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On Commutative Δ-Semigroups*

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Abstract

In this paper entitled on commutative Delta-Semigroups, we have obtained important results on commutative Δ-semigroups.

Keywords

Δ-Semigroup, Commutative Δ-Semigroup, *S*-Indecomposable Semigroup

Introduction

The concept of a commutative Δ-semigroup was introduced by a Tamura. T in his paper entitled "Commutative semigroup whose lattice of congruences is a chain" appeared in Bulletin de la S. M. F., tome 97 (1969), p. 369 - 380 [1]. A semigroup S is called a Δ-semigroup if and only if the lattice of all congruences on S is a chain with respect to inclusion relation, in fact if S is a Δ-semigroup, then all the ideals of S form a chain, hence all the principal ideals of S from a chain. It is observed that a Δ-semigroup is either an *s*-indecomposable semigroup or these tunion of two *s*-indecomposable semigroups. Further every homomorphic image of Δ-semigroup is a Δ-semigroup. A semilattice is a Δ-semigroup if and only if it is of order ≤2, further a Δ-semigroup S is a *s*-indecomposable semigroups [2]. In fact, if G is a group, then G is an abelian Δ-semigroup if and only if G is a *p*-quasicyclic for some prime p which is also equivalent for saying that all sub semigroups of G are from a chain [3]. Further an abelian group G^o with zero is a Δ-semigroup if and only if G is a *p*-quasicyclic group, p is arbitrary prime. It is also observed that an abelian group G^o with 0 is a Δ-semigroup if G is a *p*-quasicylicgroup, for an arbitrary prime p. Further in [4] Tamura stated that an ideal of semigroup S, every homomorphism of I onto a non-trivial group G can be extended to a homomorphism of S onto G. we proved that result in theorem 0.8. In fact in this paper, we gave an example to show that the above result need not valid if the word "ideal" is replaced by just "left ideal". In fact, if a semigroup S contains a proper ideal I and if S is a Δ-semigroup, then neither S nor I is homomorphic onto a non-trivial group.

First, we start with the following preliminaries:

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Definition 1 [1]: A semigroup S is called a Δ-semigroup if and only if the lattice of all congruences on S is a chain with respect to inclusion relation. That is, if ρ and σ are congruences on S, then exactly one of the following three holds $\rho \subset \sigma, \rho = \sigma, \sigma \subset \rho$.

Definition 2 [5]: If I is an ideal of a semigroup S then $\rho_I = (I \; x \; I) \cup 1S$ is a congruence on S. it is called the Rees-congruence modulo the ideal I.

Definition 3 [4]: A *s*-indecomposable semigroup is a semigroup which has no semilattice homomorphic image except trivial one (one element semigroup).

Definition 4 [3]: Let p be a prime number. If a group G is the set union of a finite or infinite ascending chain of cyclic groups c_n of order p^n, that is,

$$G = \bigcup_{n=1}^{\infty} C_n,$$

$$C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots,$$

then G is called a p-quasicyclic group, or quacyclic group if it is not necessary to specify p.

Definition 5 [5]: If a semigroup S satisfies the condition the divisibility ordering is a chain, we say then S satisfies the divisibility chain condition.

First, we start with the following Theorem.

Theorem 1: If S is a Δ-semigroup, then all the ideals of S form a chain, hence all the principal ideals of S form a chain.

Proof: If S is a Δ-semigroup by the definition of Δ-semigroup; all Rees-congruences on S form a chain. Let ρ and σ be Rees-congruences modulo ideals I and J respectively. Now we show that $\rho \subseteq \sigma$ if and only if $I \subseteq J$. Suppose $\rho \subseteq \sigma$ and let $x, y \in I$ so that $(x, y) \in \rho \subseteq \sigma$ and thus $(x, y) \in \sigma$ and hence $x, y \in J$. Thus $I \subseteq J$. Conversely assume that $I \subseteq J$, let $(x, y) \in \rho$ so that both $x, y \in I \subseteq J$ then both $x, y \in J$ an thus $(x, y) \in \sigma$ and hence $\rho \subseteq \sigma$. Therefore all the ideals of S form a chain. Since the set of all Rees-congruences of a Δ-semigroup forma infinite chain which is in fact a complete chain. In this chain every ideal is a principal ideal. Hence all the principal ideals of S form a chain.

