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On 2 - 3 Matrix Chevalley Eilenberg Cohomology

Joseph Dongho*, Epizitone Duebe-Abi, Shuntah Roland Yotcha

Department of Mathematics and Computer Science, Faculty of Sciences, University of Maroua, Maroua, Cameroon
Email: *joseph.dongho@fs.univ-maroua.cm

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Abstract

The main objective of this paper is to provide the tool rather than the classical adjoint representation of Lie algebra; which is essential in the conception of the Chevalley Eilenberg Cohomology. We introduce the notion of representation induced by a 2 - 3 matrix. We construct the corresponding Chevalley Eilenberg differential and we compute all its cohomological groups.

Keywords

Lie Algebra, Cochain, 2 - 3 Matrix Chevalley Eilenberg, Cohomology

1. Introduction

This work is included in the domain of differential geometry which is the continuation of infinitesimal calculation. It is possible to study it due to the new techniques of differential calculus and the new family of topological spaces applicable as manifold. The study of Lie algebra with classical example puts in place with so many homological materials [1]-[3] (Lie Bracket, Chevalley Eilenberg Cohomology...). The principal objective of this work is to introduce the notions of deformation of Lie algebra in the more general representation rather than the adjoint representation.

This work is base on 2 - 3 matrix Chevally Eilenberg Chohomology representation, in which our objective is to fixed a matrix representation and comes out with a representation which is different from the adjoint representation. Further, given a Lie algebra V, W respectively of dimension 2 and 3, we construct a linear map that will define a Lie algebra structure from a Lie algebra V into $EndW$ by putting the commutator structure in place.

*Corresponding author.

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This does lead us to a fundamental condition of our 2 - 3 matrix Chevalley Eilenberg Cohomology. We compute explicitly all the associated cohomological groups.

2. 2 - 3 Matrix Representation Theorem

We begin by choosing *V* to be a 2-dimensional vector space and *W* a 3-dimensional vector space, then we called our cohomology on a domain vector space *V* and codomain *W* a 2 - 3 matrix Chevalley Eilenberg Cohomology. In what follow, we denoted for all $i \in \mathbb{N}$ by $C^i(V,W)$ the space of *i*-multilinear skew symmetric map on *V* with valor in *W*; we also denoted by $(e_i)_{i=1}^2$ and $(w_i)_{i=1}^3$ respectively the basis of *V* and *W*. We also suppose that $\rho : V \to End(W)$ is a representation of the Lie algebra (V,μ) where μ is the associated Lie structure.

2.1. Description of Cochain Spaces

Since element of μ_i of $C^i(V,W)$ skew symmetric, then for all $k \in \{1,2\}$, we have $\mu_i(e_k,e_k) = 0$ [4]. Let $x \in V$ and $w \in W$, we have $x = C_1e_1 + C_2e_2$ and $w = C_1^1w^1 + C_2^1w^2 + C_3^1w^3$, where $C_1, C_2, C_1^1, C_2^1, C_3^1 \in \mathbb{K}$.

$\mu_0 \in C^0(V,W)$ implies that $\mu_0 = (w^1, w^2, w^3)$. $\mu_1 \in C^1(V,W)$ iff $\mu_1 : V \to W$ is a linear map.

Then,

$$\begin{aligned}\mu_1(x) &= \mu_1(C_1e_1 + C_2e_2) \\ &= C_1\mu_1(e_1) + C_2\mu_1(e_2), \quad \text{where } \mu_1(e_i) = C_i^1w^1 + C_i^2w^2 + C_i^3w^3, i = 1,2 \\ &= C_1\sum_{i=1}^{3} C_i^1w^i + C_2\sum_{i=1}^{3} C_i^2w^i, \\ &= C^1\left(C_1^1w^1 + C_2^1w^2 + C_3^1w^3\right) + C^2\left(C_1^2w^1 + C_2^2w^2 + C_3^2w^3\right).\end{aligned}$$

Lemma 1: *If the* $dimV = 2$ *and* $dimW = 3$, *then* $C^1(V,W) \cong M_{3\times 2}(\mathbb{K})$.

