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A Remark on Eigenfunction Estimates by Heat Flow

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Abstract

In this paper, we consider L^∞ estimates of eigenfunction, or more generally, the L^∞ estimates of equation $-\Delta u = fu$. We use heat flow to give a new proof of the L^∞ estimates for such type equations.

Keywords

L^∞ Estimates, Eigenfunction, Heat Flow

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n > 2)$ be a bounded domain. Assume $u \in C^2(\Omega)$, we consider the Laplacian equation

$$-\Delta u = fu,$$

where $|f| \in L^\infty(\Omega)$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ with $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. This is a second order differential equation. If $f = \lambda$ is a constant, then u is an eigenfunction with eigenvalue λ. By a standard Moser's iteration in [1]-[5], we have L^∞ interior estimates of u controlled by the L^p norm of u for $p > 0$. In this paper, we use heat flow to consider the L^∞ estimate and give a new proof of the L^∞ estimates without using iteration. First, we recall the definition of the heat kernel. For any $x, y \in \mathbb{R}^n$ and $t > 0$, let

$$\rho_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}$$

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be the heat kernel in $\mathbb{R}^n$. For fixed $y \in \mathbb{R}^n$, we know that

$$\left(\partial_t - \Delta_x\right)\rho_t(x, y) = 0,$$

where Δ_x is the standard Laplacian in $\mathbb{R}^n$ with respect to *x*. Our main result is the following

Theorem 1. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* $n > 2$. *Assume* $u \in C^2(\Omega)$ *and*

$$-\Delta u = fu$$

on Ω with $|f| \le A$. Then for any $p > n/2$ and any compact sub-domain $\Omega' \subset \Omega$, we have the interior L^∞ estimate

$$\sup_{x \in \Omega'} |u| \le C\left(p, n, A, dist\left(\Omega', \partial\Omega\right)\right)\left(\int_\Omega |u|^p(y)\,\mathrm{d}y\right)^{1/p}, \tag{1}$$

where $dist\left(\Omega', \partial\Omega\right)$ is the distance of Ω' and the boundary of Ω.

Remark 2. *Following from the proof, one can consider equation* $-\Delta u = fu + g$ *or* $\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j u = fu$ *by choosing appropriate kernel function* ρ_t.

2. Proving the Theorem

To estimates on $\Omega' \subset \Omega$, by the translation invariant and scaling invariant of the estimates, we only need to consider $\Omega = B_1(0)$ and $\Omega' = B_{1/2}(0)$. By using heat flow, we have the following lemma.

Lemma 1. *Let* $B_1(0) \subset \mathbb{R}^n$ *be a unite ball. Assume* $u \in C^2\left(B_1(0)\right)$ *and*

$$-\Delta u = fu$$

on $B_1(0)$ with $|f| \le A$. Then for any $y \in B_{1/2}(0)$, we have the interior L^∞ estimate

$$|u|(y) \le C(n, A)\int_{B_1(0)} \frac{|u|(x)}{|x - y|^{n-2}}\,\mathrm{d}y. \tag{2}$$

Proof. Let $\phi(x)$ be a standard smooth cutoff function with support in $B_1(0)$ and $\phi \equiv 1$ on $B_{3/4}(0)$, moreover, $|\Delta\phi| + |\nabla\phi| \le C(n)$. For any $y \in B_{1/2}(0)$, let

$$\Psi_t(y) = \int_{B_1(0)} \phi(x)u(x)\rho_t(x, y)\,\mathrm{d}x.$$

By the heat equation $\left(\partial_t - \Delta_x\right)\rho_t(x, y) = 0$, integrating by parts, we have

$$\partial_t\Psi_t(y) = \int_{B_1(0)} \phi(x)u(x)\partial_t\rho_t(x, y)\,\mathrm{d}x \tag{3}$$

$$= \int_{B_1(0)} \phi(x)u(x)\Delta_x\rho_t(x, y)\,\mathrm{d}x \tag{4}$$

$$= \int_{B_1(0)} \Delta(\phi u)\rho_t(x, y)\,\mathrm{d}x \tag{5}$$

$$= \int_{B_1(0)} \left(\Delta\phi u + \phi\Delta u + 2\langle\nabla\phi, \nabla u\rangle\right)\rho_t(x, y)\,\mathrm{d}x \tag{6}$$

$$= \int_{B_1(0)} \left(-\Delta\phi u + \phi\Delta u - 2\langle\nabla\phi, \nabla\log\rho_t(x, y)\rangle u\right)\rho_t(x, y)\,\mathrm{d}x \tag{7}$$

$$= \int_{B_1(0)} \left(-\Delta\phi u + \phi fu + \frac{2}{t}\langle\nabla\phi, x - y\rangle u\right)\rho_t(x, y)\,\mathrm{d}x, \tag{8}$$

where we use integrating by parts for term $2\langle\nabla\phi, \nabla u\rangle\rho_t(x, y)$ to get (7) from (6). By direct estimate, since $\nabla\phi(x) = 0$ for $x \in B_{3/4}(0)$ and $y \in B_{1/2}(0)$, then $|\langle\nabla\phi, x - y\rangle| \le C(n)$. Therefore, for $t \le 1$, we have

$$\left(|\Delta\phi| + t^{-1}|\langle\nabla\phi, x - y\rangle|\right)\rho_t(x, y) \le C(n)t^{-1-n/2}\mathrm{e}^{-C(n)/t} \le C(n).$$

