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Matrices Associated with Moving Least-Squares Approximation and Corresponding Inequalities

Svetoslav Nenov, Tsvetelin Tsvetkov

Department of Mathematics, University of Chemical Technology and Metallurgy, Sofia, Bulgaria
Email: nenov@uctm.edu, tstsvetkov@uctm.edu

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Abstract

In this article, some properties of matrices of moving least-squares approximation have been proven. The used technique is based on known inequalities for singular-values of matrices. Some inequalities for the norm of coefficients-vector of the linear approximation have been proven.

Keywords

Moving Least-Squares Approximation, Singular-Values

1. Statement

Let us remind the definition of the moving least-squares approximation and a basic result.

Let:

1. $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$;
2. $\boldsymbol{x}_i \in \mathcal{D}$, $i = 1, \cdots, m$; $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, if $i \neq j$;
3. $f : \mathcal{D} \to \mathbb{R}$ be a continuous function;
4. $p_i : \mathcal{D} \to \mathbb{R}$ be continuous functions, $i = 1, \cdots, l$. The functions $\{p_1, \cdots, p_l\}$ are linearly independent in $\mathcal{D}$ and let $\mathcal{P}_l$ be their linear span;
5. $W : (0, \infty) \to (0, \infty)$ be a strong positive function.

Usually, the basis in $\mathcal{P}_l$ is constructed by monomials. For example: $p_l(\boldsymbol{x}) = x_1^{k_1} \cdots x_d^{k_d}$, where $\boldsymbol{x} = (x_1, \cdots, x_d)$, $k_1, \cdots, k_d \in \mathbb{N}$, $k_1 + \cdots + k_d \le l - 1$. In the case $d = 1$, the standard basis is $\{1, x, \cdots, x^{l-1}\}$.

Following [1]-[4], we will use the following definition. The *moving least-squares approximation* of order *l* at a fixed point $\boldsymbol{x}$ is the value of $p^*(\boldsymbol{x})$, where $p^* \in \mathcal{P}_l$ is minimizing the least-squares error

$$\sum_{i=1}^{m} W(\|\boldsymbol{x}-\boldsymbol{x}_i\|)(p(\boldsymbol{x})-f(\boldsymbol{x}_i))^2$$

among all $p \in \mathcal{P}_l$.

The approximation is "local" if weight function W is fast decreasing as its argument tends to infinity and interpolation is achieved if $W(0)=\infty$. So, we define additional function $w:[0,\infty)\to[0,\infty)$, such taht:

$$w(r)=\begin{cases}\dfrac{1}{W(r)}, & \text{if } (r>0) \text{ or } (r=0 \text{ and } W(0)<\infty),\\ 0, & \text{if } (r=0 \text{ and } W(0)=\infty).\end{cases}$$

Some examples of $W(r)$ and $w(r)$, $r \ge 0$:

$$2W(r)=\mathrm{e}^{-\alpha^2 r^2} \qquad \text{exp-weight,}$$

$$W(r)=r^{-\alpha^2} \qquad \text{Shepard weights,}$$

$$w(\boldsymbol{x},\boldsymbol{x}_i)=r^2\mathrm{e}^{-\alpha^2 r^2} \qquad \text{McLain weight,}$$

$$w(\boldsymbol{x},\boldsymbol{x}_i)=\mathrm{e}^{\alpha^2 r^2}-1 \qquad \text{see Levin's works.}$$

Here and below: $\|\cdot\|=\|\cdot\|_2$ is 2-norm, $\|\cdot\|_1$ is 1-norm in $\mathbb{R}^d$; the superscript ^{t} denotes transpose of real matrix; I is the identity matrix.

We introduce the notations:

$$E=\begin{pmatrix} p_1(\boldsymbol{x}_1) & p_2(\boldsymbol{x}_1) & \cdots & p_l(\boldsymbol{x}_1)\\ p_1(\boldsymbol{x}_2) & p_2(\boldsymbol{x}_2) & \cdots & p_l(\boldsymbol{x}_2)\\ \vdots & \vdots & & \vdots\\ p_1(\boldsymbol{x}_m) & p_2(\boldsymbol{x}_m) & \cdots & p_l(\boldsymbol{x}_m)\end{pmatrix},\ \boldsymbol{a}=\begin{pmatrix}a_1\\ a_2\\ \vdots\\ a_m\end{pmatrix},$$

$$D=2\begin{pmatrix} w(\boldsymbol{x},\boldsymbol{x}_1) & 0 & \cdots & 0\\ 0 & w(\boldsymbol{x},\boldsymbol{x}_2) & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & w(\boldsymbol{x},\boldsymbol{x}_m)\end{pmatrix},\ \boldsymbol{c}=\begin{pmatrix}p_1(\boldsymbol{x})\\ p_2(\boldsymbol{x})\\ \vdots\\ p_l(\boldsymbol{x})\end{pmatrix}.$$

