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Aunu Integer Sequence as Non-Associative Structure and Their Graph Theoretic Properties

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Abstract

The generating function for generating integer sequence of Aunu numbers of prime cardinality was reported earlier by the author in [1]. This paper assigns an operator $\Theta = \Delta_{\sup} |ai + aj|, |ai - aj|$ on the function $A_{n,(123)} = \frac{P_n - 1}{2}$ for $n \geq 5$ where the operation induces addition or subtraction on the pairs of *ai*, *aj* elements which are consecutive pairs of elements obtained from a generating set $B_{n,(123)} \subset A_{n,(123)}$ of some finite order. The paper identifies that the set $B_{n,(123)}$ of the generated pairs of integer sequence is non-associative. The paper also presents the graph theoretic applications of the integers generated in which subgraphs are deduced from the main graph and adjacency matrices and incidence matrices constructed. It was also established that some of the subgraphs were found to be regular graphs. The findings in this paper can further be used in coding theory, Boolean algebra and circuit designs.

Keywords

Aunu Numbers, Nonassociative, Graph, Subgraphs, Adjacency Matrix, Incidence Matrix

1. Introduction

An overview of Aunu numbers, Aunu permutations patterns, the 123 and 132 avoiding patterns and their applications was reported by the authors in [2]. This paper considered the prime enumerative function $A_{n,(123)}, n \geq 5$ generated by the author in [3] and defined an operator on some $B_{n,(123)} \subset A_{n,(123)}$ using the addition and subtrac-

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tions as an operators such that the pairing of elements in $B_{n,(123)}$ was closed in $A_{n,(123)}$.

In simplest form, a graph is a collection of vertices that can be connected to each other by means of edges. In particular, each edge of graph joins exactly two vertices. Using a formal notation, a graph is defined as follows.

Definition 2.1: A graph *G* consists of a collection of *V* vertices and a collection of edges *E*, for which we write $G=(V,E)$. Each edge $e\in E$ is said to join two vertices, which are called its end points. If *e* joins $u,v\in V$, we write $e=\langle u,v\rangle$. Vertex *u* and *v* in this case are said to be adjacent. Each *e* is said to be incident with vertices *u* and *v* respectively.

We will often write $V(G)$ and $E(G)$ to denote the set of vertices and edges associated with graph *G* respectively. It is important to realize that an edge can actually be represented as an unordered tuple of two vertices, that is, its end points. For this reason, we make no distinction between $\langle u,v\rangle$ and $\langle u,v\rangle$: they both represent the fact that vertex *u* and *v* are adjacent [4].

Definition 2.2: A graph *H* is a subgraph of *G* if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$ such that for all $e\in E(H)$ with $e=\langle u,v\rangle$, we have that $u,v\in V(H)$. When *H* is a subgroup of *G*, we write $H\subseteq G$ [4].

Definition 2.3: Adjacency matrix is a table *A* with *n* rows and *m* columns with entry $A[i,j]$ denoting the number of edges joining vertex v_i and v_j [4].

Definition 2.4: An incidence matrix *M* of graph *G* consists of *n* rows and *m* columns such that $M[i,j]$ counts the number of times that edge e_j is incident with vertex v_i. Note that $M[i,j]$ is either 0, 1 or 2.

Theorem 2.1: For all graphs *G*, the sum of the vertex degrees is twice the number of edges [4]. That is,

$$\sum_{v\in V(G)}\delta(v)=2\cdot|E(G)|. \tag{1}$$

Corollary 2.1: For any graph *G*, the number of vertices with odd degrees is even [4].

2. Method of Construction

Let $A_n=\frac{P_n-1}{2}$ where in this case $P_n\geq 5$ (being prime numbers). The restriction $P_n\geq 5$ is deliberately put since we are only interested in enumerations involving Aunu numbers of (123)-avoiding category which, by definition begins from 5 upwards as reported in [5]. Then;

$$A_{n,(123)}=\{2,3,5,6,8,9,11,14,15,18,20,21,\cdots\}.$$

We now obtain from $A_{n,(123)}$ a restricted subset $B_{n,(123)}=\{2,3,5,6,8,9,11,14,15,18,20,21\}$. Then $B_{n,(123)}\subset A_{n,(123)}$ contains all elements of $A_{n,(123)}$ up to 21.

