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On Hom-Lie Pseudo-Superalgebras

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Abstract

The aim of this article is to introduce the notion of Hom-Lie *H*-pseudo-superalgebras for any Hopf algebra *H*. This class of algebras is a natural generalization of the Hom-Lie pseudo-algebras as well as a special case of the Hom-Lie superalgebras. We present some construction theorems of Hom-Lie *H*-pseudo-superalgebras, reformulate the equivalent definition of Hom-Lie *H*-pseudo-superalgebras, and consider the cohomology theory of Hom-Lie *H*-pseudo-superalgebras with coefficients in arbitrary Hom-modules as a generalization of Kac's result.

Keywords

Hom-Associative Pseudo-Superalgebra, Hom-Lie Pseudo-Superalgebra, Hom-Lie Conformal Superalgebra, Hom-Annihilation Superalgebra, Cohomology

1. Introduction

The notion of conformal algebras [1]-[5] was introduced by Kac as a formal language describing the singular part of the operator product expansion in two-dimensional conformal field theory, and it came to be useful for investigation of vertex algebras (see [6]-[8]). The concept of vertex algebras was derived from mathematical physics; it was first mathematically defined and considered by Borcherds in [9] to obtain his solution of the Moonshine conjecture in the theory of finite simple groups.

As a generalization of conformal algebras, Bakalov, D'Andrea and Kac [10] developed a theory of "multi-dimensional" lie conformal algebras, called Lie *H*-pseudo-algebras for any Hopf algebra *H*. Classification problems, cohomology theory and representation theory have been considered in [10]-[12]. In [13], Boyallian and Liberati studied pseudo-algebras from the point of view of pseudo-dual of classical Lie coalgebra structures by defining the notions of Lie *H*-coalgebras and Lie pseudo-bialgebras.

In [14], Sun generalized the pseudo-algebra structures to the Hom-pseudo-algebras of associative and Lie type,

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and showed some examples of the new structures and construction theorems. Hom-algebras were firstly studied by Hartwig, Larsson and Silvestrov in [15], where they introduced the structure of Hom-Lie algebras in the context of the deformations of Witt and Virasoro algebras. Later, Larsson and Silvestrov extended the notion of Hom-Lie algebras to quasi-Hom Lie algebras and quasi-Lie algebras (see [16]). Recently, Yau laid the foundation of a homology theory for Hom-Lie algebras and constructed the enveloping algebras of Hom-Lie and Hom-Leibniz algebras in [17]-[19]. Many more properties and structures of Hom-Lie algebras have been developed (see [20]-[23] and references cited therein).

In [24], Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov as a generalization of ordinary algebras and coalgebras in the following sense: the associativity of the multiplication was replaced by the Hom-associativity and similar for Hom-coassociativity. They also defined the structures of Hom-bialgebras and Hom-Hopf algebras, and described some of their properties extending properties of ordinary bialgebras and Hopf algebras in [25] and [26]. Different to Makhlouf and Silvestrov's work, Caenepeel and Goyvaerts studied the Hom-Hopf algebras from a categorical view point in [27], and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively (for more details about monoidal Hom-Hopf algebras, see references [28]-[32] and references cited therein).

In [33], Ammar and Makhlouf introduced the notion of Hom-Lie superalgebras and provided a construction theorem from which one can derive a one parameter family of Hom-Lie superalgebras deforming the orthosymplectic Lie superalgebras. The notion of Hom-Lie superalgebras is a natural and meaningful generalization of Lie superalgebras which were introduced by Kac in [3]. Motivated by [4] [10], in which Kac formulated the notion of conformal superalgebras and considered the classification theorem and representation theory of conformal superalgebras. We think whether we can extend the notions of Hom-Lie pseudo-algebras and conformal superalgebras to Hom-Lie pseudo-superalgebras.

Cohomology is an important tool in mathematics. Its range of applications contains algebra and topology as well as the theory of smooth manifolds or of holomorphic functions. The cohomology theory of Lie algebras was developed by Chevalley, Eilenberg and Cartan. Scheunert and Zhang introduced and investigated the cohomology groups of Lie superalgebras in [34]. Naturally, we think whether we can extend the notion of cohomology groups to Hom-Lie *H*-pseudo-superalgebras. This becomes our second motivation of the paper.

To give a positive answer to the questions above, we organize this paper as follows. In Section 2, we recall some basic definitions about Lie pseudo-algebras. In Section 3, we define Hom-Lie pseudo-superalgebras and introduce two construction theorems of Hom-Lie pseudo-superalgebras (see Proposition 3.12 and Theorem 3.13). In Section 4, we mainly discuss the annihilation superalgebras of Hom-pseudo-superalgebras (see Proposition 4.5). In Section 5, we determine some equivalent definitions of Hom-pseudo-superalgebras. In Section 6, we discuss the cohomology of Hom-Lie *H*-pseudo-superalgebras (see Theorem 6.1).

2. Preliminaries

In this section we recall some basic definitions and results related to our paper. Throughout the paper, all algebraic systems are supposed to be over a field k of characteristic 0, H always denotes a Hopf algebra with an antipode S. We summarize in the following the ungraded definitions of Hom-associative and Hom-Lie *H*-pseudo-algebras (see [14]). The reader is referred to Sweedler [35] about Hopf algebras, the Sweedler-type notation for the comultiplication is denoted by: $\Delta(x)=x_{(1)}\otimes x_{(2)}$.

Recall that a pseudotensor category $M^*(H)$ is a category whose objects are the same objects as in the category ${}_H M$ of left H-modules, but with a non-trivial pseudotensor structure, see [10].

A Hom-associative *H*-pseudo-algebra [14] is a triple $(A,\mu=*,\alpha)$ consisting of a linear space A in $M^*(H)$, an operation $\mu\in Hom_{H^{\otimes 2}}\left(A\otimes A,H^{\otimes 2}\otimes_H A\right)$ and a homomorphism $\alpha\in Hom_H(A,A)$ satisfying

$$\alpha(a)*(b*c)=(a*b)*\alpha(c),a,b,c\in A. \tag{2.1}$$

A Hom-Lie *H*-pseudo-algebra [14] is a triple $(L,\mu=[*],\alpha)$ consisting of a linear space L in $M^*(H)$, an operation $\mu\in Hom_{H^{\otimes 2}}\left(L\otimes L,H^{\otimes 2}\otimes_H L\right)$ and a homomorphism $\alpha\in Hom_H(L,L)$ satisfying the following axioms ($a,b,c\in L$):

1) Skew-commutativity:

$$[b*a]=-(\sigma\otimes_H id)[a*b]. \tag{2.2}$$

2) Hom-Jacobi identity:

$$\left[\alpha(a)*[b*c]\right]-\left((\sigma\otimes id)\otimes_H id\right)\left[\alpha(b)*[a*c]\right]=\left[[a*b]*\alpha(c)\right]. \quad (2.3)$$

An elementary but important property of Hom-Lie *H*-pseudo-algebra is that each Hom-associative *H*-pseudo-algebra gives rise to a Hom-Lie *H*-pseudo-algebra via the commutator bracket.

A Hom-Lie *H*-conformal algebra ([14]) is a triple $(L,[,],\alpha)$ consisting of a linear space *L* in ${}_H M$, an operation $[,]: L\otimes L\to H\otimes L$ and a homomorphism $\alpha\in Hom_H(L,L)$ satisfying the following axioms ($a,b,c\in L$ and $h\in H$):

1) *H*-sesqui-linearity:

$$[ha,b]=(h\otimes 1)[a,b],[a,hb]=\left(1\otimes h_{(2)}\right)[a,b]\left(S\left(h_{(1)}\right)\otimes 1\right). \quad (2.4)$$

2) Skew-commutativity:

$$[b,a]=-\sum_i S\left(h_{i_{(1)}}\right)\otimes h_{i_{(2)}}c_i,\ \text{if } [a,b]=\sum_i h_i\otimes c_i. \quad (2.5)$$

3) Hom-Jacobi identity:

$$\left[\alpha(a),[b,c]\right]-(\sigma\otimes id)\left[\alpha(b),[a,c]\right]=\left(F^{-1}\otimes id\right)\left[[a,b],\alpha(c)\right]. \quad (2.6)$$

Recall from Sun [14] we know that one can reformulate the definition of a Hom-Lie *H*-pseudo-algebra via a Hom-Lie *H*-conformal algebra.

3. Hom-Pseudo Superalgebras of Associative and Lie Types

In this section we will introduce the concept and construction theorems of Hom-*H*-pseudo-superalgebras of associative and Lie types, and show some examples of Hom-Lie *H*-pseudo-superalgebras that are neither Hom-Lie superalgebras nor Hom-Lie pseudo-algebras.