Proof: Let A be a Δ-semigroup that congruences of A form a chain. Let B be a homomorphic image of A, then also in B congruences form a chain. Let $f : A \to B$ be a homomorphism which is onto. Let ρ and σ be any two congruences on B, then $(f \; X \; f)^{-1}(\rho) = \{(x, y) \in A \; X \; A \,|\, (f(x), f(y)) \in \rho\}$ clearly a congruence on A containing the kernel of f. Where $\ker f = \{(x, y) \in A \; X \; A \,|\, f(x) = f(y)\}$ is congruence on A. it is observed that $\ker f \subseteq (f \; X \; f)^{-1}(\rho)$. Let $(x, y) \in \ker f \Rightarrow f(x) = f(y)$ and thus $f(x), f(y) \in \rho$ ($\therefore \rho$ is reflexive) and hence $(x, y) \in (f \; X \; f)^{-1} \rho$ and therefore $\ker f \subseteq (f \; X \; f)^{-1}(\rho)$. Now we observe that $(f \; X \; f)^{-1}(\rho)$ is a congruence. We have $(f \; X \; f)^{-1} \rho$ is reflexive. Let $(x, y) \in (f \; X \; f)^{-1} \rho$ so that $(f(x), f(y)) \in \rho$ and thus $f(y), f(x) \in \rho$ ($\therefore \rho$ is symmetric) and hence $(y, x) \in (f \; X \; f)^{-1} \rho$ and therefore $(f \; X \; f)^{-1}(\rho)$ is symmentric. Let $(x, y) \in (f \; X \; f)^{-1}(\rho)$ and $(y, x) \in (f \; X \; f)^{-1}(\rho)$ so that $(f(x), f(y)) \in \rho$ and $f(y), f(x) \in \rho$ and thus $f(x), f(z) \in \rho$ ($\therefore \rho$ is transitive) and hence $(x, z) \in (f \; X \; f)^{-1}(\rho)$ and therefore $(f \; X \; f)^{-1}(\rho)$ is transitive. Thus $(f \; X \; f)^{-1}(\rho)$ is an equivalence relation. Let $(y, x) \in (f \; X \; f)^{-1}(\rho)$ and $a \in A$ so that $(f(x), f(y)) \in \rho$ and thus $(f(a), f(x)), (f(a), f(y)) \in \rho$ and $(f(x), f(a)), (f(y), f(a)) \in \rho$ and hence $(f(ax), f(ay)) \in \rho$ and $(f(xa), f(ya)) \in \rho$ ($\therefore \rho$ is congruence) and therefore $(ax), (ay) \in (f \; X \; f)^{-1}(\rho)$ and $(xa), (ya) \in (f \; X \; f)^{-1}(\rho)$. Thus $(f \; X \; f)^{-1}(\rho)$ is a congruence containing $\ker f$] and $(f \; X \; f)^{-1}(\rho)$ is a congruence containing $\ker f$. Now the congruences on A form a chain, thus either $(f \; X \; f)^{-1}(\rho) \subseteq (f \; X \; f)^{-1}(\sigma)$ or $(f \; X \; f)^{-1}(\sigma) \subseteq (f \; X \; f)^{-1}(\rho)$. Suppose that $(f \; X \; f)^{-1}(\rho) \subseteq (f \; X \; f)^{-1}(\sigma)$. Now we claim that $\rho \subseteq \sigma$ Let $(f(x), f(y)) \in \rho$ so that $(x, y) \in (f \; X \; f)^{-1}(\rho)$ and thus $(x, y) \in (f \; X \; f)^{-1}$ and therefore $(f(x), f(y)) \in \sigma$. Thus $\rho \subseteq \sigma$. Conversely, let $(x, y) \in (f \; X \; f)^{-1}(\rho)$ so that $(f(x), f(y)) \in \rho$ thus $(f(x), f(y)) \in \sigma$ and hence $(x, y) \in (f \; X \; f)^{-1}(\sigma)$. Thus $(f \; X \; f)^{-1}(\rho) \subseteq (f \; X \; f)^{-1}(\sigma)$. Hence $(f \; X \; f)^{-1}(\rho) \subseteq (f \; X \; f)^{-1}(\sigma)$ if and only if $\rho \subseteq \sigma$. Thus every homomorphic image of a Δ-semigroup is a Δ-semigroup.

Theorem 3: A semilattice is a Δ-semigroup if and only if it is of order ≤ 2.

<u>Proof:</u> Let L be a semilattice of order ≤ 2. We define $x, y \in L$, $x \le y$ by $x = yz$ for some $z \in L$ *i.e.* ($xy = yzy = yyz = y^2 z = yz = x$ and $yx = yyz = y^2 z = yz = x$ and thus $xy = yx$). Let a, b be distinct elements of L and let $I_a = \{x \mid x \le a\}$, $I_b = \{x \mid x \le b\}$, then I_a and I_b are ideals of L. Let ρ_a and ρ_b denote the Rees-congruences modulo the ideals I_a and I_b respectively, since $I_a \ne I_b$, $\rho_a \ne \rho_b$. Suppose L is a Δ-semigroup then either $\rho_a \subseteq \rho_b$ or $\rho_b \subseteq \rho_a$. Hence either $I_a \subseteq I_b$ or $I_b \subseteq I_a$. For the first case, $a \in I_b$ namely $< b$ ($\therefore I_a \subseteq I_b$). There L is a chain. Suppose L is a chain containing at least three elements a, b, c say $a < b < c$. Let $I^+ = \{x \setminus x \ge b\}$ where I^- is an ideal of L. We defined congruences ρ^+ and ρ^- on L as follows:

$(x, y) \in \rho^+$ if and only either $x, y \in I^+$ or $x = y$,

$(x, y) \in \rho^-$ if and only either $x, y \in I^-$ or $x = y$.

