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q-Laplace Transform

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Abstract

The Fourier transformations are used mainly with respect to the space variables. In certain circumstances, however, for reasons of expedience or necessity, it is desirable to eliminate time as a variable in the problem. This is achieved by means of the Laplace transformation. We specify the particular concepts of the q-Laplace transform. The convolution for these transforms is considered in some detail.

Keywords

Time Scales, Laplace Transform, Convolution

1. Introduction

The Laplace transform provides an effective method for solving linear differential equations with constant coefficients and certain integral equations. Laplace transforms on time scales, which are intended to unify and to generalize the continuous and discrete cases, were initiated by Hilger [1] and then developed by Peterson and the authors [2].

2. The q-Laplace Transform

Definition 2.1. *A time scale T is an arbtrary nonempty closed subset of the real numbers. Thus the real numbers R, the integers Z, the natural numbers N, the nonnegative integers* N_0, *and the q-numbers* $q^{N_0}=\{q^k : k\in N_0\}$ *with fixed* $q>1$ *are examples of time scales* [2] [3].

Definition 2.2. *Assume* $f:T\to C$ *is a function and* $t\in T^k$. *Then we define* $f^{\Delta}(t)$ *to be the number with the property that given any* $\varepsilon>0$, *there is a nighbourhood U (in T) of t such that*

$$\left|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]\right|\le\varepsilon\left|\sigma(t)-s\right| \quad \text{for all } s\in U.$$

We call $f^{\Delta}(t)$ *the delta (or Hilger) derivative of f at t.*

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$f^{\Delta}(t)=\dfrac{f(qt)-f(t)}{(q-1)t}$ *is the usual Jakson derivative if* $T=q^{N_0}$.

Definition 2.3. *If* $x:q^{N_0}\to C$ *is a function, then its q-Laplace transform is defined by*

$$\tilde{x}(z)=\mathcal{L}\{x\}(z)=q'\sum_{n=0}^{\infty}\frac{q^n x(q^n)}{\prod_{k=0}^{n}(1+q'q^k z)} \tag{1}$$

for those values of $z\neq -\dfrac{1}{q'q^k}$, $k\in N_0$, for which this series converges, where $q'=q-1$.

Let us set

$$p_n(z)=\prod_{k=0}^{n}(1+q'q^k z),\quad n\in N_0, \tag{2}$$

which is a polynomial in Z of degree $n+1$. It is easily verified that the equations

$$p_n(z)-p_{n-1}(z)=zq'q^n p_{n-1}(z),\quad n\in N_0, \tag{3}$$

and

$$\frac{1}{p_{n-1}(z)}-\frac{1}{p_n(z)}=z\frac{q'q^n}{p_n(z)},\quad n\in N_0, \tag{4}$$

hold, where $p_{-1}(z)=1$. The numbers

$$\alpha_k=-\frac{1}{q'q^k},\quad k\in N_0,$$

where $q'=q-1$, belong to the real axis interval $\left[-(q-1)^{-1},0\right)$ and tend to zero as $k\to\infty$. For any $\delta>0$ and $k\in N_0$, we set

$$D_\delta^k=\left\{z\in C:|z-\alpha_k|<\delta\right\}$$

and

$$\Omega_\delta=C\setminus\bigcup_{k=0}^{\infty}D_\delta^k=\left\{z\in C:|z-\alpha_k|\geq\delta,\forall_{k\in N_0}\right\}$$

so that Ω_δ is a closed domain of the complex plane C, whose points are in distance not less than δ from the set $\{\alpha_k:k\in N_0\}$.