We are now set to carry out some algebraic theoretic investigations on $B_{n,(123)}$ being a direct subset of $A_{n,(123)}$.

First let us introduce an operator on $B_{n,(123)}$ such that:

Define an operator

$$\Theta=\Delta_{\sup}|ai+aj|,|ai-aj| \tag{2}$$

where: Θ is an operator which induces addition or subtraction on any pair $ai,aj\in B_{n,(123)}$ whereby addition or subtraction in absolute value is closed in $A_{n,(123)}$ and $\Delta_{\sup}$ implies whichever of $|ai+aj|$ or $|ai-aj|\in A_{n,(123)}$.

Then we obtain from $B_{n,(123)}$ set of pairs

$$\begin{aligned}B^p_{n,(123)}=\{&\{2,3\},\{2,5\},\{2,8\},\{3,5\},\{3,6\},\{3,8\},\{3,9\},\{5,8\},\{6,9\},\{5,6\},\{2,9\},\\&\{3,14\},\{11,3\},\{11,6\},\{11,5\},\{11,8\},\{11,9\},\{5,9\},\{6,8\},\{6,11\},\{15,6\},\\&\{15,9\},\{15,3\},\{18,3\},\{18,9\},\{15,18\},\{18,2\},\{14,6\},\{11,9\},\{15,5\},\\&\{20,2\},\{20,5\},\{20,6\},\{20,9\},\{20,14\},\{20,18\},\{21,3\},\{21,6\},\\&\{21,15\},\{21,18\},\{18,3\},\{15,6\}\}\end{aligned}$$

where the superscript p on $B_{n,(123)}$ indicates that $B^p_{n,(123)}$ is obtained from $B_{n,(123)}$ by breaking elements of $B_{n,(123)}$ into pairs such that application of Θ of (1) on $B^p_{n,(123)}$ is closed in $A_{n,(123)}$.

3. Results

3.1. Testing for Nonassociative Properties Using the Stated Pairing Scheme Yields the Following Results

1) Given $ai = \{2,3\}, bi = \{2,5\}, ci = \{2,8\}$, we note that: the operation rule is either "+ or −" as earlier defined.

$\{\{2,3\},\{2,5\}\{2,8\}\} = |2-5,3+5| = \{3,8\}\{2,8\} = \{5,6\}$.

Also $\{2,3\}\{6,3\} = \{5,9\}$

$\therefore \{5,6\} \neq \{5,9\}$, hence it is not associative.

2) $\{3,5\},\{3,6\},\{3,8\} = \{6,9\}\{2,8\} = \{2,8\}\{3,8\} = \{5,6\}$

Also, $\{3,5\}\{6,2\} = \{3,5\}$

$\therefore \{5,6\} \neq \{3,5\}$, hence it is not associative.

3) $\{3,9\},\{5,8\},\{6,9\} = \{2,5\}\{6,9\} = \{8,11\}$

Also, $\{5,8\},\{6,9\} = \{2,14\} \rightarrow \{3,9\}\{2,14\} = \{5,11\}$

$\therefore \{8,11\} \neq \{5,11\}$, hence it is not associative.

4) $\{3,5\},\{6,9\},\{11,3\} = \{9,11\}\{11,3\} = \{2,8\}$

Also, $\{3,5\}\{5,6\} = \{2,11\}$

$\therefore \{2,8\} \neq \{2,11\}$, hence it is not associative.

5) $\{3,14\},\{11,3\},\{11,6\} = \{8,3\}\{6,3\}\{11,6\} = \{2,6\}\{11,6\} = \{9,5\}$

Also, $\{11,3\}\{11,6\} = \{5,14\}$

$\{3,14\}\{5,14\} = \{8,9\}$

$\therefore \{9,5\} \neq \{8,9\}$, hence it is not associative.