Through the article, we assume the following conditions (H1):

(H1.1) $1 \in \mathcal{P}_l$;

(H1.2) $1 \le l \le m$;

(H1.3) $\operatorname{rank}(E^t)=l$;

(H1.4) w is smooth function.

Theorem 1.1. (see [2]): *Let the conditions* (*H*1) *hold true*.

Then:

1. The matrix $E^t D^{-1} E$ is non-singular;
2. The approximation defined by the moving least-squares method is

$$\hat{L}(f)=\sum_{i=1}^{m} a_i f(\boldsymbol{x}_i), \tag{1}$$

where

$$\boldsymbol{a}=A_0\boldsymbol{c} \quad \text{and} \quad A_0=D^{-1}E\left(E^t D^{-1}E\right)^{-1}. \tag{2}$$

3. If $w(\|\boldsymbol{x}_i-\boldsymbol{x}_i\|)=0$ for all $i=1,\cdots,m$, then the approximation is interpolatory.

For the approximation order of moving least-squares approximation (see [2] and [5]), it is not difficult to receive (for convenience we suppose $d=1$ and standard polynomial basis, see [5]):

$$\left|f(x)-\hat{L}(f)(x)\right| \le \left\|f(x)-p^*(x)\right\|_\infty \left[1+\sum_{i=1}^{m}|a_i|\right], \tag{3}$$

and moreover (C =const.)

$$\left\|f(x)-p^*(x)\right\|_\infty \le Ch^{l+1}\max\left\{\left|f^{(l+1)}(x)\right| : x\in\bar{\mathcal{D}}\right\}. \tag{4}$$

It follows from (3) and (4) that the error of moving least-squares approximation is upper-bounded from the 2-norm of coefficients of approximation ($\|\boldsymbol{a}\|_1 \le \sqrt{m}\|\boldsymbol{a}\|_2$). That is why the goal in this short note is to discuss a method for majorization in the form

$$\|\boldsymbol{a}\|_2 \le M\exp\left(N\|\boldsymbol{x}-\boldsymbol{x}_i\|\right).$$

Here the constants M and N depend on singular values of matrix E^t, and numbers m and l (see Section 3). In Section 2, some properties of matrices associated with approximation (symmetry, positive semi-definiteness, and norm majorization by $\sigma_{\min}(E^t)$ and $\sigma_{\max}(E^t)$) are proven.

The main result in Section 3 is formulated in the case of exp-moving least-squares approximation, but it is not hard to receive analogous results in the different cases: Backus-Gilbert wight functions, McLain wight functions, etc.

2. Some Auxiliary Lemmas

Definition 2.1. *We will call the matrices*

$$A_1 = A_0E^t = D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t \quad \text{and} \quad A_2 = A_1 - I$$

A_1*-matrix* and A_2*-matrix* of the approximation $\hat{L}$, respectively.

Lemma 2.1. *Let the conditions* (*H*1) *hold true*.

Then, the matrices A_1D^{-1} and A_2D^{-1} are symmetric.

Proof. Direct calculation of the corresponding transpose matrices.

Lemma 2.2. *Let the conditions* (*H*1) *hold true*.

Then:

1. All eigenvalues of A_1 are 1 and 0 with geometric multiplicity *l* and $m-l$, respectively;
2. All eigenvalues of A_2 are 0 and −1 with geometric multiplicity *l* and $m-l$, respectively.

Proof. Part 1: We will prove that the dimension of the null-space $\dim\left(\text{null}(A_2)\right)$ is at least *l*.