Proof. Since $\mu_1 \in C^1(V,W)$ then,

$$\mu_1(x) = C^1\left(C_1^1w^1 + C_2^1w^2 + C_3^1w^3\right) + C^2\left(C_1^2w^1 + C_2^2w^2 + C_3^2w^3\right).$$

Thus, we define an isomorphic map i_1 from $C_1(V,W)$ to $M_{3\times 2}(\mathbb{K})$ as follows;

$$i_1(\mu_1(x)) = \begin{pmatrix} C_1^1 & C_1^2 \\ C_2^1 & C_2^2 \\ C_3^1 & C_3^2 \end{pmatrix}.$$ □

$\mu_2 \in C^2(V,W)$ iff $\mu_2 : V \times V \to W$ is bilinear and antisymmetric map; then

$$\begin{aligned}\mu_2(x,y) &= \mu_2(C_{11}e_1 + C_{12}e_2, C_{21}e_1 + C_{22}e_2) \\ &= \mu_2(C_{11}e_1, C_{21}e_1) + \mu_2(C_{11}e_1, C_{22}e_2) + \mu_2(C_{12}e_2, C_{21}e_1) + \mu_2(C_{12}e_2, C_{22}e_2) \\ &= C_{11}C_{21}\mu_2(e_1,e_1) + C_{11}C_{22}\mu_2(e_1,e_2) + C_{12}C_{21}\mu_2(e_2,e_1) + C_{12}C_{22}\mu_2(e_2,e_2) \\ &= (C_{11}C_{22} - C_{12}C_{21})\mu_2(e_1,e_2) \\ &= (C_{11}C_{22} - C_{12}C_{22})\sum_{k=1}^{3} C_{12}^k w_k \\ &= (C_{11}C_{22} - C_{12}C_{22})\left(C_{12}^1\omega^1 + C_{12}^2\omega^2 + C_{12}^3\omega^3\right).\end{aligned}$$

Lemma 2: *If* $dimV = 2$ *and* $dimW = 3$ *then* $C^2(V,W) \cong M_{3\times 1}(\mathbb{K})$.

Proof. From the expression of an element μ_2 in $C^2(V,W)$ from above, μ_2 can be represented as a column matrix $\begin{pmatrix} C_{12}^1 \\ C_{12}^2 \\ C_{12}^3 \end{pmatrix}$ of the lie constant structures. □

$\mu_3 \in C^3(V,W)$ iff $\mu_3 : V \times V \times V \to W$ is a tri-linear and skew symmetric map, then

$$\begin{aligned}
\mu_3(x,y,z) &= \mu_3(C_{11}e_1 + C_{12}e_2, C_{21}e_1 + C_{22}e_2, C_{31}e_1 + C_{32}e_2) \\
&= \mu_3(C_{11}e_1, C_{21}e_1 + C_{22}e_2, C_{31}e_1 + C_{32}e_2) + \mu_3(C_{12}e_2, C_{21}e_1 + C_{22}e_2, C_{31}e_1 + C_{32}e_2) \\
&= \mu_3(C_{11}e_1, C_{21}e_1, C_{31}e_1 + C_{32}e_2) + \mu_3(C_{11}e_1, C_{22}e_2, C_{31}e_1 + C_{32}e_2) \\
&\quad + \mu_3(C_{12}e_2, C_{21}e_1, C_{31}e_1 + C_{32}e_2) + \mu_3(C_{12}e_2, C_{22}e_2, C_{31}e_1 + C_{32}e_2) \\
&= \mu_3(C_{11}e_1, C_{21}e_1, C_{31}e_1) + \mu_3(C_{11}e_1, C_{21}e_1, C_{32}e_2) + \mu_3(C_{11}e_1, C_{22}e_2, C_{31}e_1) \\
&\quad + \mu_3(C_{12}e_2, C_{21}e_1, C_{32}e_2) + \mu_3(C_{12}e_2, C_{20}e_2, C_{11}e_1) + \mu_3(C_{12}e_2, C_{22}e_2, C_{32}e_2) \\
&\quad + \mu_3(C_{12}e_2, C_{21}e_1, C_{31}e_1) + \mu_3(C_{11}e_1, C_{22}e_2, C_{32}e_2) \\
&= C_{11}C_{21}C_{31}\mu_3(e_1,e_1,e_1) + C_{11}C_{21}C_{32}(e_1,e_1,e_2) \\
&\quad + C_{11}C_{22}C_{31}\mu_3(e_1,e_2,e_1) + C_{11}C_{22}C_{32}\mu_3(e_1,e_2,e_2) \\
&= C_{12}C_{21}C_{31}\mu_3(e_2,e_1,e_1) + C_{12}C_{21}C_{32}\mu_3(e_2,e_1,e_2) \\
&\quad + C_{12}C_{12}C_{31}\mu_3(e_2,e_2,e_1) + C_{12}C_{22}C_{32}\mu_3(e_2,e_2,e_2) \\
&= 0
\end{aligned}$$

since μ_3 is a linear anti-symmetric mapping.