6) $\{11,5\},\{11,8\},\{11,9\} = \{6,3\}\{11,9\} = \{5,3\}$

Also, $\{11,5\}\{2,3\} = \{9,8\}$

$\therefore \{5,3\} \neq \{9,8\}$, hence it is not associative.

7) $\{5,9\},\{6,8\},\{6,11\} = \{11,3\}\{6,11\} = \{5,9\}$

Also, $\{5,9\}\{5,3\} = \{8,14\}$

$\therefore \{5,9\} \neq \{8,14\}$, hence it is not associative

8) $\{15,6\},\{15,9\},\{15,3\} = \{6,3\}\{15,3\} = \{9,6\}$

Also, $\{15,6\}\{18,6\} = \{3,9\}$

$\therefore \{9,6\} \neq \{3,9\}$, hence it is not associative.

9) $\{18,3\},\{18,9\},\{15,18\} = \{9,6\}\{15,18\} = \{6,14\}$

Also, $\{18,3\}\{3,6\} = \{15,9\}$

$\therefore \{6,14\} \neq \{15,9\}$, hence it is not associative.

10) $\{15,5\},\{20,2\},\{20,5\} = \{5,3\}\{20,5\} = \{15,2\}$

Also, $\{15,5\}\{18,3\} = \{3,8\}$

$\therefore \{15,2\} \neq \{3,8\}$, hence it is not associative.

11) $\{20,6\},\{20,9\},\{20,14\} = \{11,14\}\{20,14\} = \{9,6\}$

Also, $\{20,6\}\{6,11\} = \{14,9\}$

$\therefore \{9,6\} \neq \{14,9\}$, hence it is not associative

12) $\{21,3\},\{21,6\},\{21,15\} = \{15,18\}\{21,15\} = \{6,3\}$

Also, $\{21,3\}\{6,21\} = \{15,3\}$

$\therefore \{6,3\} \neq \{15,3\}$, hence it is not associative.

13) $\{21,8\},\{18,3\},\{15,6\} = \{3,11\}\{15,6\} = \{18,5\}$

Also, $\{21,8\}\{3,9\} = \{18,11\}$

$\therefore \{18,5\} \neq \{18,11\}$, hence it is not associative.

3.2. Graph Theoretic Schemes Generated Using Pairs of Elements in A_n and B_n

In what follows some graph theoretic models are presented using pairs of points of $A_{n,(123)}$ and $B_{n,(123)}$ as adjacent nodes.

Figure 6 shows some examples of regular graphs and their adjacency and incidence matrices can be constructed using the same format as outlined in **Table 1-10**.

Table 1. Adjacency matrix of **Figure 1**.

	2	3	5	6	8	9	11	14	15	18	20	21
2	0	1	1	0	1	1	0	0	0	1	1	0
3	1	0	1	1	1	1	1	1	1	1	0	1
5	1	1	0	1	1	1	1	0	1	0	1	0
6	0	1	1	0	1	1	2	1	1	0	1	1
8	1	1	1	1	0	0	1	0	0	0	0	0
9	1	1	1	1	0	0	1	0	1	1	1	0
11	0	1	1	2	1	1	0	0	0	0	0	0
14	0	1	0	1	0	0	0	0	0	0	1	0
15	0	1	1	1	0	1	0	0	0	1	0	1
18	1	1	0	0	0	1	0	0	1	0	1	1
20	1	0	1	1	0	1	0	1	0	1	0	0
21	0	1	0	1	0	0	0	0	1	1	0	0

Theorem 2.1 and corollary 2.1 has been satisfied. Note also that, every column of incidence matrix has $|e| = 2$ entries 1.

Table 2. Incidence matrix of **Figure 1**.

	e1	e2	e3	e4	e5	e6	e7	e8	e9	e10	e11	e12	e13	e14	e15	e16	e17	e18	e19	e20	e21	e22	e23	e24	e25	e26	e27	e28	e29	e30	e31	e32	e33	e34	e35	e36	e37	e38	e39
2	1	0	0	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	1	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
6	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1
8	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
9	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	1	0	0	0
18	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	1	1	0
20	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	1
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

Table 3. Adjacency matrix of subgraph in **Figure 2**.