Clearly ρ^- is the Rees-congruence modulo I^-. Now we show that ρ^+ is a Rees-congruences. Clearly ρ^+ is reflexive. Let $(x, y) \in \rho^+$ so that eighter $x = y$ or $x, y \in I^+$ and hence $(y, x) \in \rho^+$. Thus ρ^+ is symmetric. Let $(x, y) \in \rho^+$ and $(y, x) \in \rho^+$ so that either $x = y$ or $x, y \in I^+$ and either $y = z$ or $y, z \in I^+$ and thus $x = z$ or $x, z \in I^+$ and hence $(x, z) \in \rho^+$ then $(xz, yz) \in \rho^+$ and $(zx, zy) \in \rho^+$, $\forall z \in < L$. let $(x, y) \in \rho^+$ so that either $x = y$ or $x, y \in I^+$. Suppose $z \in b$ then $x \wedge z \ge b$ and $y \wedge z \ge b$ and suppose $z \ge b$ then $x \ge b \ge z$ and $y \ge b \ge z$ and thus $x \wedge z = z, y \wedge z = z$. Hence $(x \wedge z, y \wedge z) \in \rho^+$ and $(z \wedge x, z \wedge y) \in \rho^+$. Thus $(xz, yz) \in \rho^+$ and $(zx, zy) \in \rho^+$. Hence ρ^+ is a congruence modulo I^+. Now $(a, b) \in \rho^-$ but $(a, b) \notin \rho^+$. Suppose $(a, b) \in \rho^-$, then either $a, b \in I^-$ or $a = b$ since $a, b \in I^-$, so, $a \le b$, $b \le b$. Suppose $a, b \in I^+$ then $a \le b$, $\le b$. Then is contradiction ($\therefore a < b < c$ in L). thus $(a, b) \in \rho^-$ but $(a, b) \notin \rho^+$. Also $(c, b) \notin \rho^+$ but not in $(a, b) \in \rho^-$. Suppose $(a, b) \in \rho^+$, then either $c, b \in I^+$ or $c = b$ since $b \in I^+$, so $c \ge b$, $b \ge b$. Suppose $c, b \in I^-$, then $c \ge b$, $b \ge b$. this is contradiction ($\therefore a < b < c$ in L). thus $(c, b) \in \rho^+$ but $(c, b) \notin \rho^-$.

Therefore $\rho^+ \subsetneq \rho^-$ and $\rho^- \subsetneq \rho^+$. This is contradiction to our assumption. Thus L is a chain of order ≤ 2. Conversely suppose that L is a semilattice with two elements. Then $L \; x \; L$ and 1_L are congruences on L and clearly $1_L \subseteq L \; x \; L$. Thus all congruences on two elements semilattice are comparable. Thus L is a Δ-semigroup.

Theorem 4: A Δ-semigroup is either an *S*-indecomposable semigroup or the set union of two *S*-indecomposable semigroups.

Proof: We define a relation $\mathcal{N}$ on S as $a\mathcal{N}b$ if and only if $N(a) = N(b)$ that is $a\mathcal{N}b$ if and only if $a \in F$ if and only if $b \in F$ for any filter containing a. Now we show that $\mathcal{N}$ is an equivalence relation. Let $a\mathcal{N}a$ so that $N(a) = N(a)$ thus $\mathcal{N}$ is reflexive. Let $a\mathcal{N}b$ so that $N(a) = N(b)$ and thus $a \in F$ if and only if $b \in F$ and therefore $N(b) = N(a)$ and hence $b\mathcal{N}a$. Thus $\mathcal{N}$ is symmetric. Let $a\mathcal{N}b$ and $b\mathcal{N}c$ so that $N(a) = N(b)$ and $N(b) = N(c)$ and thus $a \in F$ if and only if $b \in F$ and $b \in F$ if and only if $c \in F$ and hence $a \in F$ if and only if $c \in F$ and therefore $N(a) = N(c)$. Thus $a\mathcal{N}c$. So that $\mathcal{N}$ is transitive. Thus $\mathcal{N}$ is an equivalence relation. Now we have to show that $\mathcal{N}$ is a congruence on S that is $a, b \in \mathcal{N}$ so that $(ac, bc) \in \mathcal{N}$ and $(cb, ca) \in \mathcal{N}$. Suppose $(a, b) \in \mathcal{N}$ so that $N(a) = N(b)$ and thus $N(ac) = N(bc)$ and hence $ac \in F$ if and only if $b, c \in F$ (since $N(a) = N(b)$) if and only if $bc \in F$ and therefore $N(ac) = N(bc)$. thus $(ab, ba) \in \mathcal{N}$ and similarly $(ca, cb) \in \mathcal{N}$. Thus $\mathcal{N}$ is a congruence on S. Now $(a^2, a) \in \mathcal{N}$ so that $N(a^2) = N(a)$ and thus $a^2 \in F$ if and only if $a \in F$ and also $(ab, ba) \in \mathcal{N}$ since $ab \in F$ if and only if $a, b \in F$ if only if $b, a \in F$. Thus $N(ab) = N(ba)$ so that