Lemma 3: *If* $\dim V = 2$ *and* $\dim W = 3$ *then* $C^3(V,W) \cong 0$.

Proof. Since for every $\mu_3 \in C^3(V,W)$, we have that $\mu_3 \equiv 0$ from the expression of μ_3 above. □

2.2. Diagram of a Sequence of Linear Maps

According to the above results, we have the following diagram where we shall identify and define d^0, d^1, d^2 and i_0, i_1, i_2 in order to contruct our 2 - 3 Matrix Chevalley-Eilenberg Cohomology.

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{W} & \xrightarrow{d^0} & C^1(V,W) & \xrightarrow{d^1} & C^2(V,W) & \xrightarrow{d^2} & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
0 & \to & \mathrm{W} & \xrightarrow{\tilde{d^0}} & M_{3\times 2}(\mathbb{R}) & \xrightarrow{\tilde{d^1}} & M_{3\times 1}(\mathbb{R}) & \xrightarrow{\tilde{d^2}} & 0
\end{array}$$

Expression of d^0 [1] [4]:

$$\begin{aligned}
&d^0 : W \to C^1(V,W) \\
&\quad w \mapsto d^0 w : V \to W \\
&\quad v \mapsto d^0 w(v) = \rho(v) w
\end{aligned}$$

Expression of d^1 [1] [4]:

$$d^1 : C^1(V,W) \to C^2(V,W)$$

$$\begin{aligned}
d^1\alpha(x,y) &= \delta\alpha(x,y), \text{ for all } \alpha \text{ in } C^1(V,W) \\
&= x\alpha(y) - y\alpha(x) - \alpha(\mu(x,y)) \\
&= \rho(x)\alpha(y) - \rho(y)\alpha(x) - \alpha(\mu(x,y))
\end{aligned}$$

Expression of d^2 [1] [4]:

$$d^2 : C(V,W) \to 0$$

$$\begin{aligned}
d^2\alpha(x,y,z) &= \delta\alpha(x,y,z), \text{ for all } \alpha \text{ in } C^2(V,W) \\
&= x\alpha(y,z) - y\alpha(z,x) + z\alpha(x,y) - \alpha(\mu(x,y),z) \\
&\quad + \alpha(\mu(x,z),y) - \alpha(\mu(y,z),x) \\
&\equiv 0,
\end{aligned}$$

since d^2 is mapped to the zero space. A direct computation, give us [1]

$$d^2 \circ d^1 = 0$$

Definition of i_0:

$$i_0 : W \to W$$

$$i_0(w) = w, \forall w \in W$$

i.e i_0 is the identity mappings from *W* to *W*.

Definition of i_1:

$$i_1 : C^1(V,W) \to M_{3\times 2}(\mathbb{R})$$

$$i_1(\mu_1(x)) = \begin{pmatrix} C_1^1 & C_1^2 \\ C_2^1 & C_2^2 \\ C_3^1 & C_3^2 \end{pmatrix},$$

which is the matrix of α_1, $C_j^i \in \mathbb{R}$, $i = 1,2$ and $j = 1,2,3$.

Definition of i_2:

$$i_2 : C^2(V,W) \to M_{3\times 2}(\mathbb{R})$$

$$i_2(\mu_2(e_1,e_2)) = \begin{pmatrix} C_{12}^1 \\ C_{12}^2 \\ C_{12}^3 \end{pmatrix},$$

which is the matrix of α_2, $C_{12}^l \in \mathbb{R}$, $l = 1,2,3$.

2.3. Homological Differential

In this section, we are going to determine expressions of $\tilde{d}^1, \tilde{d}^0$ and also prove that $d^1 o d^0 = 0$, $\tilde{d}^1 o \tilde{d}^0 = 0$ for us to obtain our 2 - 3 matrix Chevalley-Eilenberg differential complex. This is possible unless by stating an important hypothesis which we call 2 - 3 matrix Chevalley-Eilenberg hypothesis.

Proposition 1: *If* $\rho(x)\rho(y) - \rho(y)\rho(x) - \rho(\mu(x,y)) = 0$, *for all x, y in V, then* $d^1 o d^0 = 0$.