	3	5	6	9	11
3	0	1	1	1	0
5	1	0	1	1	1
6	1	1	0	1	2
9	1	1	1	0	1
11	0	1	2	1	0

Theorem 2.1 and corollary 2.1 has been satisfied. Note also that, every column of incidence matrix has $|e| = 2$ entries 1.

Table 4. Incidence matrix of subgraph in **Figure 2**.

	e1	e2	e3	e4	e5	e6	e7	e8	e9	e10
3	1	1	1	0	0	0	0	0	0	0
5	1	0	0	1	1	1	0	0	0	0
6	0	1	0	1	0	0	1	1	1	0
9	0	0	1	0	1	0	1	0	0	1
11	0	0	0	0	0	1	0	1	1	1

Table 5. Adjacency matrix of subgraph of **Figure 3**.

	2	3	8	9	11
2	0	1	1	1	0
3	1	0	1	1	1
8	1	1	0	0	1
9	1	1	0	0	1
11	0	1	1	1	0

Theorem 2.1 and corollary 2.1 has been satisfied. Note also that, every column of incidence matrix has $|e| = 2$ entries 1.

Table 6. Incidence matrix of subgraph of **Figure 3**.

	e1	e2	e3	e4	e5	e6	e7	e8
2	1	1	1	0	0	0	0	0
3	1	0	0	1	1	1	0	0
8	0	1	0	1	0	0	1	0
9	0	0	1	0	1	0	0	1
11	0	0	0	0	0	1	1	1

Table 7. Adjacency matrix of subgraph of **Figure 4**.

	3	5	6	9	11	14	15	18	20	21
3	0	1	0	0	0	0	1	1	0	0
5	1	0	1	1	1	0	1	0	0	0
6	0	1	0	0	1	1	0	0	0	1
9	0	1	0	0	1	0	0	0	0	0
11	0	1	1	1	0	0	0	0	0	0
14	0	0	1	0	0	0	0	0	1	0
15	1	1	0	0	0	0	0	1	0	0
18	1	0	0	0	0	1	0	0	1	1
20	0	0	0	0	0	1	0	1	0	0
21	0	0	1	0	0	0	0	1	0	0

Theorem 2.1 and corollary 2.1 has been satisfied. Note also that, every column of incidence matrix has $|e| = 2$ entries 1.

Table 8. Incidence matrix of subgraph of **Figure 4**.

	e1	e2	e3	e4	e5	e6	e7	e8	e9	e10	e11	e12	e13	e14	e15
3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
5	1	0	0	1	1	1	1	0	0	0		0	0	0	0
6	0	0	0	1	0	0	0	1	1	1	0	0	0	0	0
9	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
11	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0
14	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
15	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0
18	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1
20	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
21	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1

Table 9. Adjacency matrix of subgraph of **Figure 5**.

	3	5	6	15	18	21
3	0	1	0	1	1	1
5	1	0	1	0	0	0
6	0	1	0	1	0	1
15	1	0	1	0	1	1
18	1	0	0	1	0	1
21	1	0	1	1	1	0

Theorem 2.1 and corollary 2.1 has been satisfied. Note also that, every column of incidence matrix has $|e| = 2$ entries 1.

Table 10. Incidence matrix of subgraph of **Figure 5**.

	e1	e2	e3	e4	e5	e6	e7	e8	e9	e10
3	1	1	1	1	0	0	0	0	0	0
5	1	0	0	0	1	0	0	0	0	0
6	0	0	0	0	1	1	0	0	1	0
15	0	1	0	0	0	1	1	1	0	0
18	0	0	1	0	0	0	1	0	0	1
21	0	0	0	1	0	0	0	1	1	1