Using the definition of $A_2 = D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t - I$, we receive

$$E^tA_2 = \left(E^tD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}E^t - E^t = 0.$$

Hence,

$$\text{im}(A_2) \subseteq \text{null}\left(E^t\right).$$

Using (H1.3), E^t is $(l\times m)$-matrix with maximal rank *l* ($l<m$). Therefore, $\dim\left(\text{null}\left(E^t\right)\right) = m-l$. Moreover, $\dim\left(\text{im}(A_2)\right) = m-\dim\left(\text{null}(A_2)\right)$. That is why $m-\dim\left(\text{null}(A_2)\right) \le m-l$ or $l \le \dim\left(\text{null}(A_2)\right)$.

Part 2: We will prove that -1 is eigenvalue of A_2 with geometric multiplicity $m-l$, or the system

$$A_2\boldsymbol{\eta} = -\boldsymbol{\eta} \Leftrightarrow A_1\boldsymbol{\eta} = 0$$

has $m-l$ linearly independent solutions.

Obviously the systems

$$A_1\boldsymbol{\eta} = D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t\boldsymbol{\eta} = 0 \tag{5}$$

and

$$E^t\boldsymbol{\eta} = 0 \tag{6}$$

are equivalent. Indeed, if η_0 is a solution of (5), then

$$D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t\eta_0=0\Rightarrow E^tD^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t\eta_0=0$$
$$\Rightarrow E^t\eta_0=0,$$

i.e. η_0 is solution of (6).

On the other hand, if η_0 is a solution of (6), then

$$\left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t\right)\eta_0=\left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}\right)\left(E^t\eta_0\right)=0,$$

i.e. η_0 is solution of (5). Therefore

$$\dim\left(\mathrm{im}\left(A_1\right)\right)=\dim\left(\mathrm{im}\left(E^t\right)\right)=m-l.$$

Part 3: It follows from parts 1 and 2 of the proof that 0 is an eigenvalue of A_2 with multiplicity exactly l and -1 is an eigenvalue of A_2 with multiplicity exactly $m-l$.

It remains to prove that 1 is eigenvalue of A_1 with multiplicity at least l, but this is analogous to the proven part 1 or it follows dirctly from the definition of $A_1=A_2+I$.

The following two results are proven in [6].

Theorem 2.1 (see [6], Theorem 2.2): *Suppose U, V are* $(m\times m)$ *Hermitian matrices and either U or V is positive semi-definite. Let*

$$\lambda_1(U)\ge\cdots\ge\lambda_m(U),\quad \lambda_1(V)\ge\cdots\ge\lambda_m(V)$$

denote the eigenvalues of *U* and *V*, respectively.

Let:

1. $\pi(U)$ is the number of positive eigenvalues of *U*;
2. $\nu(U)$ is the nubver of negative eigenvalues of *U*;
3. $\xi(U)$ is the number of zero eigenvalues of *U*.

Then:

1. If $1\le k\le\pi(U)$, then

$$\min_{1\le i\le k}\left\{\lambda_i(U)\lambda_{k+1-i}(V)\right\}\ge\lambda_k(VU)\ge\min_{k\le i\le m}\left\{\lambda_i(U)\lambda_{m+k-i}(V)\right\}.$$

2. If $\pi(U)<k\le m-\nu(U)$, then

$$\lambda_k(VU)=0.$$

3. If $m-\nu(U)<k\le m$, then

$$\min_{1\le i\le k}\left\{\lambda_i(U)\lambda_{m+i-k}(V)\right\}\ge\lambda_k(VU)\ge\min_{k\le i\le m}\left\{\lambda_i(U)\lambda_{i+1-k}(V)\right\}.$$

Corollary 2.1. (see [6], Corollary 2.4): *Suppose U, V are* $(m\times m)$ *Hermitian positive definite matrices*.

Then for any $1\le k\le m$

$$\lambda_1(U)\lambda_1(V)\ge\lambda_k(VU)\ge\lambda_m(U)\lambda_m(V).$$

As a result of Lemma 2.1, Lemma 2.2 and Theorem 2.1, we may prove the following lemma.

Lemma 2.3. *Let the conditions* (*H*1) *hold true*.

1. Then A_1D^{-1} and $-A_2D^{-1}$ are symmetric positive semi-definite matrices.
2. The following inequality hods true

$$\lambda_{\max}\left(A_1D^{-1}\right)\le\frac{1}{\lambda_{\min}(D)}.$$

Proof. (1) We apply Theorem 2.1, where

$$U=D,\quad V=A_1D^{-1}.$$

Obviously, *U* is a symmetric positive definite matrix (in fact it is a diagonal matrix). Moreover $\pi(U)=m$,

$\mu(U)=\xi(U)=0$, if $\boldsymbol{x}\neq\boldsymbol{x}_i$, $i=1,\cdots,m$.