Definition 3.1. A Hom-associative H-pseudo-superalgebra is a triple $(A,\mu=*,\alpha)$ consisting of a superspace *A* in $M^*(H)$, an even operation $\mu\in Hom_{H^{\otimes 2}}\left(A\otimes A,H^{\otimes 2}\otimes_H A\right)$ and an even homomorphism $\alpha\in Hom_H(A,A)$ satisfying

$$\alpha(a)*(b*c)=(a*b)*\alpha(c) \quad (3.1)$$

in $H^{\otimes 3}\otimes_H A$ for all homogeneous elements $a,b,c\in A$.

Example 3.2. For a one dimensional Hopf algebra *H* = *k*, a Hom-associative *H*-pseudo-superalgebra is just a Hom-associative superalgebra over *k*. If $\alpha=id$, then a Hom-associative *H*-pseudo-superalgebra is an associative H-pseudo-superalgebra.

A Hom-associative *H*-pseudo-superalgebra (A,μ,α) is called multiplicative if $\left(id_{H^{\otimes 2}}\otimes_H\alpha\right)\mu=\mu(\alpha\otimes\alpha)$. For example, if $\alpha=id_A$, then the Hom-associative *H*-pseudo-superalgebra (A,μ,α) is multiplicative.

Let (A,μ_A,α_A) and (B,μ_B,α_B) be two (multiplicative) Hom-associative *H*-pseudo-superalgebras, an even homomorphism $f:A\to B$ is said to be a morphism of Hom-associative *H*-pseudo-superalgebras if

$$\left(id_{H^{\otimes 2}}\otimes_H f\right)\mu_A=\mu_B(f\otimes f),\ f\alpha_A=\alpha_B f. \quad (3.2)$$

Definition 3.3. Let $(A,*,\alpha)$ be a Hom-associative *H*-pseudo-superalgebra and *M* be a superapace in ${}_H M$. A Hom-*A*-module is a triple (M,ρ_M,α_M), where ρ_M is an even morphism in $Hom_{H^{\otimes 2}}\left(A\otimes M,H^{\otimes 2}\otimes_H M\right)$, α_M is an even morphism in $Hom_H(M,M)$ and satisfies the following properties ($a,b\in A,m\in M$):

$$\alpha_A(a)*(b*m)=(a*b)*\alpha_M(m), \quad (3.3)$$

$$\left(id_{H^{\otimes 2}}\otimes_H\alpha_M\right)(a*m)=\alpha_A(a)*\alpha_M(m), \quad (3.4)$$

where $a*m=\rho_M(a\otimes m)$.

Example 3.4. Let (A,α) be a finite dimensional Hom-associative superalgebra, *H* be a Hopf algebra. Then $(H\otimes A,*,id\otimes\alpha)$ is a Hom-associative *H*-pseudo-superalgebra with pseudoproduct $*$ given by

$$(f\otimes a)*(g\otimes b)=f\otimes g\otimes(1\otimes_H ab)$$

for all $f,g\in H$ and homogeneous elements $a,b\in A$.

Definition 3.5. A Hom-Lie H-pseudo-superalgebra is a triple $(L,\mu=[*],\alpha)$ consisting of a superspace L in $M^*(H)$, an even operation $\mu\in Hom_{H^{\otimes 2}}(L\otimes L,H^{\otimes 2}\otimes_H L)$ and an even homomorphism $\alpha\in Hom_H(L,L)$ satisfying the following axioms:

1) Skew-commutativity:

$$[b*a]=-(-1)^{|a||b|}(\sigma\otimes_H id)[a*b], \tag{3.5}$$

2) Hom-Jacobi identity:

$$[\alpha(a)*[b*c]]-(-1)^{|a||b|}((\sigma\otimes id)\otimes_H id)[\alpha(b)*[a*c]]=[[a*b]*\alpha(c)], \tag{3.6}$$

where a, b, c are homogeneous elements in L.

Here and further, $|a|$ is the parity of a.

Example 3.6. For a one dimensional Hopf algebra $H = k$, a Hom-Lie H-pseudo-superalgebra is just a Hom-Lie superalgebra over k. If $\alpha=id$, then a Hom-Lie H-pseudo-superalgebra is a Lie H-pseudo-superalgebra.

Example 3.7. Let H be a Hopf algebra and $A=A_0+A_1$ a 2-dimensional linear superspace, where A_0 is generated by x and A_1 is generated by y. Then $(L=H\{x,y\},[*],\alpha)$ is a Hom-Lie H-pseudo-superalgebra, where $H\{x,y\}$ is a free pseudo-algebra of rank 2 with pseudoproduct given by $[x*x]=[y*y]=[x*y]=0$ in $H^{\otimes 2}\otimes_H L$, α is any even homomorphism in $Hom_H(L,L)$.

Example 3.8. Let $(L,[,],\alpha)$ be a finite dimensional Hom-Lie superalgebra, H be a Hopf algebra. Then $(H\otimes L,[*],id\otimes\alpha)$ is a Hom-Lie H-pseudo-superalgebra with pseudoproduct $[*]$ given by

$$[(h\otimes x)*(l\otimes y)]=h\otimes l\otimes_H(1\otimes[x,y])$$

for all $h,l\in H$ and homogeneous elements $x,y\in L$.

Example 3.9. Let H be a Hopf algebra and $A=A_0+A_1$ a 3-dimensional linear superspace, where A_0 is generated by x, y and A_1 is generated by z. Then $(L=H\{x,y,z\},[*],\alpha)$ is a Hom-Lie H-pseudo-superalgebra defined by any even homomorphism α and operation

$$[x*y]=1\otimes h\otimes x,[y*x]=-h\otimes 1\otimes x, h\in H,$$

$$[x*x]=[y*y]=[z*z]=[x*z]=[z*x]=[y*z]=[z*y]=0.$$

In particular, if $H=k$, then the Hom-Lie H-pseudo-superalgebra $(L=H\{x,y,z\},[*],\alpha)$ is noting but the affine Hom-Lie superalgebra in [33].

A Hom-Lie H-pseudo-superalgebra $(L,[*],\alpha)$ is called multiplicative if $(id_{H^{\otimes 2}}\otimes_H\alpha)[*]=[*](\alpha\otimes\alpha)$. For example, if $\alpha=id_A$, then the Hom-Lie H-pseudo-superalgebra (A,μ,α) is multiplicative.

Let $(A,[*]_A,\alpha_A)$ and $(B,[*]_B,\alpha_B)$ be two (multiplicative) Hom-Lie H-pseudo-superalgebras. An even homomorphism $f:A\to B$ is said to be a morphism of Hom-Lie H-pseudo-superalgebras if

$$(id_{H^{\otimes 2}}\otimes_H f)[*]_A=[*]_B(f\otimes f),\ f\alpha_A=\alpha_B f. \tag{3.7}$$

Definition 3.10. Let $(L,[*],\alpha_L)$ be a Hom-Lie H-pseudo-superalgebra and M a superspace in ${}_HM$. A Hom-L-module is a triple (M,ρ_M,α_M), where ρ_M is an even morphism in $Hom_{H^{\otimes 2}}(L\otimes M,H^{\otimes 2}\otimes_H M)$, α_M is an even morphism in $Hom_H(M,M)$ and satisfies the following axioms:

$$\alpha_L(a)*(b*m)=[a*b]*\alpha_M(m),(id_{H^{\otimes 2}}\otimes_H\alpha_M)(a*m)=\alpha_L(a)*\alpha_M(m), \tag{3.8}$$

where $a*m=\rho_M(a\otimes m)$, a, b and m are homogeneous elements in L and M respectively.

In the following, we will show that the supercommutator bracket defined using the multiplication in a Hom-associative H-pseudo-superalgebra leads naturally to a Hom-Lie H-pseudo-superalgebra.

Lemma 3.11. Let $(A,*,\alpha)$ be a Hom-associative H-pseudo-superalgebra. Then

1) $((\sigma\otimes id)\otimes_H id)(\alpha(a)*(b*c))=((\sigma\otimes id)(a*b))*\alpha(c)$.

2) $((id\otimes\sigma)\otimes_H id)((a*b)*\alpha(c))=\alpha(a)*((\sigma\otimes id)(b*c))$.

3) $((\sigma\otimes id)\otimes_H id)((id\otimes\sigma)\otimes_H id)(((\sigma\otimes id)(a*b))*\alpha(c))$

$=((id\otimes\sigma)\otimes_H id)((\sigma\otimes id)\otimes_H id)(\alpha(a)*((\sigma\otimes id)(b*c)))$.