Lemma 2.4. *For any* $z\in\Omega_\delta$,

$$|p_n(z)|\geq(q'\delta)^{n+1}q^{\frac{n(n+1)}{2}},\quad n\in N_0\cup\{-1\}. \tag{5}$$

Therefore, for an arbitrary number $R>0$, *there exists a positive integer* $n_0=n_0(R,\delta,q)$ *such that*

$$|p_n(z)|\geq R^{n+1}\quad \text{for all } n\geq n_0,\ z\in\Omega_\delta. \tag{6}$$

In particular,

$$\lim_{n\to\infty}p_n(z)=\infty\quad \text{for all } z\in\Omega_\delta. \tag{7}$$

Example 2.5. *We find the q-Laplace transform of* $x(t)\equiv k$ *(k is a fixed number). We have in,*

$$\mathcal{L}\{k\}(z)=q'\sum_{n=0}^{\infty}\frac{kq^n}{p_n(z)}=\frac{k}{z}\sum_{n=0}^{\infty}\left[\frac{1}{p_{n-1}(z)}-\frac{1}{p_n(z)}\right]$$
$$=\frac{k}{z}\lim_{m\to\infty}\left[1-\frac{1}{p_m(z)}\right]=\frac{k}{z}.$$

Example 2.6. *We find the q-Laplace transform of the functions* $x(z)=\cos az$ *and* $x(z)=\sin az$ $(a\in R)$.

We have (*see* [4]),

$$\tilde{e}_\alpha(z) = \frac{1}{z-\alpha}.$$

On the other hand, we know that

$$e^{iaz} = \cos az + i\sin az$$

with respect to

$$\frac{1}{z-ia} = \frac{z}{z^2+a^2} + i\frac{a}{z^2+a^2}$$

The q-Laplace transform of the functions $x(z) = \cos az$ *and* $x(z) = \sin az$, *would be*

$$\cos az = \frac{z}{z^2+a^2}$$

and

$$\sin az = \frac{a}{z^2+a^2},$$

respectively.

Theorem 2.7. *If the function* $x : q^{N_0} \to C$ *satisfies the condition*

$$\left|x\left(q^n\right)\right| \le cR^n \quad \text{for all } n \in N_0, \tag{8}$$

where c and R are some positive constants, then the series in (1) *converges uniformly with respect to z in the region* Ω_δ *and therefore its sum* $\tilde{x}(z)$ *is an analytic* (*holomorphic*) *function in* Ω_δ.

Proof. By Lemma 2.4, for the number R given in (8) we can choose an $n_0 \in N$ such that

$$\left|p_n(z)\right| \ge \left[q(1+R)\right]^{n+1} \quad \text{for all } n \ge n_0, z \in \Omega_\delta.$$

Then for the general term of the series in (1), we have the estimate

$$\left|\frac{q^n x\left(q^n\right)}{p_n(z)}\right| \le \frac{c}{q(1+R)}\left(\frac{R}{1+R}\right)^n \quad \text{for all } n \ge n_0, z \in \Omega_\delta.$$

Hence the proof is completed.

A larger class of functions for which the q-Laplace transform exists is the class $\mathcal{F}_\delta$ of functions $x : q^{N_0} \to C$ satisfying the condition

$$\sum_{n=0}^{\infty} (q'\delta)^{-n} q^{-\frac{n(n-1)}{2}} \left|x\left(q^n\right)\right| < \infty. \tag{9}$$

Theorem 2.8. *For any* $x \in \mathcal{F}_\delta$, *the series in* (1) *converges uniformly with respect to z in the region* Ω_δ, *and therefore its sum* $\tilde{x}(z)$ *is an analytic function in* Ω_δ.

Proof. By using the reverse (5), hence

$$\frac{1}{\left|p_n(z)\right|} \le (q'\delta)^{-(n+1)} q^{-\frac{n(n+1)}{2}}$$

and comparison test to get the desired result.

