1_i\cdots1_p\right)\\&=v1_i\cdots v1_p\\&=v\end{aligned}$$

4) Let $\lambda\in\mathbb{F},v\in V,\omega_i\in\mathcal{G}_p$ then

$$\left(\lambda v\right)\omega_i=u\left(\omega_i\right)\quad\text{where } \lambda v=u,u\in V$$

$$\text{hence } u\omega_i=\lambda v\omega_i=\lambda\left(v\omega_i\right)$$

5) Let $u,v\in V$ and $\omega_i\in\mathcal{G}_p$ and given that $\omega_i=\left(\alpha_1\cdots\alpha_p\right)$ then

$$\begin{aligned}\left(u+v\right)\omega_i&=\left(u+v\right)\left(\alpha_1\cdots\alpha_p\right)\\&=\left(u+v\right)\alpha_1\left(u+v\right)\alpha_2\cdots\left(u+v\right)\alpha_p\\&=\left(u\alpha_1+v\alpha_1\right)\left(u\alpha_2+v\alpha_2\right)\cdots\left(u\alpha_p+v\alpha_p\right)\\&=\left(u\alpha_1u\alpha_2\cdots u\alpha_p\right)+\left(v\alpha_1v\alpha_2\cdots v\alpha_p\right)\\&=u\left(\alpha_1\cdots\alpha_p\right)+v\left(\alpha_1\cdots\alpha_p\right)\\&=u\omega_i+v\omega_i\end{aligned}$$

hence, our group $\mathcal{G}_p$ is an $\mathbb{F}\mathcal{G}_p$-module.

□

3.3. Corollary

Let $\mathcal{G}_p$ be a subgroup of S_p ($p\ge5$; p a prime) so that $\mathcal{G}_p$ is a Γ_1-permutation group of S_p. Let

$\rho:\mathcal{G}_p \to GL(p,\mathbb{F})$ be a representation $\mathcal{G}_p$ define as $\rho\omega_i = v$ ($\rho\omega_i = \rho(\alpha_1\alpha_2\cdots\alpha_p) = v_i$ where $\alpha_i \in \omega_i$), for all $\omega_i \in \mathcal{G}_p$ and V a p-dimensional vector space over $\mathbb{F}$, with bases $v_1 \cdots v_p$ for each $1 \le i \le p$. Then $\mathcal{G}_p$ is a permutation module over F with natural bases $v_1,\cdots,v_p$ of V.

Example

Let $\mathcal{G}_p = \mathcal{G}_5$ and let B denote the basis v_1, v_2, v_3, v_4, and v_5 of V. If $\omega_i = (2354) \in \mathcal{G}_5$ then

$$\rho\omega_i = v_i$$

$$\rho\alpha_1 = v_1,\ \rho\alpha_2 = v_3,\ \rho\alpha_3 = v_5,\ \rho\alpha_4 = v_2,\ \rho\alpha_5 = v_4$$

And if $\omega_j = (3245) \in S_5$ then

$$\rho\alpha_1 = v_1,\ \rho\alpha_2 = v_4,\ \rho\alpha_3 = v_2,\ \rho\alpha_4 = v_5,\ \rho\alpha_5 = v_3$$

We have

$$[\omega_i]_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$[\omega_j]_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

3.4. Character of a Representation

Suppose that $\rho: G \to GL(n,C)$ is a representation of a finite group G. With each $n \times n$ matrix $g\rho\,(g \in G)$, we associate the complex number given by adding all the diagonal entries of the matrix, and call this number $\chi(g)$. The function $\chi: G \to C$ is called the character of the representation ρ.

Suppose that V is an CG-module with basis β. Then the character of V is the function $\chi: G \to C$ define by

$$\chi(g) = tr[g]_\beta \text{ where } g \in G.$$

Naturally enough, we define the character of a representation $\rho: G \to GL(n,C)$ to be the character of the corresponding CG-module C^n namely

$$\chi(g) = tr(g\rho) \text{ where } g \in G.$$

Similarly, suppose that V is an $\mathbb{F}\mathcal{G}_p$-module with basis β. Then the character of V is the function $\chi: \mathcal{G}_p \to R$ define by

$$\chi(\omega_i) = tr[\omega_i]_\beta \text{ where } \omega_i \in \mathcal{G}_p.$$

Naturally enough, we can also define the character of our representation $\rho: \mathcal{G}_p \to GL(p,\mathbb{R})$ to be the character of the corresponding $\mathbb{R}\mathcal{G}_p$-module $\mathbb{R}^p$ namely

$$\chi(\omega_i) = tr[\omega_i]_\beta \text{ where } \omega_i \in \mathcal{G}_p.$$

3.5. Theorem

Let $\mathcal{G}_p$ be a Γ_1-non deranged permutation group, (p = prime and $p \ge 5$), the character $\chi(\omega_i)$ of $\omega_i \in \mathcal{G}_p$ is never zero.

$$\chi(\omega_i) \neq 0, \omega_i \in \mathcal{G}_p, p \geq 5$$

Proof:

To prove that $\chi(\omega_i) \neq 0, \omega_i \in \mathcal{G}_p, p \geq 5$, then it's sufficient enough if we can show at least one diagonal element of $[\rho\omega_i] \neq 0$ since $\chi(\omega_i) = \sum_{i=1}^{p} a_{ii}$.

Suppose that $\rho : \mathcal{G}_p \to GL(p, \mathbb{R})$ and recall that $\omega_i \in \mathcal{G}_p$ is defined as

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & \cdots & (1+(p-1)i)_{mp} \end{pmatrix}$$

$$\Rightarrow \text{the leading diagonal element for } [\rho\omega_i] = 1$$

Therefore the character of every $[\rho\omega_i]$ is at least 1 $(1 \leq i \leq p)$. □

3.6. Corollary

Every $\omega_i \in \mathcal{G}_p$ (where $i \neq 1$) has a trivial character.

3.7. Theorem

Let $\mathcal{G}_p$ be a permutation group, (p = prime and $p \geq 5$) and $\omega_1 = e$, then the character χ of $\omega_i \in \mathcal{G}_p$ is

$$\chi(\omega_i) = \begin{cases} 1, & \text{if } \omega_i \neq \omega_1; \\ p, & \text{if } \omega_i = \omega_1. \end{cases}$$

Proof:

From **Corollary 3.6** above the first part of the proof is obvious, for the second part.

Let $\mathcal{G}_p$ be a subgroup of S_n, so that $\mathcal{G}_p$ is a group of permutations of $\{1, 2, \cdots, p\}$. Let V be a p-dimensional vector space over $\mathbb{F}$, with basis $\omega_i = (v_1\ v_2 \cdots v_p)$ from 2.1 it implies that

$$\omega_1 = e$$

then for all $p \geq 5$, the representation ω_1 will be

$$\alpha_1 \to v_1, \alpha_2 \to v_2, \cdots, \alpha_p \to v_p.$$

Applying the definition 2.2 and corollary 3.5, then taking the summation of the diagonal elements will give p as the character. □

4. Conclusion

This paper has extended the one line permutation pattern of Abor and Ibrahim (2010) to a two-line notation and hence $\mathcal{G}_p$ is a Γ_1-non deranged permutation group with a natural identity. The representation of $\mathcal{G}_p$ as a finite permutation group was done and the character of $\mathcal{G}_p$ was computed to be 1 if $\omega_i = e$ otherwise p.

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