The matrix V is symmetric (see Lemma 2.1).

From the cited theorem, for any index k $(k=1,\cdots,m=\pi(U))$ we have

$$\lambda_k(A_1)=\lambda_k(A_1D^{-1}D)=\lambda_k(VU)\leq\min_{1\leq i\leq k}\{\lambda_i(U)\lambda_{m+i-k}(V)\}.$$

In particular, if $k=m$:

$$\lambda_m(A_1)\leq\min_{1\leq i\leq m}\{\lambda_i(U)\lambda_i(V)\}. \tag{7}$$

Let us suppose that there exists index i_0 $(i_0=1,\cdots,m-1)$ such that

$$\lambda_1(V)\geq\cdots\geq\lambda_{i_0}(V)\geq 0>\lambda_{i_0+1}(V)\geq\cdots\geq\lambda_m(V). \tag{8}$$

It fowollws from (8) and positive definiteness of U, that

$$\min_{1\leq i\leq m}\{\lambda_i(U)\lambda_i(V)\}\leq\lambda_{i_0+1}(U)\lambda_{i_0+1}(V)<0.$$

Therefore (see (7)), $\lambda_m(A_1)<0$. This contradiction (see Lemma 2.2) proves that the matrix A_1D^{-1} is positive semi-definite.

If we set $U=D$, $V=-A_2D^{-1}$ then by analogical arguments, we see that the matrix $-A_2D^{-1}$ is positive semi-definite.

(2) From the first statement of Lemma 2.3, $V=A_1D^{-1}$ is positive semi-definite. Therefore (see Corollary 2.1 and Lemma 2.2):

$$1\geq\lambda_k(A_1)=\lambda_k(VU)\geq\max\{\lambda_m(U)\lambda_k(V),\lambda_m(V)\lambda_k(U)\}$$

for all $k=1,\cdots,m$. Moreover, all numbers $\lambda_k(U)$, $\lambda_k(V)$ are non-negative and

$$\lambda_{\max}(D)=\lambda_1(U)\geq\cdots\geq\lambda_m(U)=\lambda_{\min}(D),\quad\lambda_1(V)\geq\cdots\geq\lambda_m(V).$$

Therefore

$$1\geq\max\{\lambda_m(U)\lambda_1(V),\lambda_m(V)\lambda_1(U)\},$$

or

$$\lambda_{\max}(A_1D^{-1})=\lambda_1(V)\leq\frac{1}{\lambda_m(U)}=\frac{1}{\lambda_{\min}(D)}. \qquad \square$$

In the following, we will need some results related to inequalities for singular values. So, we will list some necessary inequalities in the next lemma.

Lemma 2.4. (see [7] [8]): *Let U be an $(d_1\times d_2)$-matrix, V be an $(d_3\times d_4)$-matrix.*

Then:

$$2\sigma_{\max}(UV)\leq\sigma_{\max}(U)\sigma_{\max}(V), \tag{9}$$

$$\sigma_{\max}(U^{-1})=\frac{1}{\sigma_{\min}(U)},\quad\text{if } d_1=d_2,\det U\neq 0, \tag{10}$$

$$\sigma_{\max}(V)\sigma_{\min}(U)\leq\sigma_{\max}(UV),\quad\text{if } d_1\geq d_2=d_3, \tag{11}$$

$$\sigma_{\max}(U)\sigma_{\min}(V)\leq\sigma_{\max}(UV),\quad\text{if } d_4\geq d_3=d_2. \tag{12}$$

If $d_1=d_2$ and U is Hermitian matrix, then $\|U\|=\sigma_{\max}(U)$, $\sigma_i(U)=|\lambda_i(U)|$, $i=1,\cdots,d_1$.

Lemma 2.5. *Let the conditions* ($H1$) *hold true and let* $\boldsymbol{x}\neq\boldsymbol{x}_i$, $i=1,\cdots,m$.