Proof. We only prove (3), and similarly for (1), (2). For any homogeneous elements $a,b,c \in A$, let

$$a*b=\sum_i f_i \otimes g_i \otimes e_i, e_i * \alpha(c)=\sum_{i,j} f_{i,j} \otimes g_{i,j} \otimes e_{i,j},$$

$$b*c=\sum_i h_i \otimes l_i \otimes d_i, \alpha(a)*d_i=\sum_{i,j} h_{i,j} \otimes l_{i,j} \otimes d_{i,j}.$$

On one hand we have

$$\begin{aligned}
&((\sigma\otimes id)\otimes_H id)((id\otimes\sigma)\otimes_H id)(((\sigma\otimes id)(a*b))*\alpha(c))\\
&=\sum_{i,j}((\sigma\otimes id)\otimes_H id)((id\otimes\sigma)\otimes_H id)\left(\left(g_i f_{ij_{(1)}}\otimes f_i f_{ij_{(2)}}\otimes g_{ij}\right)\otimes_H e_{ij}\right)\\
&=\sum_{i,j}((\sigma\otimes id)\otimes_H id)\left(\left(g_i f_{ij_{(1)}}\otimes g_{ij}\otimes f_i f_{ij_{(2)}}\right)\otimes_H e_{ij}\right)\\
&=\sum_{i,j}\left(g_{ij}\otimes g_i f_{ij_{(1)}}\otimes f_i f_{ij_{(2)}}\right)\otimes_H e_{ij}=\sum_{i,j}\left(g_{ij}\otimes g_i f_{ij_{(2)}}\otimes f_i f_{ij_{(1)}}\right)\otimes_H e_{ij},
\end{aligned}$$

since H is cocommutative. Similarly, we have

$$\begin{aligned}
&((id\otimes\sigma)\otimes_H id)((\sigma\otimes id)\otimes_H id)(\alpha(a)*((\sigma\otimes id)(b*c)))\\
&=((id\otimes\sigma)\otimes_H id)((\sigma\otimes id)\otimes_H id)\left(\left(h_{ij}\otimes l_i l_{ij_{(1)}}\otimes h_i l_{ij_{(2)}}\right)\otimes_H d_{ij}\right)\\
&=\sum_{i,j}((id\otimes\sigma)\otimes_H id)\left(\left(l_i l_{ij_{(1)}}\otimes h_{ij}\otimes h_i l_{ij_{(2)}}\right)\otimes_H d_{ij}\right)\\
&=\sum_{i,j}\left(l_i l_{ij_{(1)}}\otimes h_i l_{ij_{(2)}}\otimes h_{ij}\right)\otimes_H d_{ij}=\sum_{i,j}\left(l_i l_{ij_{(2)}}\otimes h_i l_{ij_{(1)}}\otimes h_{ij}\right)\otimes_H d_{ij},
\end{aligned}$$

as required. So (3) holds since A is Hom-associative. □

Proposition 3.12. Given any Hom-associative H-pseudo-superalgebra $(A,*,\alpha)$, one can define the bracket pseudoproduct on homogeneous elements by

$$[a*b]=a*b-(-1)^{|a||b|}(\sigma\otimes_H id)(b*a) \tag{3.9}$$

and then extending by linearity to all elements. Then $(A,[*],\alpha)$ is a Hom-Lie H-pseudo-superalgebra.

Proof. We shall show that the condition (3.9) leads A to be a Hom-Lie H-pseudo-superalgebra, in the sense of Definition 3.5. For this purpose, we first claim that the bracket pseudoproduct is both H-bilinear and skew-commutative, but these are easy to check. It remains to verify that the conditions (2) of Definition 3.5 are satisfied by the condition (3.9). Now we have the following calculations:

$$\begin{aligned}
[\alpha(a)*[b*c]]&=\left[\alpha(a)*\left(b*c-(-1)^{|b||c|}(\sigma\otimes_H id)(c*b)\right)\right]\\
&=[\alpha(a)*(b*c)]-(-1)^{|b||c|}[\alpha(a)*((\sigma\otimes_H id)(c*b))]\\
&=\alpha(a)*(b*c)-(-1)^{|b||c|}\alpha(a)*((\sigma\otimes_H id)(c*b))\\
&\quad-(-1)^{|a|(|b|+|c|)}((\sigma\otimes id)\otimes_H id)((id\otimes\sigma)\otimes_H id)((b*c)*\alpha(a))\\
&\quad+(-1)^{|b||c|+|a|(|b|+|c|)}((\sigma\otimes id)\otimes_H id)((id\otimes\sigma)\otimes_H id)(((\sigma\otimes id)(c*b))*\alpha(a))\\
&=\alpha(a)*(b*c)-(-1)^{|b||c|}((id\otimes\sigma)\otimes_H id)((a*c)*\alpha(b))\\
&\quad-(-1)^{|a|(|b|+|c|)}((\sigma\otimes id)\otimes_H id)(\alpha(b)*(\sigma\otimes id)(c*a))\\
&\quad+(-1)^{|b||c|+|a|(|b|+|c|)}((\sigma\otimes id)\otimes_H id)((id\otimes\sigma)\otimes_H id)(((\sigma\otimes id)(c*b))*\alpha(a)).
\end{aligned}$$

Immediately, we can obtain $[\alpha(b)*[a*c]]$, then

$$
\begin{aligned}
&(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left[\alpha(b)*[a*c]\right]\\
&=(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left(\alpha(b)*(a*c)\right)\\
&\quad-(-1)^{|b||c|}\left((id\otimes\sigma)\otimes_H id\right)\left((a*c)*\alpha(b)\right)\\
&\quad-(-1)^{|a|(|b|+|c|)}\left((\sigma\otimes id)\otimes_H id\right)\left(\alpha(b)*(\sigma\otimes id)(c*a)\right)\\
&\quad+(-1)^{|c|(|a|+|b|)}\left((id\otimes\sigma)\otimes_H id\right)\left((\sigma\otimes id)(c*a)*\alpha(b)\right).
\end{aligned}
$$

It follows from Lemma 3.12 that

$$
\begin{aligned}
&\left[\alpha(a)*[b*c]\right]-(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left[\alpha(b)*[a*c]\right]\\
&=(a*b)*\alpha(c)-(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left((b*a)*\alpha(c)\right)\\
&\quad-(-1)^{|c|(|a|+|b|)}\left((id\otimes\sigma)\otimes_H id\right)\left((\sigma\otimes id)(c*a)*\alpha(b)\right)\\
&\quad+(-1)^{|a||b|+|c||a|+|b||c|}\left((\sigma\otimes id)\otimes_H id\right)\left((id\otimes\sigma)\otimes_H id\right)\left((\sigma\otimes id)(c*b)*\alpha(a)\right).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\left[[a*b]*\alpha(c)\right]&=\left[(a*b)*\alpha(c)\right]-(-1)^{|a||b|}\left[(\sigma\otimes id)(a*b)*\alpha(c)\right]\\
&=(a*b)*\alpha(c)-(-1)^{|a||b|}(\sigma\otimes id)(b*a)*\alpha(c)\\
&\quad-(-1)^{|c|(|a|+|b|)}\left((id\otimes\sigma)\otimes_H id\right)\left((\sigma\otimes id)\otimes_H id\right)\left((c*a)*\alpha(b)\right)\\
&\quad+(-1)^{|a||b|+|c||a|+|b||c|}\left((id\otimes\sigma)\otimes_H id\right)\left((\sigma\otimes id)\otimes_H id\right)\left(\alpha(c)*(\sigma\otimes id)(b*a)\right)\\
&=(a*b)*\alpha(c)-(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left((b*a)*\alpha(c)\right)\\
&\quad-(-1)^{|c|(|a|+|b|)}\left((id\otimes\sigma)\otimes_H id\right)\left((\sigma\otimes id)(c*a)*\alpha(b)\right)\\
&\quad+(-1)^{|a||b|+|c||a|+|b||c|}\left((\sigma\otimes id)\otimes_H id\right)\left((id\otimes\sigma)\otimes_H id\right)\left((\sigma\otimes id)(c*b)*\alpha(a)\right).
\end{aligned}
$$

Together with the above results, we finally obtain

$$
\begin{aligned}
&\left[\alpha(a)*[b*c]\right]-(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left[\alpha(b)*[a*c]\right]\\
&=\left[[a*b]*\alpha(c)\right].
\end{aligned}
$$

The proof is completed. □

Next we will construct Hom-Lie H-pseudo-superalgebras from Lie H-pseudo-superalgebras and even Hom-Lie superalgebra endomorphisms, generalizing the results for Hom-Lie H-pseudo-algebras in [14] and Hom-Lie superalgebras in [33].

Theorem 3.13. Let $(L,[*])$ be a Lie H-pseudo-superalgebra and α an even endomorphisms of L. Defining $[*]_\alpha\in Hom_{H^{\otimes 2}}\left(L^{\otimes 2},H^{\otimes 2}\otimes_H L\right)$ by $[x*y]_\alpha=\left[\alpha(x)*\alpha(y)\right]$ for all homogeneous elements x, y in L, then $\left(L,[*]_\alpha,\alpha\right)$ is a Hom-Lie H-pseudo-superalgebra.