$(ab,ba)\in\mathcal{N}$. Thus $\mathcal{N}$ is a semilattice congruence on *S*. Now we have to show that $\mathcal{N}$ is the least semilattice congruence on *S*. If ρ is any semilattice congruence on *S*. Now we claim that $\mathcal{N}\subseteq\rho$. Suppose $a,b\in\mathcal{N}$ so that $N(a)=N(b)$ we have $(ab,ba)\in\rho$ and $(a^2,a)\in\rho$. Now we have to show that $(a,b)\in\rho$ so that $a\rho=b\rho$. Let *K* be a filter in *S*|*ρ* such that $a\rho\in K$ we have $\Pi:S\to s\,|\,\rho$ by $\Pi(x)=x\rho$ is the natural homomorphic $a\in\{x\in S\,|\,x\rho\in K\}$ is a filter of s so that $b\rho\in K$. Thus we have for any filter *K* of *S*|*ρ* we have $a\rho\in K$ if and only if $b\rho\in K$ so that $a\rho=b\rho$ and thus $(a,b)\in\rho$. Thus $\mathcal{N}\subseteq\rho$. Thus $\mathcal{N}$ is a least semilattice congruence on *S*. we have $f:Na\to Y$ is onto homomorphism. Here $Na\,|\ker f\approx Y$.

We have to show that *ker f = Na X Na.*

Let be the family of completely prime ideals of *S* such that $(a,b)\in\sigma$ so for any either $a,b\in I$ or $a,b\notin I$ where $ab\in I$ wherever either $a\in I$ or $b\in I$. Here $Na=\{x\in S\,|\,N(x)=N(a)\}$. Suppose $(x,y)\in Na\ X\ Na$ then $N(x)=N(y)=N(a)$. Now *I* is a completely prime ideal such that $x\in I$ and $y\notin I$ that is $y\in S\setminus I$ is a filter and $x\notin S\setminus I$. We have to show that $S\setminus I$ is a filter. Let $a,b\in S\setminus I$ so that $a\notin I$ and $b\notin I$ and thus $ab\notin I$ and hence $a\,b\in S\setminus I$ if $a\,b\in S\setminus I$ then $b\in S\setminus I$ otherwise either $a\in I$ or $b\in I$ we have $a\in I$ so that $ab\in I$ or $b\in I$ so that $ab\in I$. This is contradiction, so that $a\,b\notin I$. Thus $a\,b\in S\setminus I$. Hence $S\setminus I$ is a filter. Thus each Na is a *s*-indecomposable semigroup.

Now let τ be a family of completely prime ideals of *S*. Define $(a,b)\in\sigma$ by for any $I\in\tau$ we have either $a,b\in I$ or $a,b\notin I$. Let $(a,a)\in\sigma$ so that either *a*, $a\in I$ or $a\notin I$ so *σ* is a reflexive. Let $(a,b)\in\sigma$ so that either $a,b\in I$ or $a,b\notin I$ so that $(b,a)\in\sigma$ and hence *σ* is a symmetric. Let $(a,b)\in\sigma$ and $(b,a)\in\sigma$ so that either $a,b\in I$ or $a,b\notin I$ and either $b,c\in I$ or $b,c\notin I$.

Case (i): Let $a,b\in I$ and $b,c\in I$, so $a,c\in I$.

Case (ii): Let $a,b\in I$ and $b,c\notin I$, so $a,c\in I$ and thus $a\in I$ or $c\in I$ and hence $c\in I$.

Case (iii): Let $a,b\in I$ and $b,c\in I$ so that $ac\in I$ and thus $a\in I$ or $c\in I$ and hence a $a\in I$ thus $(a,c)\in\sigma$.

Thus *σ* is an equivalence relation on *S*.

Now we have $(a,c)\in\sigma$ we have to show that $(ac,bc)\in\sigma$ that is $ac,bc\in I$ or $ac,bc\notin I$. For any $I\in\tau$ since $(a,b)\in\sigma$, so $(a,b)\in I$ or $a,b\notin I$. Now we take $ac\in I$ and $bc\notin I$. So that either $a\in I$ or $c\in I$ and thus $c\in I$ and hence $bc\in I$ (since $b\in I$). Thus $(ac,bc)\in\sigma$ and similarly $(ca,cb)\in\sigma$. Thus *σ* is a congruence on *S*. Also clearly $(a^2,a)\in\sigma$ (since $a\in I$, so $a^2\in I$). Now we claim that $(ab,ba)\in\sigma$ that is both either $ab,ba\in I$ or $ab,ba\notin I$. Now take $ab\in I$ and $ba\notin I$ so that either $a\in I$ or $b\in I$ and thus $b\in I$ and hence $ba\in I$. Thus $(ab,ba)\in\sigma$. Thus *σ* is a semilattice congruence on *S*.

Conversely given any semilattice congruence *ρ* we have to show that $\rho=\sigma$ we have $\Pi:S\to s\,|\,\rho$ where *s*|*ρ* is a semilattice so that $a\rho\le b\rho$ if and only if $a\rho\cdot b\rho=b\rho$. Let *J* be an ideal of *s*|*ρ* so that $a\notin\Pi^{-1}(J)=\{x\in S\,|\,xp\in J\}$ is completely prime ideal. Let $a\notin\Pi^{-1}(J)$ so and $b\notin\Pi^{-1}(J)$ so that $a\rho\notin J$ and $b\rho\notin J$. Now $(ab)\rho=(a\rho)(b\rho)=a\rho\vee b\rho\notin J$

So $ab\notin\Pi^{-1}(J)$.