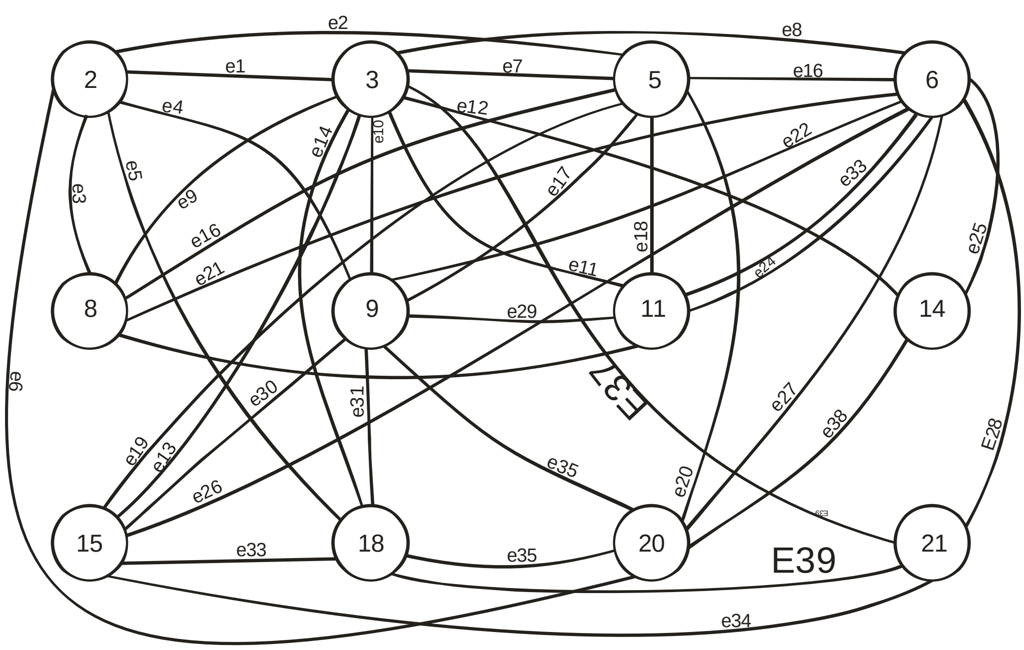

Figure 1. Graph network constructed from elements of A_n and B_n.

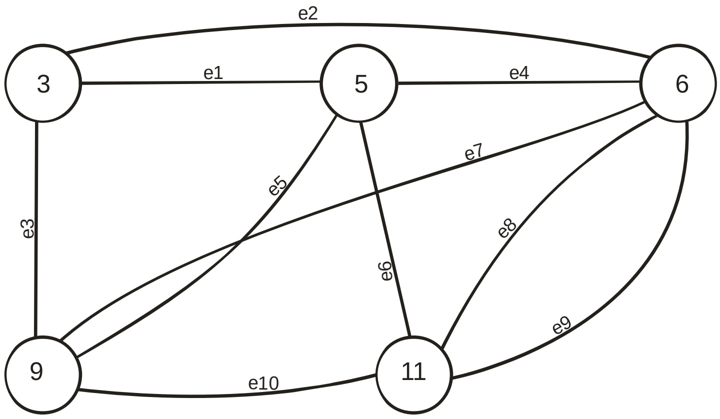

Figure 2. Subgraph of the network of **Figure 1**.

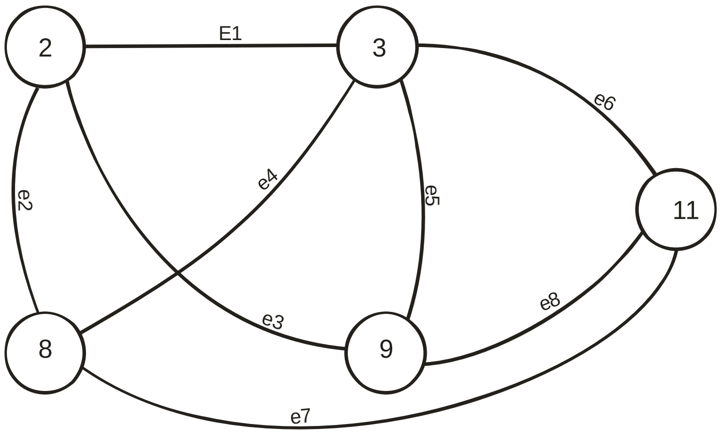

Figure 3. Subgraph of the network of **Figure 1**.