Moreover, suppose that $\left(L',[*]'\right)$ is another Lie H-pseudo-superalgebra and α' is an even endomorphisms of L'. If $f:L\to L'$ is a morphism of Lie H-pseudo-superalgebras that satisfies $f\alpha=\alpha' f$, then

$$
f:\left(L,[*]_\alpha,\alpha\right)\to\left(L,[*]'_{\alpha'},\alpha'\right) \tag{3.10}
$$

is a morphism of Hom-Lie H-pseudo-superalgebras.

Proof. We shall show that $\left(L,[*]_\alpha,\alpha\right)$ satisfies the skew-commutativity and the Hom-Jacobi identity. For any homogeneous elements x,y,z in L,

$$\begin{aligned}[x*y]_\alpha &= \left(id_{H^{\otimes 2}} \otimes_H \alpha\right)[x*y] \\ &= -(-1)^{|x||y|}\left(id_{H^{\otimes 2}} \otimes_H \alpha\right)(\sigma \otimes_H id)[y*x] \\ &= -(-1)^{|x||y|}(\sigma \otimes_H id)\left(id_{H^{\otimes 2}} \otimes_H \alpha\right)[y*x] \\ &= -(-1)^{|x||y|}(\sigma \otimes_H id)[y*x]_\alpha .\end{aligned}$$

Since α is an endomorphism of L,

$$\begin{aligned}\left[\alpha(x)*[y*z]_\alpha\right]_\alpha &= \left[\alpha(x)*\left(id_{H^{\otimes 2}} \otimes_H \alpha\right)[y*z]\right]_\alpha \\ &= \left(id_{H^{\otimes 3}} \otimes_H \alpha^2\right)\left[x*[y*z]\right].\end{aligned}$$

Therefore we have

$$\begin{aligned}&\left[\alpha(x)*[y*z]_\alpha\right]_\alpha - \left[[x*y]_\alpha * \alpha(z)\right]_\alpha - (-1)^{|x||y|}\left((\sigma\otimes id)\otimes_H id\right)\left[\alpha(y)*[x*z]_\alpha\right]_\alpha \\ &= \left(id_{H^{\otimes 3}} \otimes_H \alpha^2\right)\left(\left[x*[y*z]\right]-\left[[x*y]*z\right]-(-1)^{|x||y|}\left((\sigma\otimes id)\otimes_H id\right)\left[y*[x*z]\right]\right) \\ &= 0,\end{aligned}$$

as needed. To show that f is a morphism of Hom-Lie H-pseudo-superalgebras, we do the calculations:

$$\begin{aligned}&\left(id_{H^{\otimes 2}} \otimes f\right)\left([x*y]_\alpha\right) = \left(id_{H^{\otimes 2}} \otimes f\right)\left(\left(id_{H^{\otimes 2}} \otimes_H \alpha\right)[x*y]\right) \\ &= \left(\left(id_{H^{\otimes 2}} \otimes_H f\alpha\right)[x*y]\right) = \left(\left(id_{H^{\otimes 2}} \otimes_H \alpha' f\right)[x*y]\right) \\ &= \left(id_{H^{\otimes 2}} \otimes \alpha'\right)\left(\left(id_{H^{\otimes 2}} \otimes_H f\right)[x*y]\right) \\ &= \left(id_{H^{\otimes 2}} \otimes \alpha'\right)\left[f(x)*f(y)\right]' \\ &= \left[f(x)*f(y)\right]'_{\alpha'} .\end{aligned}$$

The proof is completed. □

To provides another way to construct Hom-Lie H-pseudo-superalgebras and Hom-associative H-pseudo-superalgebras, we first recall the definition of current H-pseudo-algebras in [10].

Let H' be a Hopf subalgebra of H and A an H'-pseudo-algebra. Then define the current H-pseudo-algebra $CurA = H \otimes_{H'} A$ by extending the pseudoproduct $a*b$ of A using the H-bilinearity. Explicitly, for any $a,b \in A$, define

$$\begin{aligned}(f \otimes_{H'} a)*(g \otimes_{H'} b) &= (f \otimes g \otimes_H 1)(a*b) \\ &= \sum_i (ff_i \otimes gg_i) \otimes_H (1 \otimes_{H'} e_i)\end{aligned} \tag{3.11}$$

if $a*b = f_i \otimes g_i \otimes_{H'} e_i$. Then $CurA = H \otimes_{H'} A$ is an H-pseudo-algebra which is Lie or associative when A is so.

Proposition 3.14. Let H' be a Hopf subalgebra of H and $(L,[*],\alpha)$ a Hom-Lie H'-pseudo-superalgebra. Then $(CurL, \gamma, \beta = id_H \otimes_{H'} \alpha)$ is a Hom-Lie H-pseudo-superalgebra, where γ is the multiplication of $CurL$. Moreover, there is a similar result in the case of Hom-associative H'-pseudo-superalgebras as well.

Proof. We only prove the case of Hom-Lie H'-pseudo-superalgebras, the Hom-associative case is similar. We denote

$$\gamma\left((f \otimes_{H'} a)\otimes(g \otimes_{H'} b)\right) = \left[(f \otimes_{H'} a)*(g \otimes_{H'} b)\right]. \tag{3.12}$$

It is obviously that the skew-commutativity holds since $(L,[*],\alpha)$ is a Hom-Lie H'-pseudo-superalgebra. So it is sufficient to verify the Hom-Jacobi identity. For any $f \otimes_{H'} a, g \otimes_{H'} b, l \otimes_{H'} c \in CurL$, suppose

$$b*c = \sum_i g_i \otimes l_i \otimes_{H'} d_i, \alpha(a)*d_i = \sum_j g_{ij} \otimes l_{ij} \otimes_{H'} d_{ij},$$

$$a*c=\sum_i m_i\otimes n_i\otimes_{H'} e_i,\alpha(b)*e_i=\sum_j m_{ij}\otimes n_{ij}\otimes_{H'} e_{ij},$$

$$a*b=\sum_i s_i\otimes t_i\otimes_{H'} u_i,u_i*\alpha(c)=\sum_j s_{ij}\otimes t_{ij}\otimes_{H'} u_{ij}.$$

Since $(L,[*],\alpha)$ is a Hom-Lie H'-pseudo-superalgebra, we have

$$[\alpha(a)*[b*c]]-(-1)^{|a||b|}((\sigma\otimes id)\otimes_{H'} id)[\alpha(b)*[a*c]]=[[a*b]*\alpha(c)],$$

that is,

$$\sum_{i,j} g_{ij}\otimes g_i l_{ij_{(1)}}\otimes g_i l_{ij_{(2)}}\otimes_H d_{ij}-(-1)^{|a||b|}\sum_{i,j} m_i n_{ij_{(1)}}\otimes m_{ij}\otimes n_i n_{ij_{(2)}}\otimes_H e_{ij}$$
$$=\sum_{i,j} s_i s_{ij_{(1)}}\otimes t_i s_{ij_{(2)}}\otimes t_{ij}\otimes_H u_{ij}.$$

By the multiplication of $CurA$, we obtain

$$\begin{aligned}
&[\beta(f\otimes_{H'} a)*[(g\otimes_{H'} b)*(l\otimes_{H'} c)]]\\
&-(-1)^{|a||b|}((\sigma\otimes id)\otimes_H id)[\beta(g\otimes_{H'} b)*[(f\otimes_{H'} a)*(l\otimes_{H'} c)]]\\
&=\sum_{i,j} fg_{ij}\otimes gg_i l_{ij_{(1)}}\otimes lg_i l_{ij_{(2)}}\otimes_H (1\otimes_{H'} d_{ij})\\
&\quad-(-1)^{|a||b|}\sum_{i,j} fm_i n_{ij_{(1)}}\otimes gm_{ij}\otimes ln_i n_{ij_{(2)}}\otimes_H (1\otimes_{H'} e_{ij})\\
&=\sum_{i,j} fs_i s_{ij_1}\otimes gt_i s_{ij_2}\otimes lt_{ij}\otimes_H (1\otimes_{H'} u_{ij})\\
&=[[(f\otimes_{H'} a)*(g\otimes_{H'} b)]*\beta(l\otimes_{H'} c)].
\end{aligned}$$

Hence $(CurL,\gamma,\beta)$ is a Hom-Lie *H*-pseudo-superalgebra. This ends the proof. □

4. Hom-Annihilation Superalgebras

In this section we will study the annihilation superalgebras of Hom-*H*-pseudo-superalgebras. First of all we will give the definition of *H*-differential superalgebras.

Definition 4.1. An associative superalgebra *Y* is called an associative *H*-differential superalgebra if it is a left *H*-module such that $h(yy')=(h_{(1)}y)(h_{(2)}y')$, for all $h\in H$ and homogeneous elements $y,y'\in Y$.