Proof. We assume that $\rho(x)\rho(y) - \rho(y)\rho(x) - \rho(\mu(x,y)) = 0$, for all *x*, *y* in *V*.

By definition, we have that

$$d^0 w(x) = \rho(x) w \tag{1}$$

$$d^1 \alpha(x,y) = \rho(x)\alpha(y) - \rho(y)\alpha(x) - \alpha(\mu(x,y)). \tag{2}$$

Then by substituting equation (1) into (2),we have

$$\begin{aligned} d^1 d^0 w(x,y) &= \rho(x) d^0 w(y) - \rho(y) d^0 w(x) - d^0 w(\mu(x,y)) \\ &= \rho(x)\rho(y) w - \rho(y)\rho(x) w - \rho(\mu(x,y)) w \\ &= \left[\rho(x)\rho(y) - \rho(y)\rho(x) - \rho(\mu(x,y))\right] w \\ &= 0 \end{aligned}$$

by hypothesis.

Expression of $\tilde{d}^1$:

Let *V* be a two dimensional Lie-algebra with basis (e_1, e_2) and the Lie's bracket $\mu(x,y)$ where $\mu(e_1,e_2) = C_{12}^1 e_1 + C_{12}^2 e_2$ and *W* a three dimensional vector space with basis (w^1, w^2, w^3). We define $\rho : V \to End(W)$ by

$v \mapsto \rho(v) := f_{M_v}$, where $M_v = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$ and f_{M_v} is a linear mapping associated to the matrix M_v.

Let $M=\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \in M_{3\times 2} \mapsto f_M : V \to W$ defined by

$$f_M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1x + a_2y \\ b_1x + b_2y \\ c_1x + c_2y \end{pmatrix} := M\begin{pmatrix} x \\ y \end{pmatrix}.$$

Therefore;

$$d^1 f_M(\boldsymbol{v}_1, \boldsymbol{v}_2) = -\boldsymbol{v}_1 \cdot f_M(\boldsymbol{v}_2) + \boldsymbol{v}_2 \cdot f_M(\boldsymbol{v}_1) - f_M(\mu(\boldsymbol{v}_1, \boldsymbol{v}_2)).$$

Since

$$\begin{aligned}
\mu(\boldsymbol{v}_1, \boldsymbol{v}_2) &= \mu\left(v_1^1 e_1 + v_2^1 e_2, v_1^2 e_1 + v_2^2 e_2\right) \\
&= \mu\left(v_1^1 e_1, v_1^2 e_2\right) + \mu\left(v_1^1 e_1, v_2^2 e_2\right) + \mu\left(v_2^1 e_2, v_1^2 e_1\right) + \mu\left(v_2^1 e_2, v_2^2 e_2\right) \\
&= \left(v_1^1 v_2^2 - v_2^1 v_1^2\right)\mu(e_1, e_2) = \left(v_1^1 v_2^2 - v_2^1 v_1^2\right)\left(C_{12}^1 e_1 + C_{12}^2 e_2\right) \\
&= C_{12}^1\left(v_1^1 v_2^2 - v_2^1 v_1^2\right)e_1 + C_{12}^2\left(v_1^1 v_2^2 - v_2^1 v_1^2\right)e_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_M(\mu(\boldsymbol{v}_1, \boldsymbol{v}_2)) &= f_M\left[\left(C_{12}^1\left(v_1^1 v_2^2 - v_2^1 v_1^2\right)e_1 + C_{12}^2\left(v_1^1 v_2^2 - v_2^1 v_1^2\right)e_2\right)\right] \\
&= \left(a_1 C_{12}^1\left(v_1^1 v_2^2 - v_2^1 v_1^2\right) + a_2 C_{12}^2\left(v_1^1 v_2^2 - v_2^1 v_1^2\right)\right)\boldsymbol{w}_1 \\
&\quad + \left(b_1 C_{12}^1\left(v_1^1 v_2^2 - v_2^1 v_1^2\right) + b_2 C_{12}^2\left(v_1^1 v_2^2 - v_2^1 v_1^2\right)\right)\boldsymbol{w}_2 \\
&\quad + \left(c_1 C_{12}^1\left(v_1^1 v_2^2 - v_2^1 v_1^2\right) + c_2 C_{12}^2\left(v_1^1 v_2^2 - v_2^1 v_1^2\right)\right)\boldsymbol{w}_3.
\end{aligned}$$