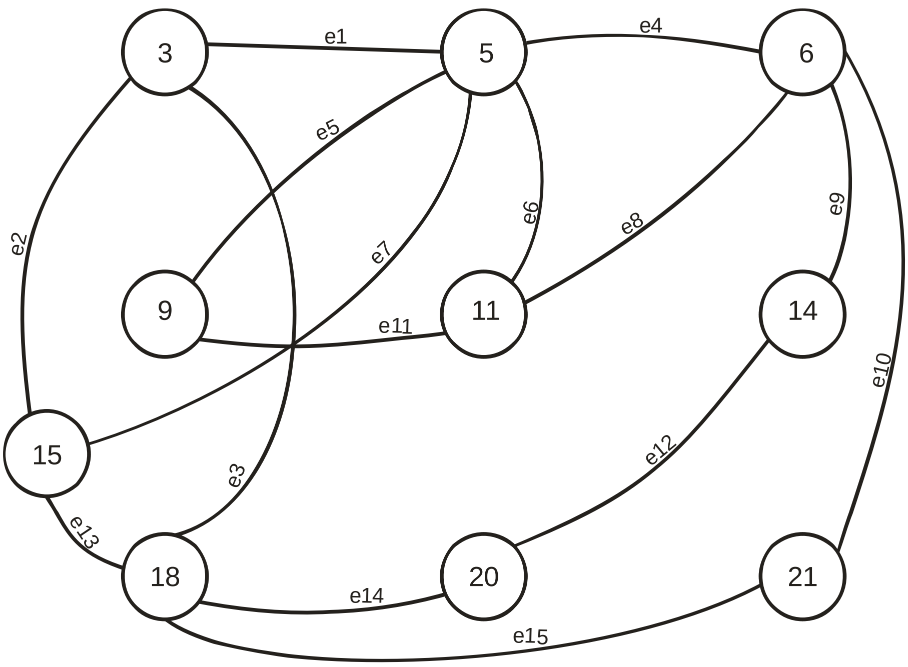

Figure 4. Subgraph of the network of **Figure 1**.

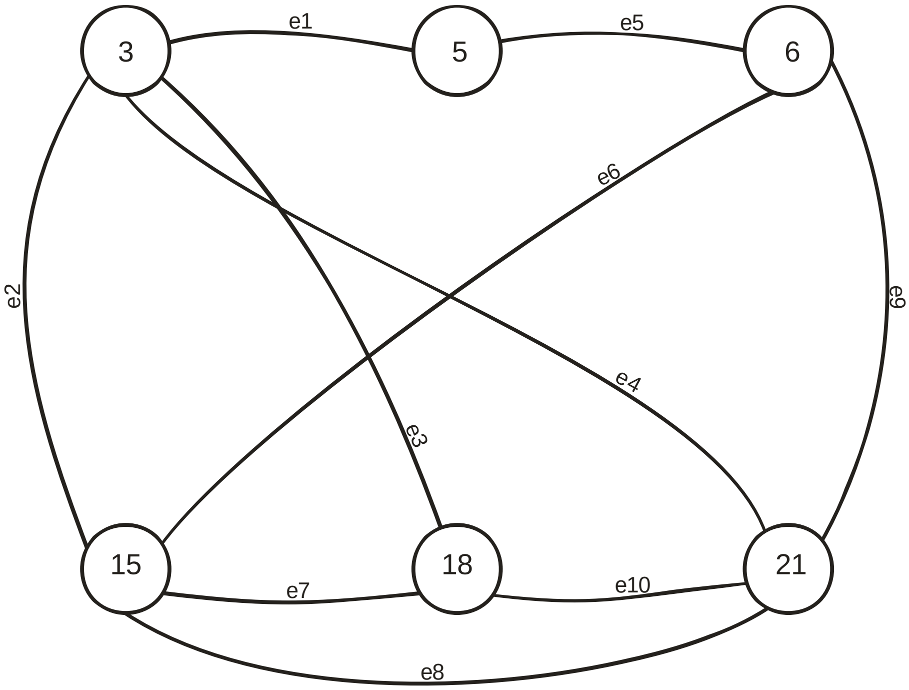

Figure 5. Subgraph of the network of **Figure 1**.