Let *Y* be an *H*-bimodule which is a commutative associative *H*-differential superalgebra. For a left *H*-module *L*, it is easy to see that $A_Y L=Y\otimes_H L$ is a left *H*-module via $h(y\otimes_H l)=(hy)\otimes_H l$, for all $h\in H$ and $y\otimes_H l\in A_Y L$.

The definition of Hom-Lie *H*-differential-superalgebras can be obtained similarly.

Proposition 4.2. Let *Y* be a Hom-Lie *H*-differential-superalgebra and $(L,[*],\alpha_L)$ a Hom-Lie *H*-pseudo-superalgebra. Then $A_Y L$ is a Hom-Lie *H*-differential superalgebra, where the bracket and the action are given by

$$[y\otimes_H l,y'\otimes_H l']=\sum_i (yf_i)(y'g_i)\otimes e_i, \tag{4.1}$$

$$h[y\otimes_H l,y'\otimes_H l']=[h_{(1)}(y\otimes_H l),h_{(2)}(y'\otimes_H l')], \tag{4.2}$$

for all $h\in H$ and $y\otimes_H l,y'\otimes_H l'\in A_Y L$, where $[l*l']=\sum_i f_i\otimes g_i\otimes e_i$.

Proof. First we shall show that $A_Y L$ is an H-module, but this is easy to check. It remains to verify that the conditions (1) and (2) in Definition 3.5 are satisfied. For this purpose, we take $x\otimes_H a,y\otimes_H b,z\otimes_H c\in A_Y L$, and suppose

$$[b*c]=\Sigma_i g_i\otimes l_i\otimes_H a_i,[\alpha_L(a)*a_i]=\Sigma_j g_{ij}\otimes l_{ij}\otimes_H a_{ij},$$

$$[a*c]=\Sigma_i m_i\otimes n_i\otimes_H b_i,[\alpha_L(b)*b_i]=\Sigma_j m_{ij}\otimes n_{ij}\otimes_H b_{ij},$$

$$[a*b]=\Sigma_i f_i\otimes k_i\otimes_H c_i, [c_i*\alpha_L(c)]=\Sigma_j f_{ij}\otimes k_{ij}\otimes_H c_{ij}.$$

Since L is a Hom-Lie H-pseudo-superalgebra, then $[b*a]=-(-1)^{|a||b|}(\sigma\otimes_H id)[a*b]=-(-1)^{|a||b|}\sum_i k_i\otimes f_i\otimes_H c_i$, therefore we have

$$\begin{aligned}[y\otimes_H b, x\otimes_H a] &= -(-1)^{|a||b|}\sum_i (yk_i)(xf_i)\otimes c_i\\ &= -(-1)^{|a||b|}\sum_i (xf_i)(yk_i)\otimes c_i\\ &= -(-1)^{|a||b|}[x\otimes_H a, y\otimes_H b],\end{aligned}$$

as required. Next we verify the Hom-Jacobi identity by the following calculations:

$$\begin{aligned}[\beta(x\otimes_H a),[y\otimes_H b, z\otimes_H c]] &= \sum_i [x\otimes_H \alpha(a),(yg_i)(zl_i)\otimes a_i]\\ &= \sum_{i,j}(xg_{ij})(((yg_i)(zl_i))l_{ij})\otimes_H a_{ij}\\ &= \sum_{i,j}(xg_{ij})\left(yg_i l_{ij_{(1)}}\right)\left(zl_i l_{ij_{(2)}}\right)\otimes_H a_{ij}.\end{aligned}$$

Similarly, by exchanging the status of the element $x\otimes_H a, y\otimes_H b, z\otimes_H c\in A_Y L$, we have

$$[\beta(y\otimes_H b),[x\otimes_H a, z\otimes_H c]] = \sum_{i,j}(ym_{ij})\left(xm_i n_{ij_{(1)}}\right)\left(zn_i n_{ij_{(2)}}\right)\otimes_H b_{ij},$$

$$[[x\otimes_H a, y\otimes_H b],\beta(z\otimes_H c)] = \sum_{i,j}\left(xf_i f_{ij_{(1)}}\right)\left(yk_i f_{ij_{(2)}}\right)(zk_{ij})\otimes_H c_{ij}.$$

By the Hom-Jacobi identity of L, we have

$$\begin{aligned}&[[a*b]*\alpha(c)]-[\alpha(a)*[b*c]]+(-1)^{|a||b|}((\sigma\otimes id)\otimes_H id)[\alpha(b)*[a*c]]\\ &= \sum_{i,j} f_i f_{ij_{(1)}}\otimes k_i f_{ij_{(2)}}\otimes k_{ij}\otimes_H c_{ij} - g_{ij}\otimes g_i l_{ij_{(1)}}\otimes l_i l_{ij_{(2)}}\otimes_H a_{ij}\\ &\quad +(-1)^{|a||b|} m_i n_{ij_{(1)}}\otimes m_{ij}\otimes n_i n_{ij_{(2)}}\otimes_H b_{ij}\\ &= 0.\end{aligned}$$

Hence

$$\begin{aligned}&[[x\otimes_H a, y\otimes_H b],\beta(z\otimes_H c)]-[\beta(x\otimes_H a),[y\otimes_H b, z\otimes_H c]]\\ &\quad +(-1)^{|a||b|}((\sigma\otimes id)\otimes_H id)[\beta(y\otimes_H b),[x\otimes_H a, z\otimes_H c]]\\ &= \left(xf_i f_{ij_{(1)}}\right)\left(yk_i f_{ij_{(2)}}\right)(zk_{ij})\otimes_H c_{ij} - (xg_{ij})\left(yg_i l_{ij_{(1)}}\right)\left(zl_i l_{ij_{(2)}}\right)\otimes_H a_{ij}\\ &\quad +(-1)^{|a||b|}\left(xm_i n_{ij_{(1)}}\right)(ym_{ij})\left(zn_i n_{ij_{(2)}}\right)\otimes_H b_{ij}\\ &= 0.\end{aligned}$$

So $A_Y L$ is a Hom-Lie H-differential superalgebra. This completes the proof. □

Remark 4.3. In particular, when $Y = X$, $A(L) = A_X L = X\otimes_H L$ is a Hom-Lie H-differential superalgebra, we call it Hom-annihilation superalgebra of the Hom-Lie H-pseudo-algebra L and write $a_x = x\otimes_H a$ for any homogeneous elements $a\in L$ and $x\in X$.

Remark 4.4. A similar statement holds for Hom-associative H-pseudo-superalgebras and Hom-modules as well. For example, if (M,ρ_M,α_M) is a Hom-L-module, then $(A_Y M,\rho_{A_Y M},\beta_M = id_Y\otimes\alpha_M)$ is a Hom-$A_Y L$-module with a compatible H-action, where

$$\rho_{A_Y M}((x\otimes_H a)\otimes(y\otimes_H m)) = \sum_i (xf_i)(yg_i)\otimes_H e_i, \tag{4.3}$$

if $\rho_M(a\otimes m)=\sum_i f_i\otimes g_i\otimes_H e_i$ for any homogeneous elements $x\otimes_H a\in A_Y L$ and $y\otimes_H m\in A_Y M$.

Proposition 4.5. Let $(L,[*],\alpha_L)$ be a Hom-Lie H-pseudo-superalgebra and Y a commutative associative H-differential superalgebra with a right action of H. Then $(Y\otimes_H L,[*],id_Y\otimes\alpha_L)$ is a Hom-Lie H-pseudo-superalgebra with bracket pseudoproduct

$$\left[(x\otimes_H a)*(y\otimes_H b)\right]=\sum_i f_{i_{(1)}}\otimes k_{i_{(1)}}\otimes_H\left(\left(xf_{i_{(2)}}\right)\left(yk_{i_{(2)}}\right)\otimes c_i\right), \tag{4.4}$$

if $[a*b]=\sum_i f_i\otimes k_i\otimes_H c_i$ for any homogeneous elements $x\otimes_H a, y\otimes_H b\in Y\otimes_H L$.