Then:

$$\|A_1D^{-1}\|\leq\frac{1}{\lambda_{\min}(D)}, \tag{13}$$

$$\sigma_{\max}(A_1)\sigma_{\min}(D^{-1}) \le \sigma_{\max}(A_1 D^{-1}), \tag{14}$$

$$1 \le \|A_1\| \le \sqrt{\frac{\sigma_{\max}(D)}{\sigma_{\min}(D)}}. \tag{15}$$

Proof. The matrix $A_1 D^{-1}$ is simmetric and positive semi-definite (see Lemma 2.3 (1)). Using the second statement of Lemma 2.3 and Lemma 2.4, we receive

$$\|A_1 D^{-1}\| = \sigma_{\max}(A_1 D^{-1}) = \lambda_{\max}(A_1 D^{-1}) \le \frac{1}{\lambda_{\min}(D)}.$$

The inequality (14) follows from (12) ($d_4 = d_3 = m$).

From (14) and (10), we receive

$$\sigma_{\max}(A_1) \le \frac{\sigma_{\max}(A_1 D^{-1})}{\sigma_{\min}(D^{-1})} = \frac{\sigma_{\max}(D)}{\sigma_{\min}(D)}.$$

Therefore, the equality $\|A_1\| = \sqrt{\sigma_{\max}(A_1)}$ implies the right inequality in (15).

Using $E^t = E^t A_1$ and inequality (9), we receive

$$\sigma_{\max}(E^t) \le \sigma_{\max}(E^t)\sigma_{\max}(A_1),$$

or $1 \le \sigma_{\max}(A_1) = \|A_1\|^2$, *i.e.* the left inequality in (15).

The lemma has been proved. □

3. An Inequality for the Norm of Approximation Coefficients

We will use the following hypotheses (H2):

(H2.1) The hypotheses (H1) hold true;

(H2.2) $d = 1$, $x_1 < \cdots < x_m$;

(H2.3) The map $\boldsymbol{c}$ is C^1-smooth in $[x_1, x_m]$;

(H2.4) $w(|x - x_i|) = \exp\left(\alpha (x - x_i)^2\right)$, $i = 1, \cdots, m$.

Theorem 3.1. *Let the following conditions hold true*:

1. Hypotheses (H2);
2. Let $x \in [x_1, x_m]$ be a fixed point;
3. The index $k_0 \in \{1, \cdots, m\}$ is choosen such that

$$|x - x_{k_0}| = \min\{|x - x_i| : i = 1, \cdots, m\}.$$

Then, there exist constants $M_1, M_2 > 0$ such that

$$\|\boldsymbol{a}(x)\| \le \left(\|\boldsymbol{a}(x_{k_0})\| + M_1 |x - x_{k_0}|\right) \exp\left(M_2 |x - x_{k_0}|\right).$$

Proof. Part 1: Let

$$H = \begin{pmatrix} 2\alpha(x - x_1) & 0 & \cdots & 0 \\ 0 & 2\alpha(x - x_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 2\alpha(x - x_m) \end{pmatrix},$$

then

$$\frac{\mathrm{d}D}{\mathrm{d}x} = HD, \quad \frac{\mathrm{d}D^{-1}}{\mathrm{d}x} = -HD^{-1}.$$

We have (obviously $D = D(x)$, $H = H(x)$, and $\boldsymbol{c} = \boldsymbol{c}(x)$)

$$\frac{\mathrm{d}\boldsymbol{a}(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c}\right)$$
$$= \left(\frac{\mathrm{d}}{\mathrm{d}x}D^{-1}\right)E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c} + D^{-1}E\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c} + D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{c}$$
$$= -HD^{-1}E\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c} + D^{-1}E\left(-\left(E^tD^{-1}E\right)^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}E^tD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}\right)\boldsymbol{c} + D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{c}$$
$$= -H\boldsymbol{a} + D^{-1}E\left(E^tD^{-1}E\right)^{-1}\left(E^tHD^{-1}E\right)\left(E^tD^{-1}E\right)^{-1}\boldsymbol{c} + D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{c}$$
$$= \left(D^{-1}E\left(E^tD^{-1}E\right)^{-1}E^t - I\right)H\boldsymbol{a} + D^{-1}E\left(E^tD^{-1}E\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{c}$$
$$= A_2H\boldsymbol{a} + A_0\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{c}.$$

Therefore, the function $\boldsymbol{a}(x)$ satisfies the differential equation

$$\frac{\mathrm{d}\boldsymbol{a}(x)}{\mathrm{d}x} = A_2H\boldsymbol{a} + A_0\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{c}. \tag{16}$$