Also, we have

$$f_M(\boldsymbol{v}_1) = \begin{pmatrix} a_1 v_1^1 + a_2 v_2^1 \\ b_1 v_1^1 + b_2 v_2^1 \\ c_1 v_1^1 + c_2 v_2^1 \end{pmatrix}, \quad f_M(\boldsymbol{v}_2) = \begin{pmatrix} a_1 v_1^2 + a_2 v_2^2 \\ b_1 v_1^2 + b_2 v_2^2 \\ c_1 v_1^2 + c_2 v_2^2 \end{pmatrix}.$$

$\rho(\boldsymbol{v}_2) \cdot f_M(\boldsymbol{v}_1) = M_{\boldsymbol{v}_2} \cdot f_M(\boldsymbol{v}_1)$, where

$$M_{\boldsymbol{v}_2} = \begin{pmatrix} v_{11}^2 & v_{12}^2 & v_{13}^2 \\ v_{21}^2 & v_{22}^2 & v_{23}^2 \\ v_{31}^2 & v_{32}^2 & v_{33}^2 \end{pmatrix}$$

$\rho(\boldsymbol{v}_1) \cdot f_M(\boldsymbol{v}_2)$, where

$$M_{\boldsymbol{v}_1} = \begin{pmatrix} v_{11}^1 & v_{12}^1 & v_{13}^1 \\ v_{21}^1 & v_{22}^1 & v_{23}^1 \\ v_{31}^1 & v_{32}^1 & v_{33}^1 \end{pmatrix}.$$

So,

$$\begin{aligned}
M_{\boldsymbol{v}_2} \cdot f_M(\boldsymbol{v}_1) &= \left(\left(a_1 v_1^1 + a_2 v_2^1\right)v_{11}^2 + \left(b_1 v_1^1 + b_2 v_2^1\right)v_{21}^2 + \left(c_1 v_1^1 + c_2 v_2^1\right)v_{31}^2\right)\boldsymbol{w}_1 \\
&\quad + \left(\left(a_1 v_1^1 + a_2 v_2^1\right)v_{12}^2 + \left(b_1 v_1^1 + b_2 v_2^1\right)v_{22}^2 + \left(c_1 v_1^1 + c_2 v_2^1\right)v_{32}^2\right)\boldsymbol{w}_2 \\
&\quad + \left(\left(a_1 v_1^1 + a_2 v_2^1\right)v_{13}^2 + \left(b_1 v_1^1 + b_2 v_2^1\right)v_{23}^2 + \left(c_1 v_1^1 + c_2 v_2^1\right)v_{33}^2\right)\boldsymbol{w}_3.
\end{aligned}$$