Proof. According to the bracket pseudoproduct defined above, it is easy to see that H-bilinearity holds. To verify the Skew-commutativity and Hom-Jacobi identity, take $x\otimes_H a, y\otimes_H b, z\otimes_H c\in Y\otimes L$ and suppose

$$[b*c]=\Sigma_i g_i\otimes l_i\otimes_H a_i, \left[\alpha_L(a)*a_i\right]=\Sigma_j g_{ij}\otimes l_{ij}\otimes_H a_{ij},$$

$$[a*c]=\Sigma_i m_i\otimes n_i\otimes_H b_i, \left[\alpha_L(b)*b_i\right]=\Sigma_j m_{ij}\otimes n_{ij}\otimes_H b_{ij},$$

$$[a*b]=\Sigma_i f_i\otimes k_i\otimes_H c_i, \left[c_i*\alpha_L(c)\right]=\Sigma_j f_{ij}\otimes k_{ij}\otimes_H c_{ij}.$$

Since L is a Hom-Lie H-pseudo-superalgebra, $[b*a]=-(-1)^{|a||b|}(\sigma\otimes_H id)[a*b]=-(-1)^{|a||b|}\sum_i k_i\otimes f_i\otimes_H c_i$, therefore we have

$$\begin{aligned}&\left[(y\otimes_H b)*(x\otimes_H a)\right]\\&=-(-1)^{|a||b|}\sum_i k_{i_{(1)}}\otimes f_{i_{(1)}}\otimes_H\left(\left(yf_{i_{(2)}}\right)\left(xk_{i_{(2)}}\right)\otimes c_i\right)\\&=-(-1)^{|a||b|}\sum_i k_{i_{(1)}}\otimes f_{i_{(1)}}\otimes_H\left(\left(xk_{i_{(2)}}\right)\left(yf_{i_{(2)}}\right)\otimes c_i\right)\\&=-(-1)^{|a||b|}\left((\sigma\otimes_H id)\otimes id\right)\sum_i f_{i_{(1)}}\otimes k_{i_{(1)}}\otimes_H\left(\left(xf_{i_{(2)}}\right)\left(yk_{i_{(2)}}\right)\otimes c_i\right)\\&=-(-1)^{|a||b|}\left((\sigma\otimes_H id)\otimes id\right)\left[(x\otimes_H a)*(y\otimes_H b)\right].\end{aligned}$$

That is, the skew-commutativity holds. So it is sufficient to verify the Hom-Jacobi identity. Since

$$\left[(x\otimes_H a)*(y\otimes_H b)\right]=\sum_i f_{i_{(1)}}\otimes k_{i_{(1)}}\otimes_H\left(\left(xf_{i_{(2)}}\right)\left(yk_{i_{(2)}}\right)\otimes c_i\right),$$

we have

$$\begin{aligned}&\left[\left[(x\otimes_H a)*(y\otimes_H b)\right]*\beta(z\otimes_H c)\right]\\&=\sum_i\left[\left(f_{i_{(1)}}\otimes k_{i_{(1)}}\otimes_H\left(\left(xf_{i_{(2)}}\right)\left(yk_{i_{(2)}}\right)\otimes_H c_i\right)\right)*\left(z\otimes_H\alpha(c)\right)\right]\\&=\sum_{i,j} f_{i_{(1)}}f_{ij_{(1)}}\otimes k_{i_{(1)}}f_{ij_{(2)}}\otimes k_{ij_{(1)}}\otimes_H\left(xf_{i_{(2)}}f_{ij_{(3)}}\right)\left(yk_{i_{(2)}}f_{ij_{(4)}}\right)\left(zk_{ij_{(2)}}\right)\otimes_H c_{ij}\\&=\sum_{i,j}\left(f_{i_{(1)}}\otimes k_{i_{(1)}}\right)f_{ij_{(1)}}\otimes k_{ij_{(1)}}\otimes_H\left(\left(\left(xf_{i_{(2)}}\right)\left(yk_{i_{(2)}}\right)\right)f_{ij_{(2)}}\right)\left(zk_{ij_{(2)}}\right)\otimes_H c_{ij}.\end{aligned}$$

Similarly, by exchanging the status of the element $x\otimes_H a, y\otimes_H b, z\otimes_H c\in A_Y L$, we have

$$\begin{aligned}&\left[\beta(x\otimes_H a)*\left[(y\otimes_H b)*(z\otimes_H c)\right]\right]\\&=\sum_{i,j} g_{ij_{(1)}}\otimes\left(g_{i_{(1)}}\otimes l_{i_{(1)}}\right)l_{ij_{(1)}}\otimes_H\left(xg_{ij_{(2)}}\right)\left(\left(\left(yg_{i_{(2)}}\right)\left(zl_{i_{(2)}}\right)\right)l_{ij_{(2)}}\right)\otimes_H a_{ij},\end{aligned}$$

$$\begin{aligned}&\left[\beta(y\otimes_H b)*\left[(x\otimes_H a)*(z\otimes_H c)\right]\right]\\&=\sum_{i,j} m_{ij_{(1)}}\otimes\left(m_{i_{(1)}}\otimes n_{i_{(1)}}\right)n_{ij_{(1)}}\otimes_H\left(ym_{ij_{(2)}}\right)\left(\left(\left(xm_{i_{(2)}}\right)\left(zn_{i_{(2)}}\right)\right)n_{ij_{(2)}}\right)\otimes_H b_{ij}.\end{aligned}$$

By the Hom-Jacobi identity of L, we have

$$\begin{aligned}&\left[\left[a*b\right]*\alpha(c)\right]-\left[\alpha(a)*\left[b*c\right]\right]+(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left[\alpha(b)*\left[a*c\right]\right]\\&=\sum_{i,j} f_i f_{ij_{(1)}}\otimes k_i f_{ij_{(1)}}\otimes k_{ij}\otimes_H c_{ij}-g_{ij}\otimes g_i l_{ij_{(1)}}\otimes l_i l_{ij_{(2)}}\otimes_H a_{ij}\\&\quad+(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left(m_{ij}\otimes m_i n_{ij_{(1)}}\otimes n_i n_{ij_{(2)}}\otimes_H b_{ij}\right)\\&=\sum_{i,j}(f_i\otimes k_i)f_{ij}\otimes k_{ij}\otimes_H c_{ij}-g_{ij}\otimes(g_i\otimes l_i)l_{ij}\otimes_H a_{ij}\\&\quad+(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left(m_{ij}\otimes(m_i\otimes n_i)n_{ij}\otimes_H b_{ij}\right)\\&=0,\end{aligned}$$

it follows that

$$\begin{aligned}&\left[\left[x\otimes_H a, y\otimes_H b\right],\beta(z\otimes_H c)\right]-\left[\beta(x\otimes_H a),\left[y\otimes_H b, z\otimes_H c\right]\right]\\&+(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left[\beta(y\otimes_H b),\left[x\otimes_H a, z\otimes_H c\right]\right]=0.\end{aligned}$$

So $A_Y L$ is a Hom-Lie H-pseudo-superalgebra. This completes the proof. □

5. Hom-Lie Conformal Superalgebras

In this section we will reformulate the definition of Hom-Lie (or Hom-associative) H-pseudo-superalgebras. The resulting notion, equivalent to that of Hom-H-pseudo-superalgebras, will be called Hom-H-conformal superalgebras.

Let us start by racalling the definitions of the Fourier transform and the x-brackets in [10]. For an arbitrary Hopf algebra H, the Fourier transform $F: H\otimes H\to H\otimes H$ is defined by $F(f\otimes g)=fS(g_{(1)})\otimes g_{(2)}$. F is an isomorphism with an inverse given by $F(f\otimes g)=fg_{(1)}\otimes g_{(2)}$. The significance of Fourier transform F is the identity

$$f\otimes g=F^{-1}F(f\otimes g)=fS(g_{(1)})\otimes g_{(2)}. \tag{5.1}$$

In order to reformulate the definition of a Lie (or associative) H-pseudo-algebra, Bakalov, D'Andrea and Kac introduced the bracket $[a,b]\in H\otimes L$ as the Fourier transform of $[a*b]$:

$$[a,b]=\sum_i F(f_i\otimes g_i)(1\otimes e_i)=f_iS(g_{i_{(1)}})\otimes g_{i_{(2)}}e_i.$$

That is,

$$[a,b]=\sum_i h_i\otimes c_i,\ \text{if}\ [a*b]=\sum_i(h_i\otimes 1)\otimes_H c_i.$$

Then for $x\in X=H^*$, the x-bracket is defined in [3] as follows:

$$[a_x b]=(\langle S(x),\cdot\rangle)[a,b]=\sum_i\langle S(x),h_i\rangle c_i.$$

Let $(L,[*],\alpha)$ be a Hom-Lie H-pseudo-superalgebra. For any homogeneous elements $a,b,c\in L$, suppose

$$[b*c]=\sum_i l_i\otimes 1\otimes_H a_i,\ [\alpha(a)*a_i]=\sum_j l_{ij}\otimes 1\otimes_H a_{ij},$$

$$[a*c]=\sum_i g_i\otimes 1\otimes_H b_i,\ [\alpha(b)*b_i]=\sum_j g_{ij}\otimes 1\otimes_H b_{ij},$$

$$[a*b]=\sum_i f_i\otimes 1\otimes_H c_i,\ [c_i*\alpha(c)]=\sum_j f_{ij}\otimes 1\otimes_H c_{ij}.$$