Part 2: Obviously

$$\|A_2H\| = \|(A_1 - I)H\| \le (\|A_1\| + 1)\|H\|.$$

It follows from (15) that

$$\|A_1\| \le \sqrt{\frac{\sigma_{\max}(D)}{\sigma_{\min}(D)}}.$$

Here $\sigma_{\max}(D) \le 2\exp\left(\alpha r^2\right)$, $r = x_m - x_1$, and $\sigma_{\min}(D) \ge 2$. Hence

$$\|A_1\| \le \sqrt{\exp\left(\alpha r^2\right)}.$$

For the norm of diagonal matrix *H*, we receive

$$\|H\| \le 2\alpha r.$$

Therefore $\|A_2H\| \le M_2$, where

$$M_2 = 2\alpha r\left(1 + \sqrt{\exp\left(\alpha r^2\right)}\right).$$

We will use Lemma 2.4 to obtain the norm of A_0.
Obviously, $A_0E^t = A_1$. Therefore by (12) ($m = d_4 \ge d_3 = l$), we have

$$\sigma_{\max}(A_0)\sigma_{\min}\left(E^t\right) \le \sigma_{\max}(A_1),$$

i.e.

$$\|A_0\| \le \frac{1}{\sigma_{\min}\left(E^t\right)}\sqrt{\frac{\sigma_{\max}(D)}{\sigma_{\min}(D)}}.$$

Therefore, if we set $M_{11} = \dfrac{M_2}{\sigma_{\min}\left(E^t\right)}$, then $\|A_0\| \le M_1$.

Let the constant M_{12} be choosen such that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{c}(x)\right\| \le M_{12}, \quad x \in [x_1, x_m]$$

and let $M_1 = M_{11} M_{12}$.

Part 3: On the end, we have only to apply Lemma 4.1 form [9] to the Equation (16):

$$\|\boldsymbol{a}(x)\| \le \left(\left\|\boldsymbol{a}\left(x_{k_0}\right)\right\| + \left|\int_{x_{k_0}}^{x} \left\|A_0 \frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{c}\right\| \mathrm{d}x\right|\right) \exp\left|\int_{x_{k_0}}^{x} \|A_2 H\| \mathrm{d}x\right|$$
$$\le \left(\left\|\boldsymbol{a}\left(x_{k_0}\right)\right\| + M_1 \left|x - x_{k_0}\right|\right) \exp\left(M_2 \left|x - x_{k_0}\right|\right).$$

Remark 3.1. *Let the hypotheses* $(H2)$ *hold true and let moreover*

$$p_1(x) = 1,\ p_2(x) = x, \cdots,\ p_l(x) = x^{l-1}, \quad l \ge 1.$$

In such a case, we may replace the differentiation of vector-fuction

$$\boldsymbol{c}(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_l(x) \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{l-1} \end{pmatrix}$$

by left-multiplication:

$$\frac{\mathrm{d}\boldsymbol{c}(x)}{\mathrm{d}x} = \begin{pmatrix} 0 \\ 1 \\ 2x \\ 3x^2 \\ \vdots \\ (l-2)x^{l-3} \\ (l-1)x^{l-2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & l-2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & l-1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{l-2} \\ x^{l-1} \end{pmatrix} = \bar{\partial}\boldsymbol{c}(x).$$

The singular values of the matrix $\bar{\partial}$ are: $0, 1, \cdots, l-1$. Therefore $\|\bar{\partial}\| = \sqrt{l-1}$.
That is why, we may chose

$$M_{22} = \sqrt{(l-1)} \max_{1 \le i \le l} \left\{ \max_{x_1 < x < x_m} \left|p_i(x)\right| \right\}.$$

Additionally, if we supose $|x_1| \le |x_m|$, then

$$\max_{x_1 < x < x_m} \left|p_i(x)\right| = \left|p_i(x_m)\right|, \quad i = 1, \cdots, l.$$

Therefore, in such a case:

$$M_{22} = \sqrt{(l-1)} \max_{1 \le i \le l} \left\{ \left|p_i(x_m)\right| \right\}.$$

If we suppose $-1 \le x_1 \le x \le x_m \le 1$, then obviously, we may set

$$M_{22} = \sqrt{l-1}.$$

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