Let τ be the set of all completely prime ideals of the form $\Pi^{-1}(J)$ where *J* is an ideal of *S*|*ρ*. Let *σ* be the induced semilattice congruence on *S*. Now $(a,b)\in\sigma$ if and only if for any completely prime ideals $\notin\Pi^{-1}(J)$ where *J* is an ideal of *S*|*ρ* so that $a,b\in\Pi^{-1}(J)$ or $a,b\notin\Pi^{-1}(J)$ and thus $a\rho,b\rho\in J$ or $a\rho,b\rho\notin J$ and hence $(a,b)\in\rho$. Thus $\subseteq\rho$. Now we claim $\subseteq\sigma$. Let $(a,b)\in\rho$ so that $a\rho=b\rho$. Suppose $a\rho\ne b\rho$ so that $a\rho\nleq b\rho$ and thus $a\rho\notin(b\rho]$. Then there is an ideal *J* of *S*|*ρ* such that $b\rho\in J$ and $a\rho\notin J$. Then $b\in\Pi^{-1}(J)$ and $a\notin\Pi^{-1}(J)$. This is contradiction since $a\rho\ne b\rho$. Thus $a\rho\in J$ and $b\rho\in J$. Then $a,b\in\Pi^{-1}(J)$. Hence $(a,b)\in\sigma$. Thus $\rho\subseteq\sigma$. Hence $\rho=\sigma$.

Since *S* is a Δ-semigroup and every homomorphic image of a Δ-semigroup is a Δ-semigroup, So $S/\mathcal{N}$ is a Δ-semigroup, which is also a semilattice Δ-semigroup if and only if is of order less than 2, so $|S/\mathcal{N}|\le 2$. thus $|S/\mathcal{N}|=1$ or $|S/\mathcal{N}|\le 2$. If $|S/\mathcal{N}|=1$ we are through. If $|S/\mathcal{N}|=2$ then we have $S=S_0\bigcup S_1$ where $S_0S_1\subseteq S_0$, and $S_1S_0\subseteq S_0$, and $S_0\bigcap S_1=\varnothing, S_0,S_1\ne\varnothing$.

Theorem 5: If *G* is a group then the following statements are equivalent:

(1) *G* is an abelian group which is a Δ-semigroup;

(2) *G* is a group in which all subgroups from a chain;

(3) For any two elements *a* and *b* of a group *G*, either $a=b^n$ or $b=a^n$ for some positive integer *n*;

(4) *G* is a *p*-quasicyclic group for some prime *p*;

(5) *G* is a group in which all subsemigroups from a chain.

Proof: (1) ⇒ (2) Let *G* be an abelian group such that *G* is a Δ-semigroup. Since *G* be an abelian, so *G* is a group such that G is a Δ-semigroup. Let H be any subgroup of G then we have the congruence on H is defined by $a \equiv b(\bmod H)$ if and only if $b^{-1} \in H$. This relation is an equivalence relation. Clearly reflexive, since a is always congruent to a.

It is also symmetric. Since a is congruent to b, so b is also congruent to a. Let $a \equiv b(\bmod H)$ and $b \equiv c(\bmod H)$ then $b^{-1}a \in H$ and $c^{-1} \in H$. Now we show that $a \equiv c(\bmod H)$ if and only if $c^{-1} \in H$. Now $c^{-1}bb^{-1}a = c^{-1}a \in H$. Now we so that $ac \equiv bc(\bmod H)$ so that $(bc)^{-1}(ac) = c^{-1}b^{-1}ac = b^{-1}c^{-1}ca = b^{-1} \in H$. Thus $ac \equiv bc(\bmod H)$ similarly $ac \equiv bc(\bmod H)$. Thus the relation is congruence. Let $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n$ then $H_1 \cup H_2 \cup \cdots \cup H_n = \{H_n\}$. Hence the set of all subgroups of a group G from a chain.

(2)$\Longrightarrow$ (3) Let G be a group satisfying the condition that subgroups from a chain. Then G is periodic and all cyclic subgroups from a chain. *i.e.* $\langle a\rangle \subseteq \langle b\rangle$ or $\langle b\rangle \subseteq \langle a\rangle$. If $\langle a\rangle \subseteq \langle b\rangle$ then $a = b^n$ for some positive integer n, or if $\langle b\rangle \subseteq \langle a\rangle$. Then $b = a^n$ for some positive integer n.