Then we have

$$[\alpha(a)*[b*c]]=\sum_i[\alpha(a)*(l_i\otimes 1\otimes_H a_i)]=\sum_{i,j} l_{ij}\otimes l_i\otimes 1\otimes_H a_{ij},$$

$$\left[\alpha(a),[b,c]\right]=\sum_i(\sigma\otimes id)\left(id\otimes\left[\alpha(a),\cdot\right]\right)\left([b,c]\right)=\sum_{i,j}l_{ij}\otimes l_i\otimes a_{ij}.$$

Similarly, we can obtain $\left[\alpha(b)*[a*c]\right],\left[\alpha(b),[a,c]\right],\left[[a*b]*\alpha(c)\right],\left[[a,b],\alpha(c)\right]$, thus

$$\begin{aligned}&\left[\alpha(a)*[b*c]\right]-\left[[a*b]*\alpha(c)\right]-(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left[\alpha(b)*[a*c]\right]\\&=\sum_{i,j}l_{ij}\otimes l_i\otimes 1\otimes_H a_{ij}-\sum_{i,j}f_if_{ij_1}\otimes f_{ij_2}\otimes 1\otimes_H a_{ij}-(-1)^{|a||b|}\sum_{i,j}g_i\otimes g_{ij}\otimes 1\otimes_H c_{ij},\end{aligned}$$

$$\begin{aligned}&\left[\alpha(a),[b,c]\right]-\left(F^{-1}\otimes id\right)\left[[a,b],\alpha(c)\right]-(-1)^{|a||b|}(\sigma\otimes id)\left[\alpha(b),[a,c]\right]\\&=\sum_{i,j}l_{ij}\otimes l_i\otimes a_{ij}-\sum_{i,j}(f_i\otimes 1)\Delta(f_{ij})\otimes a_{ij}-(-1)^{|a||b|}\sum_{i,j}g_i\otimes g_{ij}\otimes c_{ij}\\&=\sum_{i,j}l_{ij}\otimes l_i\otimes a_{ij}-\sum_{i,j}f_if_{ij_1}\otimes f_{ij_2}\otimes a_{ij}-(-1)^{|a||b|}\sum_{i,j}g_i\otimes g_{ij}\otimes c_{ij}.\end{aligned}$$

Therefore

$$\left[\alpha(a)*[b*c]\right]-(-1)^{|a||b|}\left((\sigma\otimes id)\otimes_H id\right)\left[\alpha(b)*[a*c]\right]=\left[[a*b]*\alpha(c)\right]$$

is equivalent to

$$\left[\alpha(a),[b,c]\right]-(-1)^{|a||b|}(\sigma\otimes id)\left[\alpha(b),[a,c]\right]=\left(F^{-1}\otimes id\right)\left[[a,b],\alpha(c)\right].$$

So the definition of Hom-Lie *H*-pseudo-superalgebra can be equivalently reformulated as follows.

Definition 5.1. A Hom-Lie *H*-conformal superalgebra is a triple $(L,[,],\alpha)$ consisting of a superspace L in ${}_HM$, an even operation $[,]:L\otimes L\to H\otimes L$ and an even homomorphism $\alpha\in Hom_H(L,L)$ satisfying the following axioms:

1) *H*-sesqui-linearity:

$$[ha,b]=(h\otimes 1)[a,b],[a,hb]=\left(1\otimes h_{(2)}\right)[a,b]\left(S\left(h_{(1)}\right)\otimes 1\right); \tag{5.2}$$

2) Skew-commutativity:

$$[b,a]=-(-1)^{|a||b|}\sum_i S\left(h_{i_{(1)}}\right)\otimes h_{i_{(2)}}c_i,\ \text{if } [a,b]=\sum_i h_i\otimes c_i; \tag{5.3}$$

3) Hom-Jacobi identity:

$$\left[\alpha(a),[b,c]\right]-(-1)^{|a||b|}(\sigma\otimes id)\left[\alpha(b),[a,c]\right]=\left(F^{-1}\otimes id\right)\left[[a,b],\alpha(c)\right]; \tag{5.4}$$

where a, b, c are homogeneous elements in L and $h\in H$.

One can also reformulate Definition 4.1 in terms of x-brackets $[a_xb]$ as below.

Definition 5.2. A Hom-Lie *H*-conformal superalgebra is a triple $(L,[,],\alpha)$ consisting of a superspace L in ${}_HM$, an even operation $[_x]:L\otimes L\to L$ and an even homomorphism $\alpha\in Hom_H(L,L)$ satisfying the following axioms:

1) Locality:

$$codim\left\{x\in X\,\middle|\,[a_xb]=0\right\}<\infty\ \text{ for any } a,b\in L; \tag{5.5}$$

2) *H*-sesqui-linearity:

$$[ha_xb]=[a_{xh}b],[a_xhb]=h_2\left[a_{S(h_1)x}b\right]; \tag{5.6}$$

3) Skew-super commutativity:

$$[b_xa]=-(-1)^{|a||b|}\sum_i\left\langle x,S\left(h_{i_1}\right)\right\rangle S\left(h_{i_2}\right)\left[b_{x_i}a\right]; \tag{5.7}$$

4) Hom-super Jacobi identity:

$$\left[\alpha(a)_x\left[b_y c\right]\right]-(-1)^{|a||b|}\left[\alpha(b)_y\left[a_x c\right]\right]=\left[\left[a_{x_2} b\right]_{yx_1}\alpha(c)\right]. \tag{5.8}$$

where x_i and y_i are dual bases of X and H, a, b, c are homogeneous elements in L, $h \in H$ and $x, y \in X$.

In the following we will show that there is a simple relationship between the x-bracket of a Hom-Lie H-conformal superalgebra and the commutator in its annihilation Hom-Lie H-pseudo-superalgebra $A(L)$ defined in Proposition 4.5. Let h_i, x_i be dual linear basis of H and X. Then we have

$$[a,b]=\sum_i S(h_i)\otimes\left[a_{x_i}b\right],[a*b]=\sum_i\left(S(h_i)\otimes 1\right)\otimes_H\left[a_{x_i}b\right].$$

According to Proposition 4.2, we obtain

$$\left[a_x,b_y\right]=\sum_i\left[a_{x_i}b\right]_{(xS(h_i))y}=\left[a_{x_{(2)}}b\right]_{x(1)y}.$$

In other words,

$$[a_x b]_y=\left[a_{x_{(2)}},b_{S(x(1))y}\right]=\sum_i\left[a_{x_i},b_{(h_iS(x))y}\right].$$

Below we give one way of constructing Hom-modules over Hom-Lie H-pseudo-algebras, whose proofs are similar to that in [10].

Proposition 5.3. Any Hom-module (M,ρ_M,α_M) over a Hom-Lie H-pseudo-superalgebra $(L,[*],\alpha_L)$ has a natural structure of a Hom-$A(L)$-module, given by $(x\otimes_H a)\cdot m=a_x m$, where

$$a_x m=\left\langle S(x), f_i S\left(g_{i_{(1)}}\right)\right\rangle g_{i_{(2)}}e_i, \text{if } a*m=\sum_i\left(f_i\otimes g_i\right)\otimes_H e_i, \tag{5.9}$$

for all homogeneous elements $a\in L, x\in X$ and $m\in M$. This action is compatible with the action of H, that is, $h(bm)=\left(h_{(1)}b\right)\left(h_{(2)}m\right)$ for all homogeneous elements $b\in A(L), m\in M$ and $h\in H$, and satisfies the locality condition: $codim\left\{x\in X\,|\left[a_x M\right]=0\right\}<\infty$ for any homogeneous elements $a\in L$ and $m\in M$.

Conversely, any Hom-$A(L)$-module (M,ρ_M,α_M) satisfying the above conditions has a natural structure of an Hom-L-module, given by

$$a*m=\sum_i\left(S(h_i)\otimes 1\right)\otimes_H a_{x_i}m, \tag{5.10}$$

where h_i and x_i are dual linear basis of H and X.

6. Cohomology of Hom-Lie *H*-Pseudo-Superalgebras

In this section, we will consider the cohomology of Hom-Lie H-pseudo-superalgebras, generalizing the results of Hom-Lie H-pseudoalgebras and Lie superalgebras.