Hence, for $t \le 1$ and noting that $|\phi| \le 1$, we have

$$\left|\partial_t \Psi_t(y)\right| \le C(n)\int_{B_1(0)} |u|(x) + C(n)\int_{B_1(0)} |f|(x)\cdot|u|(x)\rho_t(x,y)\,\mathrm{d}x.$$

Since $|f| \le A$, then we have

$$\left|\partial_t \Psi_t(y)\right| \le C(n)\int_{B_1(0)} |u|(x) + C(n,A)\int_{B_1(0)} |u|(x)\rho_t(x,y)\,\mathrm{d}x.$$

By the property of heat kernel, we have $\Psi_0(u) = u(y)$. Then we have

$$\left|u(y) - \Psi_1(y)\right| \le \int_0^1 \left|\partial_t \Psi_t(y)\right|\mathrm{d}t \le C(n)\int_{B_1(0)} |u|(x) + C(n,A)\int_0^1\int_{B_1(0)} |u|(x)\rho_t(x,y)\,\mathrm{d}x\mathrm{d}t.$$

On the other hand, as $n > 2$, we have

$$\int_0^1 \rho_t(x,y)\,\mathrm{d}t = \int_0^1 (4\pi t)^{-n/2}\,\mathrm{e}^{-|x-y|^2/4t}\mathrm{d}t = (4\pi)^{-n/2}\int_1^\infty s^{-2+\frac{n}{2}}\mathrm{e}^{-s\frac{|x-y|^2}{4}}\mathrm{d}s \le C(n)|x-y|^{2-n}. \tag{9}$$

Combining with $\left|\Psi_1(y)\right| \le C(n)\int_{B_1(0)} |u|(x)\,\mathrm{d}x$, we have

$$|u|(y) \le C(n,A)\int_{B_1(0)} \frac{|u|(x)}{|x-y|^{n-2}}\,\mathrm{d}x.$$

Hence we finish the proof.

The following lemma is fundamental.

Lemma 2. *For any* $y \in B_1(0)$ *and any* $0 < p < n$, *we have*

$$\int_{B_1(0)} \frac{1}{|x-y|^p}\,\mathrm{d}x \le C(n,p).$$

Proof. Let $r_i = 2^{-i}$ and $A_i = B_{r_{i-1}}(y) \setminus B_{r_i}(y)$. Then

$$\int_{B_1(0)} \frac{1}{|x-y|^p}\,\mathrm{d}x \le \sum_{i=0}^{\infty}\int_{A_i} \frac{1}{|x-y|^p}\,\mathrm{d}x \le \sum_{i=0}^{\infty} r_i^{-p}\int_{A_i}\mathrm{d}x \tag{10}$$

$$\le \sum_{i=0}^{\infty} r_i^{-p}C(n)r_{i-1}^n \le C(n)\sum_{i=0}^{\infty} r_i^{n-p} \le C(n,p). \tag{11}$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. *Refmaintheorem.* For any compact subset $\Omega' \subset \Omega$, let $2r := dist(\Omega', \partial\Omega)$. For any $x \in \Omega'$, we have $B_r(x) \subset \Omega$. Consider equation

$$-\Delta u = fu,$$

on $B_r(x)$. By Lemma 1, since the estimates are scaling invariant, we have

$$\left|u(x)\right| \le C(r,n,A)\int_{B_r(x)} \frac{|u|(y)}{|x-y|^{n-2}}\,\mathrm{d}y \le C(r,n,A)\left(\int_{B_r(x)} |u|^p(y)\,\mathrm{d}y\right)^{1/p}\left(\int_{B_r(x)} |x-y|^{-\frac{p(n-2)}{p-1}}\right)^{(p-1)/p}.$$

If $p > n/2$, then $p(n-2)/(p-1) < n$. By Lemma 2, we have

$$|u|(x) \le C(r,n,A,p)\left(\int_{B_r(x)} |u|^p(y)\,\mathrm{d}y\right)^{1/p} \le C(r,n,A,p)\left(\int_{\Omega} |u|^p(y)\,\mathrm{d}y\right)^{1/p}.$$

Hence we finish the proof.

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