(3)$\Longrightarrow$ (4) By the periodicity of G it follows that all cyclic subgroups of G form a chain with respect to inclusion. According the order of every element, hence of every cyclic subgroup is a power of a same prime number p. let $C(x)$ denote the cyclic subgroup generated by x. Let F_n be the set of all elements of order p^n in G. We have a finite or infinite sequence $\{F_n\}$ and

$$G = \bigcup_{n=1}^{\infty} F_n \tag{1}$$

Let $x, y \in \{F_n\}$ by (3), we have either $x = y^m$ or $y = x^m$ for some $m > 0$. Assume that $x = y^m$, so $C(x) \subseteq C(y)$ since $|C(x)| = |C(y)| = p^n$, so we have $C(x) = C(y)$ similarly $y = x^m$, we have $C(x) = C(y)$. Conversely, suppose $C(x) = C(y)$ we have $|C(x)| = |C(y)| = p^n$. since $x \in C(x)$ so that $x = p^n$ for some p and $x = q^m$ for some q and thus $p^n = q^m$ (since $|C(x)| = |C(y)|$). Thus $x \in F_n$ and similarly $y \in F_n$ so that $x, y \in F_n$. Thus $C(x) = C(y)$ if and only if x, y are in a same F_n. Choose one element a_n in F_n. then we have a finite or infinite sequence $C(a_1) \subset C(a_2) \subset \cdots \subset C(a_n) \subset \cdots$ $\rightarrow$ (2) where $|C(a_n)| = p^n$ and $F_n \subset C(a_n)$ by (1).

$$G = \bigcup_{n=1}^{\infty} C(a_n)$$

If the sequence (2) is finite $G = C(a_n)$ for some n. that is G cyclic subgroup of order p^n. Thus we have G is a p-quasicyclic group of some prime p.

(4) $\Longrightarrow$ (5) Let G be a p-quasicyclic group, that is G is $\bigcup_{n=1}^{\infty} C(a_n)$ where $C(a_n)$ is cyclic group of order p. Let H be a subsemigroup of G and let $H'_n = F_n \cap H$ where F_n is the set of all elements of order p_n in G, also $F_n \subset C(a_n)$. Here

$$H = \bigcup_{n=1}^{\infty} H'_n$$

Let $x \in H'_n$ by the definition of F_n, $C(a_n) = C(x) \subseteq H$. If the set $\{n_i \mid H'_{n_i} \neq \phi\}$ is infinite, then $H = G$. If the set is finite, and if n_m its maximum, $H = C(a_{nm})$. Consequently G has no proper subsemigroup, hence no proper subgroup except $C(a_n), n = 1, 2, \cdots$ in (2). Thus $C(a_1) \subset C(a_2) \subset \cdots \subset C(a_{n_m})$. Thus we have all subsemigroups of G form a chain.

(4) $\Rightarrow$ (1) Since cyclic groups are abelian, so G is abelian which is a Δ-semigroup.

(5) $\Rightarrow$ (1) It follows that G is periodic, therefore every subsemigroup is a subgroup. Hence all subgroups form a chain.

Theorem 6: A group G^0 with zero is a Δ-semigroup if and only if the group G is a Δ-semigroup.

Proof: Let G be a group and G^0 be the group G with zero element adjoined. Let ρ be any congruence on G. A congruence ρ^0 on G^0 is associated with as follows:

$(a,b) \in \rho^0$ if and only if either $a = b = 0$ or $(a,b) \in \rho$, $a, b \in G$. Clearly ρ^0 is reflexive, since $(a,a) \in \rho$ and ρ is reflexive. Let $(a,b) \in \rho^0$ so that either $a = b = 0$ or $(a,b) \in \rho$ and thus $a = b = 0$ or $(b,a) \in \rho$ ($\therefore \rho$ is symmetric). Thus $(b,a) \in \rho^0$ and hence ρ is symmetric. Let $(a,b) \in \rho^0$ and $(b,c) \in \rho^0$ so that either $a=b=0$ or $(a,b)\in\rho$ and either $b=c=0$ or $(b,c)\in\rho$ and thus $a=c=0$ or $(a,c)\in\rho$ ($\therefore\rho$ is transtive). Thus $(a,c)\in\rho^0$ and hence ρ^0 is transtive. Hence ρ^0 is an equivalence relation. Let $(a,b)\in\rho^0$ and $c\in G$ so that either $a=b=0$ or $(a,b)\in\rho$ an also $ac=bc=0$ or $(ac,bc)\in\rho$ ($\therefore\rho$ is congruence). Thus $(ac,bc)\in\rho^0$ and similarly $(ca,cb)\in\rho^0$. Thus ρ^0 is a congruence on G^0. Clearly the mapping $\rho^0\to\rho$ is a one to one. Now we have to show that $\rho\subseteq\sigma$ if and only if $\rho^0\subseteq\sigma^0$. Assume that $\rho\subseteq\sigma$ and let $(a,b)\in\rho^0$ so that either $a=b=0$ or $(a,b)\in\rho$, $a,b\in G$ and thus either $a=b=0$ or $(a,b)\in\sigma$ ($\therefore\rho\subseteq\sigma$) and hence $(a,b)\in\sigma^0$. Thus $\rho^0\subseteq\sigma^0$. Conversely suppose that $\rho^0\subseteq\sigma^0$ let $(a,b)\in\rho^0\subseteq\rho^0$ and thus $(a,b)\in\sigma^0$ ($\therefore\rho^0\subseteq\sigma^0$) and hence $(a,c)\in\sigma$ ($\therefore\sigma^0\subseteq\sigma^0$) so that $\rho\subseteq\sigma$. Hence $\rho\subseteq\sigma$ if and only if $\rho^0\subseteq\sigma^0$. Let ω_G and ω_{G^0} denote the universal relation on G and G^0 respectively. Now we will prove that every congruence on G^0 is either ω_{G^0} or ρ^0, acongruence associated with ρ on G. let σ be a congruence on G^0 so that $(a,0)\in\sigma$ for some $a\in G$ and multiflying the both sides by $a^{-1}x$, $x\in G^0$ we have $(x,0)\in\sigma$, for all $x\in G^0$. Therefore $\sigma=\omega_{G^0}$. Thus every congruence on G^0 is either ω_{G^0} or ρ^0 for some congruence ρ on G, and also clearly $\omega_G\subseteq\omega_{G^0}$. Hence a group G^0 with zero is a Δ-semigroup if and only if the group G is a Δ-semigroup.