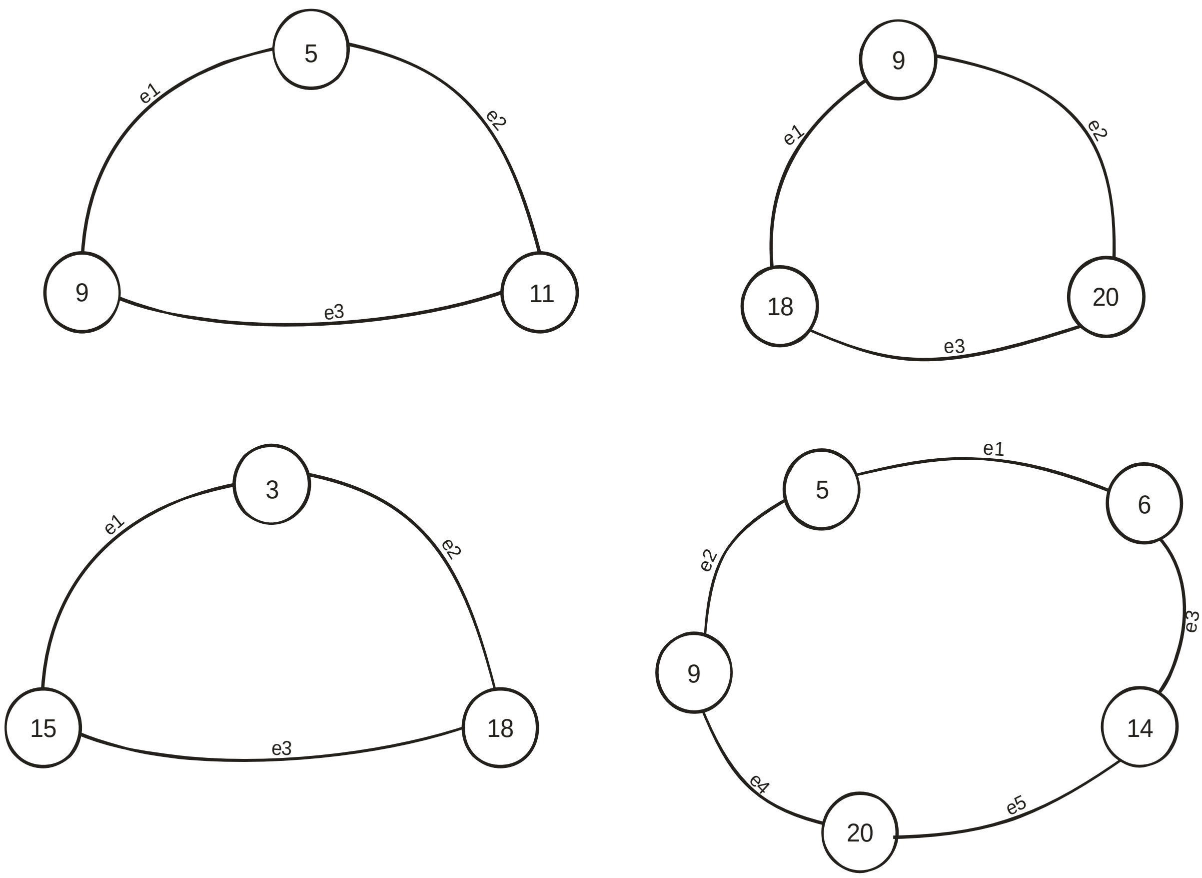

Figure 6. Subgraph of the network of **Figure 1**.