$$
\begin{aligned}
M_{v_1}\cdot f_M\left(\boldsymbol{v}_2\right)=&\Big(\left(a_1v_1^2+a_2v_2^2\right)v_{11}^1+\left(b_1v_1^2+b_2v_2^2\right)v_{21}^1+\left(c_1v_1^2+c_2v_2^2\right)v_{31}^1\Big)\boldsymbol{w}_1\\
&+\Big(\left(a_1v_1^2+a_2v_2^2\right)v_{12}^1+\left(b_1v_1^2+b_2v_2^2\right)v_{22}^1+\left(c_1v_1^2+c_2v_2^2\right)v_{32}^1\Big)\boldsymbol{w}_2\\
&+\Big(\left(a_1v_1^2+a_2v_2^2\right)v_{13}^1+\left(b_1v_1^2+b_2v_2^2\right)v_{23}^1+\left(c_1v_1^2+c_2v_2^2\right)v_{33}^1\Big)\boldsymbol{w}_3.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&d^1 f_M\left(\boldsymbol{v}_1,\boldsymbol{v}_2\right)\\
&=-M_{v_1}\cdot f_M\left(\boldsymbol{v}_2\right)+M_{v_2}\cdot f_M\left(\boldsymbol{v}_1\right)-f_M\left(\mu\left(\boldsymbol{v}_1,\boldsymbol{v}_2\right)\right)\\
&=\Big[\left(a_1v_1^1+a_2v_2^1\right)v_{11}^2-\left(a_1v_1^2+a_2v_2^2\right)v_{11}^1+\left(b_1v_1^1+b_2v_2^1\right)v_{21}^2-\left(b_1v_1^2+b_2v_2^2\right)v_{21}^1\\
&\quad+\left(c_1v_1^1+c_2v_2^1\right)v_{31}^2-\left(c_1v_1^2+c_2v_2^2\right)v_{31}^1-a_1C_{12}^1\left(v_1^1v_2^2-v_2^1v_1^2\right)-a_2C_{12}^2\left(v_1^1v_2^2-v_2^1v_1^2\right)\Big]\boldsymbol{w}_1\\
&\quad+\Big[\left(a_1v_1^1+a_2v_2^1\right)-\left(a_1v_1^2+a_2v_2^2\right)v_{12}^1+\left(b_1v_1^1+b_2v_2^1\right)v_{22}^2-\left(b_1v_1^2+b_2v_2^2\right)v_{22}^1\\
&\quad-\left(c_1v_1^1+c_1v_2^1\right)v_{32}^2-\left(c_1v_1^2+c_2v_2^2\right)v_{32}^1-b_1C_{12}^1\left(v_1^1v_2^2-v_2^1v_1^2\right)-b_2C_{12}^1\left(v_1^1v_2^2-v_2^1v_1^2\right)\Big]\boldsymbol{w}_2\\
&\quad+\Big[\left(a_1v_1^1+a_2v_2^1\right)v_{13}^2-\left(a_1v_1^2+a_2v_2^2\right)v_{13}^1+\left(b_1v_1^1+b_2v_2^1\right)v_{23}^2-\left(b_1v_1^2+b_2v_2^2\right)v_{23}^1\\
&\quad+\left(c_1v_1^1+c_2v_2^1\right)v_{33}^2-\left(c_1v_1^2+c_2v_2^2\right)v_{33}^1-c_1C_{12}^1\left(v_1^1v_2^2-v_2^1v_1^2\right)-c_2C_{12}^2\left(v_1^1v_2^2-v_2^1v_1^2\right)\Big]\boldsymbol{w}_3.
\end{aligned}
$$

Now, we compute $d^1 f_M\left(e_1,e_2\right)$ where e_1 and e_2 are basis vectors of V.
By replacing the constants $v_1^1=1, v_2^1=0, v_1^2=0$ and $v_2^2=1$, we obtain $d^1 f_M\left(e_1,e_2\right)$ which is given as;

$$
\begin{aligned}
d^1 f_M\left(e_1,e_2\right)=&\left(a_1e_{11}^2-a_2e_{11}^1+b_1e_{21}^2-b_2e_{21}^1+c_1e_{31}^2-c_2e_{31}^1-a_1C_{12}^1-a_2C_{12}^2\right)\boldsymbol{w}_1\\
&+\left(a_1e_{12}^2-a_2e_{12}^1+b_1e_{22}^2-b_2e_{22}^1+c_1e_{32}^2-c_2e_{32}^1-b_1C_{12}^1-b_2C_{12}^1\right)\boldsymbol{w}_2\\
&+\left(a_1e_{13}^2-a_2e_{13}^1+b_1e_{23}^2-b_2e_{23}^1+c_1e_{33}^2-c_2e_{33}^1-c_1C_{12}^1-c_2C_{12}^2\right)\boldsymbol{w}_3.
\end{aligned}
$$

Thus,

$$
\tilde{d}^1 f_M\left(e_1,e_2\right)=\begin{pmatrix}
a_1e_{11}^2-a_2e_{11}^1+b_1e_{21}^2-b_2e_{21}^1+c_1e_{31}^2-c_2e_{31}^1-a_1C_{12}^1-a_2C_{12}^2\\
a_1e_{12}^2-a_2e_{12}^1+b_1e_{22}^2-b_2e_{22}^1+c_1e_{32}^2-c_2e_{32}^1-b_1C_{12}^1-b_2C_{12}^2\\
a_1e_{13}^2-a_2e_{13}^1+b_1e_{23}^2-b_2e_{23}^1+c_1e_{33}^2-c_2e_{33}^1-c_1C_{12}^1-c_2C_{12}^2
\end{pmatrix}.
$$