Let $(L,[,],\alpha_L)$ be a Hom-Lie H-pseudo-superalgebra, (M,ρ_M,α_M) is a Hom-L-module. Let $n\ge 1$ be a natural number and let $C^n_{Hom}(L,M)$ be the superspace of all homogeneous skew-symmetric cochains $\gamma\in Hom_{H^{\otimes n}}\left(L^{\otimes n},H^{\otimes n}\otimes_H M\right)$ satisfies

$$\begin{aligned}&\left(id_H^n\otimes_H\alpha_M\right)\gamma\left(a_1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes a_n\right)\\&=\gamma\left(\alpha_L(a_1)\otimes\alpha_L(a_2)\otimes\cdots\otimes\alpha_L(a_{n-1})\otimes\alpha_M(a_n)\right).\end{aligned} \tag{6.1}$$

Explicitly, γ has the following defining properties:

1) H-polylinearity: For any $h\in H$ and $a_i\in L$,

$$\begin{aligned}&\gamma\left(h_1a_1\otimes h_2a_2\otimes\cdots\otimes h_{n-1}a_{n-1}\otimes h_na_n\right)\\&=\left(\left(h_1\otimes h_2\otimes\cdots\otimes h_{n-1}\otimes h_n\right)\otimes_H\otimes 1\right)\gamma\left(a_1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes a_n\right).\end{aligned} \tag{6.2}$$

2) Skew-supersymmetry: For any $a_i\in L$,

$$\gamma\left(a_1\otimes\cdots\otimes a_{i+1}\otimes a_i\otimes\cdots\otimes a_n\right) = -(-1)^{|a_i||a_{i+1}|}\left(\sigma_{i,i+1}\otimes_H id\right)\gamma\left(a_1\otimes\cdots\otimes a_i\otimes a_{i+1}\otimes\cdots\otimes a_n\right), \tag{6.3}$$

where $\sigma_{i,i+1}:H^{\otimes n}\to H^{\otimes n}$ is the transposition of the ith and $(i+1)$ st factors.

The map γ is called even (resp. odd) when $\gamma(a_1\otimes a_2\otimes\cdots\otimes a_n)\in M_0$ (resp. $\gamma(a_1\otimes a_2\otimes\cdots\otimes a_n)\in M_1$) for all even (resp. odd) elements $a_1\otimes a_2\otimes\cdots\otimes a_n\in H^{\otimes n}$, where the parity of the element $a_1\otimes a_2\otimes\cdots\otimes a_n$ is $|a_1|+|a_2|+\cdots+|a_n|$. We denote the parity of the map γ by $|\gamma|$.

For $n\geq 1$, the map $d:C^n_{Hom}(L,M)\to C^{n+1}_{Hom}(L,M)$ is defined as follows:

$$\begin{aligned}&(d\gamma)\left(a_1\otimes\cdots\otimes a_{i+1}\otimes a_i\otimes\cdots\otimes a_n\otimes a_{n+1}\right)\\&=\sum_{1\leq i\leq n+1}(-1)^{i+1}(-1)^{|a_i|(|\gamma|+|a_1|+\cdots+|a_{i-1}|)}\left(\sigma_{1\to i}\otimes_H id\right)\alpha_L^{n-1}(a_i)*\gamma\left(a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes a_{n+1}\right)\\&\quad+\sum_{1\leq i<j\leq n+1}(-1)^{i+j}(-1)^{|a_i|(|a_1|+\cdots+|a_{i-1}|)}(-1)^{|a_j|(|a_1|+\cdots+|a_{i-1}|+|a_{i+1}|+\cdots+|a_{j-1}|)}\left(\sigma_{1\to i,2\to j}\otimes_H id\right)\\&\quad\times\gamma\left(\left[a_i*a_j\right]\otimes\alpha_L(a_1)\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes\hat{a}_j\otimes\cdots\otimes\alpha(a_{n+1})\right),\end{aligned} \tag{6.4}$$

where $\sigma_{1\to i}$ is the permutation $h_i\otimes h_1\otimes\cdots\otimes h_{i-1}\otimes h_{i+1}\otimes\cdots\otimes h_{n+1}\mapsto h_1\otimes\cdots\otimes h_{n+1}$, , $\sigma_{1\to i,2\to j}$ is the permutation $h_i\otimes h_j\otimes h_1\otimes\cdots\otimes h_{i-1}\otimes h_{i+1}\otimes\cdots\otimes h_{j-1}\otimes h_{j+1}\otimes\cdots\otimes h_{n+1}\mapsto h_1\otimes\cdots\otimes h_{n+1}$, and the sign ^ indicates that the element below it must be omitted. In particular, for $n=1$ we have

$$(d\gamma)(a_1,a_2)=-\gamma\left(\left[a_1*a_2\right]\right)+(-1)^{|\gamma||a_1|}a_1*\gamma(a_2)-(-1)^{|a_2|(|\gamma|+|a_1|)}\left(\sigma_{1\to 2}\otimes_H id\right)\alpha_2*\gamma(a_1), \tag{6.5}$$

and for $n=2$ we obtain

$$\begin{aligned}&(d\gamma)(a_1\otimes a_2\otimes a_3)=(-1)^{|\gamma||a_1|}\alpha_L(a_1)*\gamma(a_2\otimes a_3)\\&-(-1)^{|a_2|(|\gamma|+|a_1|)}\left(\sigma_{1\to 2}\otimes_H id\right)\alpha_L(a_2)*\gamma(a_1\otimes a_3)\\&+(-1)^{|a_3|(|\gamma|+|a_1|+|a_2|)}\left(\sigma_{1\to 3}\otimes_H id\right)\alpha_L(a_3)*\gamma(a_1\otimes a_2)\\&-(-1)^{|a_3||a_2|}\left(\sigma_{1\to 2,2\to 3}\otimes_H id\right)\gamma\left(\left[a_2*a_3\right]\otimes\alpha_L(a_1)\right)\\&+(-1)^{|a_3||a_1|}\left(\sigma_{1\to 1,2\to 3}\otimes_H id\right)\gamma\left(\left[a_1*a_3\right]\otimes\alpha_L(a_2)\right)-\gamma\left(\left[a_1*a_2\right]\otimes\alpha_L(a_3)\right).\end{aligned} \tag{6.6}$$

The fact that $d^2=0$ is most easily checked and the same argument is in the usual Lie superalgebra case in [26] [36] [37] and Hom-Lie H-pseudoalgebra case in [34]. The cohomology of the resulting complex $C^\bullet_{Hom}(L,M)$ is called the cohomology of $(L,[*],\alpha_L)$ with coefficients in (M,ρ_M,α_M) and is denoted by $H^\bullet_{Hom}(L,M)$.

One can also modify the above definition by replacing everywhere $\otimes_H$ by $\otimes$. Let $\tilde{C}^n_{Hom}(L,M)$ consist of all skew-symmetric cochains $\gamma\in Hom_{H^{\otimes n}}\left(L^{\otimes n},H^{\otimes n}\otimes M\right)$. Then we can define a differential $\tilde{d}:\tilde{C}^n_{Hom}(L,M)\to\tilde{C}^{n+1}_{Hom}(L,M)$ by (6.1) with $\otimes_H$ replaced by $\otimes$ everywhere; then again $\tilde{d}^2=0$. The corresponding cohomology $\tilde{H}^\bullet_{Hom}(L,M)$ will be called the basic cohomology of $(L,[*],\alpha_L)$ with coefficients in (M,ρ_M,α_M). In contrast, $H^\bullet_{Hom}(L,M)$ is sometimes called the reduced cohomology.

In the following we will show that the cohomology theory of Hom-Lie H-pseudo-superalgebras describes extensions and deformations, just as any cohomology theory.

Theorem 6.1. Let $(L,[*],\alpha_L)$ be a multiplicative Hom-Lie H-pseudo-superalgebra, and (M,ρ_M,α_M) be a Hom-L-module, considering a Hom-Lie H-pseudo-superalgebra with respect to the zero pseudobracket, then the equivalence classes of H-split abelian extensions

$$0\to M\to\hat{L}\to L\to 0 \tag{6.7}$$

of the Hom-Lie H-pseudo-superalgebra $(L,[*],\alpha_L)$ correspond bijectively to $H^2_{Hom}(L,M)_0$, the homogeneous component of degree zero of the reduced cohomology $H^2_{Hom}(L,M)$.

Proof. Let $0\to M\to\hat{L}\to L\to 0$ be an extension of L-modules, which is split over H. Choosing a splitting $\hat{L}=L\oplus M=\{l+m\mid l\in L,m\in M\}$ as an H-module, and denoting the pseudobracket of $\hat{L}$ by $\left[a\,\hat{*}\,b\right]$, we have for all $a,b\in L,m,n\in M$:

$$\left[a \hat{*} m\right] = \left[a * m\right], \left[m \hat{*} n\right] = 0, \left[a \hat{*} b\right] - \left[a * b\right] =: \gamma\left(a \otimes b\right) \in H^{\otimes 2} \otimes_H M. \quad (6.8)$$

It is not hard to verify that γ is a homogeneous 2-cochain of degree zero, *i.e.*, $\gamma \in C^2_{Hom}\left(L, M\right)_0$. The Hom-super Jacobi identity of L and $\hat{L}$ implies $d\gamma = 0$ in the sense of (6.1).

Conversely, given an element of $H^2_{Hom}\left(L, M\right)_0$, we can choose a representative $\gamma \in C^2_{Hom}\left(L, M\right)_0$ and define an action $\left[\hat{*}\right]$ by (6.2). Then $\left[\hat{*}\right]$ depends only on the γ. □

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