Theorem 7: An abelian group G^0 with zero is a Δ-semigroup if and only if G is a p-quasicycli group, p is arbitrary prime.

Proof: From theorem 6 we have G is a p-quasicycli group for some prime p if and only if G is an abelian group which is a Δ-semigroup. From theorem 6 we have a group G^0 with zero is a Δ-semigroup if and only if the group G is a Δ-semigroup. Since G is an abelian group which is a Δ-semigroup. So G^0 is an abelian group with zero is a Δ-semigroup.

Theorem 8: Let I be an ideal of a semigroup S. If f is a homomorphism of I onto a non-trivial group G, then there is a homomorphism g of S onto G such that f is the restriction of g to I.

Proof: Let $f:I\to G$ is an onto homomorphism $g:S\to G$ defined by $g(x)=f(x)$, if $x\in I$, $g(x)=f(xa)f(a)^{-1}$, If $x\notin I$ (choose $a\in I$) Now we show that g is a homomorphism.

Case(i): If both $x\in I$, $y\in I$. $g(xy)=f(xy)=f(x)f(y)=g(x)g(y)$. Thus g is a homomorphism.

Case(ii): If $x\notin I$, $y\in I$.

We have $g(xy)=f(xy)$ and $g(x)g(y)=f(xa)f(a)^{-1}f(y)$. Now $f(xy)=f(xa)f(a)^{-1}f(y)$ since $f(a)f(xy)=f(a)f(xa)f(a)^{-1}f(y)$ and since $f(a)f(xy)=f(a(xy))=f(ax)f(a)f(a)^{-1}f(y)$ and $f(a)f(xa)f(a)^{-1}f(y)=f(ax)f(y)$ and $f(a(xy))=f((ax)y)$. Thus $g(xy)=g(y)$ and hence g is homomorphism.

Case(iii): If $x,y\notin I$, $xy\in I$.

We have $g(xy)=f(xy)$ and $g(x)g(y)=f(xa)f(a)^{-1}f(ya)f(a)^{-1}$. Now we have to show that $f(xy)=f(xa)f(a)^{-1}f(ya)f(a)^{-1}$. Put $f(a)^{-1}=f(b)$ $\therefore f$ is onto) so that $f(a)f(a)^{-1}=f(a)f(b)$ and thus $f(ab)=e$. Then to show that $f(xy)=f((xab)(yab))$.Now we claim that

$f((xab)(yab))f(xy)^{-1}=e$. Put $f(xy)^{-1}=f(z)$ ($\therefore f$ is onto) so that $f(xy)f(xy)^{-1}=f(xy)f(z)$ and hence $f(xyz)=e$. Then we prove that $f((xab)(yab))f(z)=e$. Now $f((xab)(yab))f(z)$ $=f((xab)(yab)(z))=f(xa)f(by)f(ab)f(z)=f(xa)f(by)f(z)=f(xabyz)=f(xab)f(yz)$. and now we show that $f(xab)=f(x)$. *i.e.* $f(x)^{-1}f(xab)=e$ put $f(x)^{-1}=f(t)$ so that

$f(x)f(x)^{-1}=f(x0)f(t)$ and thus $f(xt)=e$ then $f(t)f(xab)=f(txab)=f(tx)f(ab)=e\cdot e=e$ and hence $f(x)^{-1}f(xab)=e$ and therefore $f(xab)=f(x)$. Now we have

$f(xab)f(yz)=f(x)f(yz)=f(xyz)=e.$ and thus $f((xab)(yab))f(z)=e$ and hence

$f((xab)(yab))f(xy)^{-1}=e$ and therefore put $f(xab)f(yab)=f(xy)$. Thus $g(xy)=g(x)g(y)$. Hence g is a homomorphism.

Case(iv): If $x, y \notin I$, $xy \notin I$.

We have $g(xy) = f(xya) f(a)^{-1}$ by definition and $g(x)g(y) = f(xa) f(a)^{-1} f(ya) f(a)^{-1}$. Now we show that $f(xya) f(a)^{-1} = f(xa) f(a)^{-1} f(ya) f(a)^{-1}$. we have $f(xya) = f(xa) f(a)^{-1} f(ya)$ and hence $f(a) f(xya) = f(a) f(xa) f(a)^{-1} f(ya)$ and therfore $f(axya) = f(ax) f(a) f(a)^{-1} f(ya)$ thus $f(axya) = f(ax) f(ya)$. Now we show that $f((ax)(ya)) = f(ax) f(ya)$.