Hence, $\tilde{d}^1: M_{3\times 2}\to M_{3\times 1}$ is defined by

$$
\begin{pmatrix} a_1 & a_2\\ b_1 & b_2\\ c_1 & c_2\end{pmatrix}\mapsto\begin{pmatrix}
a_1e_{11}^2-a_2e_{11}^1+b_1e_{21}^2-b_2e_{21}^1+c_1e_{31}^2-c_2e_{31}^1-a_1C_{12}^1-a_2C_{12}^2\\
a_1e_{12}^2-a_2e_{12}^1+b_1e_{22}^2-b_2e_{22}^1+c_1e_{32}^2-c_2e_{32}^1-b_1C_{12}^1-b_2C_{12}^2\\
a_1e_{13}^2-a_2e_{13}^1+b_1e_{23}^2-b_2e_{23}^1+c_1e_{33}^2-c_2e_{33}^1-c_1C_{12}^1-c_2C_{12}^2
\end{pmatrix}\quad\text{with}
$$

$$
M_{e_1}=\begin{pmatrix} e_{11}^1 & e_{12}^1 & e_{13}^1\\ e_{21}^1 & e_{22}^1 & e_{23}^1\\ e_{31}^1 & e_{32}^1 & e_{33}^1\end{pmatrix},\quad M_{e_2}=\begin{pmatrix} e_{11}^2 & e_{12}^2 & e_{13}^2\\ e_{21}^2 & e_{22}^2 & e_{23}^2\\ e_{31}^2 & e_{32}^2 & e_{33}^2\end{pmatrix}.
$$

Corollary 1: *If*

$$
\begin{aligned}
&d^1 f_M\left(e_1,e_2\right)\\
&=\left(a_1e_{11}^2-a_2e_{11}^1+b_1e_{21}^2-b_2e_{21}^1+c_1e_{31}^2-c_2e_{31}^1-a_1C_{12}^1-a_2C_{12}^2\right)\boldsymbol{w}_1\\
&\quad+\left(a_1e_{12}^2-a_2e_{12}^1+b_1e_{22}^2-b_2e_{22}^1+c_1e_{32}^2-c_2e_{32}^1-b_1C_{12}^1-b_2C_{12}^1\right)\boldsymbol{w}_2\\
&\quad+\left(a_1e_{13}^2-a_2e_{13}^1+b_1e_{23}^2-b_2e_{23}^1+c_1e_{33}^2-c_2e_{33}^1-c_1C_{12}^1-c_2C_{12}^2\right)\boldsymbol{w}_3
\end{aligned}
$$

then $\tilde{d}^1: M_{3\times 2}\to M_{3\times 1}$ is defined by

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1e_{11}^2 - a_2e_{11}^1 + b_1e_{21}^2 - b_2e_{21}^1 + c_1e_{31}^2 - c_2e_{31}^1 - a_1C_{12}^1 - a_2C_{12}^2 \\ a_1e_{12}^2 - a_2e_{12}^1 + b_1e_{22}^2 - b_2e_{22}^1 + c_1e_{32}^2 - c_2e_{32}^1 - b_1C_{12}^1 - b_2C_{12}^2 \\ a_1e_{13}^2 - a_2e_{13}^1 + b_1e_{23}^2 - b_2e_{23}^1 + c_1e_{33}^2 - c_2e_{33}^1 - c_1C_{12}^1 - c_2C_{12}^2 \end{pmatrix}.$$

2.4. Fundametal Condition of 2 - 3 Matrix Chevalley-Eilenberg Cohomology

We now state the main hypothesis for our 2 - 3 matrix Chevalley-Eilenberg Cohomology, which we suppose that $\rho(e_1)\rho((e_2) - \rho(e_2)\rho(e_1) - \rho(\mu(e_1,e_2)) = 0$.

i.e $M_{e_1}M_{e_2} - M_{e_2}M_{e_1} - C^1M_{e_1} - C_{12}^2M_{e_2} = 0$

$$\begin{aligned} i.e \quad & \Big[\left(e_{11}^1e_{11}^2 + e_{12}^1e_{21}^2 + e_{13}^1e_{31}^2\right) - \left(e_{11}^2e_{11}^1 + e_{12}^2e_{21}^1 + e_{13}^2e_{31}^1\right) + \left(e_{11}^1e_{12}^2 + e_{12}^1e_{22}^2 + e_{13}^1e_{32}^2\right) \\ & - \left(e_{11}^2e_{12}^1 + e_{12}^2e_{22}^1 + e_{13}^2e_{32}^1\right) + \left(e_{11}^1e_{13}^2 + e_{12}^1e_{23}^2 + e_{13}^1e_{33}^2\right) - \left(e_{11}^2e_{13}^1 + e_{12}^2e_{23}^1 + e_{13}^2e_{33}^1\right), \\ & - C_{12}^1e_{11}^1 - C_{12}^2e_{11}^2 - C_{12}^1e_{12}^1 - C_{12}^2e_{12}^2 - C_{12}^1e_{13}^1 - C_{12}^2e_{13}^2\Big] = 0 \end{aligned}$$