Theorem 2.9. (*Initial Value and Final Value Theorem*). *We have the following*:

a) *If* $x \in \mathcal{F}_\delta$ *for some* $\delta > 0$, *then*

$$x(1) = \lim_{z\to\infty}\left\{z\tilde{x}(z)\right\}. \tag{10}$$

b) *If* $x \in \mathcal{F}_\delta$ *for all* $\delta > 0$, *then*

$$\lim_{n\to\infty} x\left(q^n\right) = \lim_{z\to 0}\left\{z\tilde{x}(z)\right\}. \tag{11}$$

Proof. Assume $x \in \mathcal{F}_\delta$ for some $\delta > 0$. It follows from (1) that

$$\tilde{x}(z) = \frac{q'x(1)}{1+q'z} + \frac{q'qx(q)}{(1+q'z)(1+q'qz)} + \frac{q'q^2x(q)}{(1+q'z)(1+q'qz)(1+q'q^2z)} + \cdots \tag{12}$$

and

$$(1+q'z)\tilde{x}(z) = q'x(1) + \frac{q'qx(q)}{(1+q'qz)} + \frac{q'q^2x(q)}{(1+q'qz)(1+q'q^2z)} + \cdots \tag{13}$$

Hence

$$\lim_{z\to\infty} \tilde{x}(z) = 0 \quad \text{and} \quad \lim_{z\to\infty} \{(1+q'z)\tilde{x}(z)\} = q'x(1),$$

Multiplying $z \neq 0$, on both sides of the relation of (12) and by using equivalence relation, which yields (10). Note that we have taken a term-by-term limit due to the uniform convergence (Theorem 2.8) of the series in the region Ω_δ.

3. Convolutions

Definition 3.1. *Let T be a time scale. We define the forward jump operator* $\sigma : T \to T$ *by*

$$\sigma(t) = \inf\{s \in T : s > t\} \text{ for } t \in T.$$

Definition 3.2. *For a given function* $f : [t_0, \infty) \to C$, *its shift (or delay)* $\hat{f}(t,s)$ *is defined as the solution of the problem*

$$\begin{aligned} &\hat{f}^{\Delta t}(t, \sigma(s)) = -\hat{f}^{\Delta s}(t,s), \quad t,s \in T; \quad t \ge s \ge t_0, \\ &\hat{f}(t,t_0) = f(t), \quad t \in T, t \ge t_0. \end{aligned} \tag{14}$$

Definition 3.3. *For given functions* $f, g : [t_0, \infty)_T \to C$, *their convolution* $f * g$ *is defined by*

$$(f*g)(t) = \int_{t_0}^{t} \hat{f}(t, \sigma(s)) g(s) \Delta s, \quad t \in T, t \ge t_0, \tag{15}$$

where $\hat{f}$ *is the shift of f introduced in Definition* 3.2 [4].

Definition 3.4. *For given functions* $f, g : q^{N_0} \to C$, *their convolution* $f * g$ *is defined by*

$$\begin{aligned} (f*g)(q^n) &= (q-1)\sum_{k=0}^{n-1} q^k \hat{f}(q^n, q^{k+1}) g(q^k) \\ &= (q-1)\sum_{k=0}^{n-1} q^k \left\{ \sum_{\nu=0}^{n-k-1} f(q^\nu) \binom{n-k-1}{\nu} q^{(k+1)\nu} \prod_{j=0}^{n-k-\nu-2} (1-q^{k+j+1}) \right\} g(q^k) \end{aligned}$$

with $(f*g)(q^0) = 0$, *where* $n \in N_0$.

Theorem 3.5. (*Convolution Theorem*). *Assume that* $\mathcal{L}\{f\}(z)$, $\mathcal{L}\{g\}(z)$, *and* $L\{f*g\}(z)$ *exist for a given* $z \in C$. *Then at the point z,*

$$L\{f*g\}(z) = \mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z). \tag{16}$$

4. Concluding Remarks

1) We can see from Theorem 2.9(a) that no function has its q-Laplace transform equal to the constant function 1.

2) Finally, we note that most of the results concerning the Laplace transform on q^{N_0} can be generalized appropriately to an arbitrary isolated time scale $T = \{t_n\}_{n\in N_0}$ such that

$$\lim_{n\to\infty} t_n = \infty \qquad \inf\{t_{n+1} - t_n : n \in N_0\} > 0.$$

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