We have $f((ax)(ya)) = g(ax) f(ya) = g(ax) g(ya) = f(ax) f(ya)$ and thus $f((ax)(ya)) = f(ax) f(ya)$ and hence $g(xy) = g(x) g(y)$. Thus g is a homomorphism. Since $f: I \to G$ is onto, so that $f(x) = y, y \in G$. Now $g: S \to G$ by $g(x) = f(x)$, If $x \in I$ then $g(x) = y$, for some $y \in G$. Thus g is onto. Thus $g: S \to G$ is an onto-homomorphism.

Example 1: Let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / a > 0, b, c, d \geq 0, a, b, c, d \in \mathbb{R} \right\}$.

Let $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_1 & b_2 \\ c_2 & d_2 \end{pmatrix} \in S$.

Now $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_1 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix} \in S$.

Where $a_1 a_2 + b_1 c_2 \geq a_1 a_2 > 0$ because $b_1 c_2 \geq 0$ and since a_1 and a_2 are positive and also matrix multiplication is associative. Thus S is a semigroup.

Now write $I = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} / a > 0, b \geq 0 \right\}$.

We verify I is left ideals of S.

Let $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a_1 a + b_1 b & 0 \\ c_1 a + d_1 b & 0 \end{pmatrix} \in I$ Thus I is a left ideal.

Now $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in S$ and $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in I$.

Thus I is not a right ideal.

Now we have ($\mathbb{R}^+$) is a group.

Define $f: I \to \mathbb{R}^+$ by $f \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = a$.

Now let $f\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \right) = \begin{pmatrix} ac & 0 \\ bc & 0 \end{pmatrix} = ac$.

And let $f\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \right) f\left(\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \right) = ac$.

Thus f is a homomorphism which is also onto.

Now we claim that f can't be extended to homomorphism $g: S \to \mathbb{R}^+$ such that $\frac{g}{I} = f$.

We have $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is an idempotent.

Since g is a homomorphism, so $g\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right) = 1$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$.

Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} a & a \\ c & c \end{pmatrix}$.

An let $g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)g\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right)=g\left(\begin{pmatrix} a & a \\ c & c \end{pmatrix}\right)$.

And hence $g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=g\left(\begin{pmatrix} a & a \\ c & c \end{pmatrix}\right) \rightarrow (1)$.

Put $b=0, d=0$ in (1)

Then $g\left(\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right)=g\left(\begin{pmatrix} a & a \\ c & c \end{pmatrix}\right)$.

And hence $f\left(\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right)=g\left(\begin{pmatrix} a & a \\ c & c \end{pmatrix}\right)$.

Since $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\in I$, we have $g\left(\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right)=f\left(\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right)=a$.

And therefore $g\left(\begin{pmatrix} a & a \\ c & c \end{pmatrix}\right)=a$.

From (1), $g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=g\left(\begin{pmatrix} a & a \\ c & c \end{pmatrix}\right)=a$.

This is contradiction.

Consider $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}=\begin{pmatrix} 5 & 11 \\ 10 & 18 \end{pmatrix}$.

So, $g\left(\begin{pmatrix} 5 & 11 \\ 10 & 18 \end{pmatrix}\right)$.

And $g\left(\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\right)g\left(\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\right)=1\times 1=1$.

Here $1\neq 5$. Thus *g* is not homomorphism.

Theorem 9: If a semigroup *S* contains a proper ideal *I* and *I* and if *S* is a Δ-semigroup, then neither *S* nor *I* is homomorphic onto a nono-trivial group.

Proof: Suppose there is a homomorphism *f* of *S* onto *G*, so that $f(s)=G$, $|G|>1$. Since *G*, contains no ideal except *G*, so $f(I)=G$. Hence $|I|>1$. Let ρ be the congruence on *S* induced by *f*. For each $a\in S\ I$, there is an element *b* in *I* such that $(a,b)\in\rho$. Let σ be the Rees-congruence on *S* modulo *I*. Then $(a,b)\in\rho$ but $(a,b)\notin\sigma$ then both $(a,b)\in I$. Now since $|G|>1\ni(x,y)\in\sigma$ but $(x,y)\notin\rho$, for some $(x,y)\in I$. Since $(x,y)\in\sigma$ so that both $(x,y)\in I$. But if $(x,y)\in\rho$ then $x\notin I$. Thus $\rho\nsubseteq\sigma$ and $\rho\nsupseteq\sigma$. Which is contradiction to assumption, since *S* is a Δ-semigroup. Therefore a *S* is not homomorphic onto a group *G*, $|G|>1$. Suppose that *I* is homorphic onto *G*, $|G|>1$. Then by the above theorem there is a homomorphic of *S* onto *G*. This leads to the same contradiction above. Therefore *I* is not homomorphic onto *G*.

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