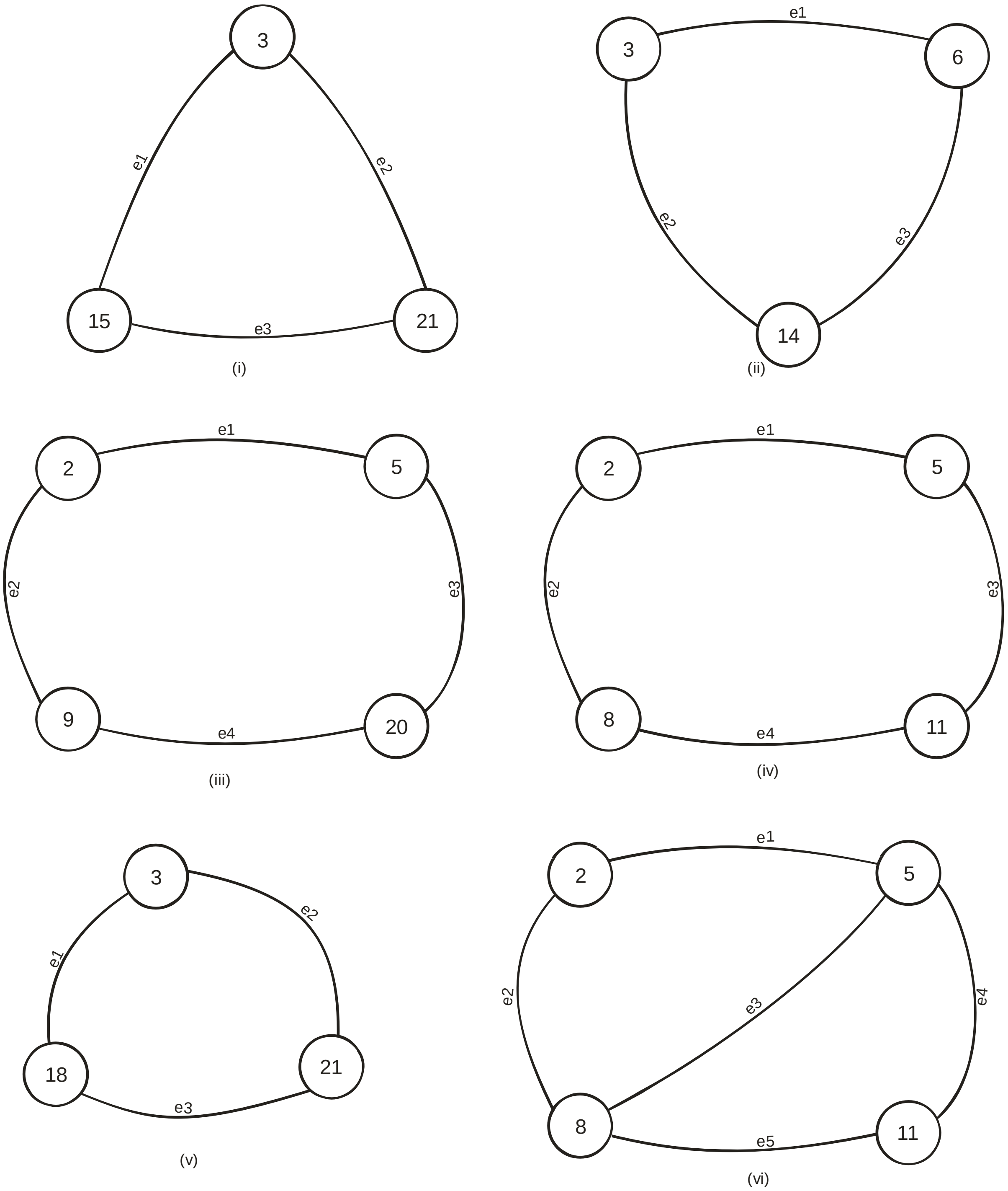

Figure 7. Subgraph of the network of **Figure 1**.

Figure 7(i)-(vi) also shows some examples of regular graphs and their adjacency and incidence matrices can be constructed using the same format as outlined in **Table 1-10** and can also be viewed as Eulerian circuits.

4. Conclusion

After establishing the non-associativity of the finite sets $A_{n(123)}$ and $B_{n(123)}$ under the action of an operator Δ_{sup} we have also established some good applications in graph network analysis. This, we have achieved by generating some Eulerian circuits which are of some consequences in the study of network theory and in circuits theory. Our results would thus have some promising applications in both the communication and in the signal processing formalisms. Also the results involving adjacency and incidence matrices could be used in communication and coding theory which could be investigated in further researches.

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