$$\begin{aligned} & \Big[\left(e_{21}^1e_{11}^2 + e_{22}^1e_{21}^2 + e_{23}^1e_{31}^2\right) - \left(e_{21}^2e_{11}^1 + e_{22}^2e_{21}^1 + e_{23}^2e_{31}^1\right) + \left(e_{21}^1e_{12}^2 + e_{22}^1e_{22}^2 + e_{23}^1e_{32}^2\right) \\ & - \left(e_{21}^2e_{12}^1 + e_{22}^2e_{22}^1 + e_{23}^2e_{32}^1\right) + \left(e_{21}^1e_{13}^2 + e_{22}^1e_{23}^2 + e_{23}^1e_{33}^2\right) - \left(e_{21}^2e_{13}^1 + e_{22}^2e_{23}^1 + e_{23}^2e_{33}^1\right) \quad \text{and} \\ & - C_{12}^1e_{21}^1 - C_{12}^2e_{21}^2 - C_{12}^1e_{22}^1 - C_{12}^2e_{22}^2 - C_{12}^1e_{23}^1 - C_{12}^2e_{23}^2\Big] = 0, \end{aligned}$$

$$\begin{aligned} & \Big[\left(e_{31}^1e_{11}^2 + e_{32}^1e_{21}^2 + e_{33}^1e_{31}^2\right) - \left(e_{31}^2e_{11}^1 + e_{32}^2e_{21}^1 + e_{33}^2e_{31}^1\right) + \left(e_{31}^1e_{12}^2 + e_{32}^1e_{22}^2 + e_{33}^1e_{32}^2\right) \\ & - \left(e_{31}^2e_{12}^1 + e_{32}^2e_{22}^1 + e_{33}^2e_{32}^1\right) + \left(e_{31}^1e_{13}^2 + e_{32}^1e_{23}^2 + e_{33}^1e_{33}^2\right) - \left(e_{31}^2e_{13}^1 + e_{32}^2e_{23}^1 + e_{33}^2e_{33}^1\right). \\ & - C_{12}^1e_{31}^1 - C_{12}^2e_{31}^2 - C_{12}^1e_{32}^1 - C_{12}^2e_{32}^2 - C_{12}^1e_{33}^1 - C_{12}^2e_{33}^2\Big] = 0 \end{aligned}$$

This is an important tool in the construction of our 2 - 3 matrix cohomology differential complex.

2.5. Expression of $\tilde{d}^0$

From the diagram,

$$\tilde{d}^0 = i_1 \circ d^0 \circ i_0^{-1}$$

$$i_0 : W \to W$$

$$w \mapsto i(w) = 1_w(w) = w \quad \forall w \in W.$$

$$d^0 : W \to C^1(V,W)$$

$$w \mapsto \rho(w)v = f_{M_v} := \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$$

where

$$\begin{aligned} d^0M(v) &= \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} \\ &= \begin{pmatrix} v_{11}w^1 + v_{12}w^2 + v_{13}w^3 \\ v_{21}w^1 + v_{22}w^2 + v_{23}w^3 \\ v_{31}w^1 + v_{32}w^2 + v_{33}w^3 \end{pmatrix} \end{aligned}$$

$$\left[\left(\tilde{d}^0(M)(e_1) \quad \tilde{d}^0(M)(e_2)\right)\right]=\left(\begin{pmatrix} e_{11}^1 & e_{12}^1 & e_{13}^1 \\ e_{21}^1 & e_{22}^1 & e_{23}^1 \\ e_{31}^1 & e_{32}^1 & e_{33}^1 \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} \quad \begin{pmatrix} e_{11}^2 & e_{12}^2 & e_{13}^2 \\ e_{21}^2 & e_{22}^2 & e_{23}^2 \\ e_{31}^2 & e_{32}^2 & e_{33}^2 \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}\right)$$

$$=\begin{pmatrix} e_{11}^1 w^1+e_{12}^1 w^2+e_{13}^1 w^3 & e_{11}^2 w^1+e_{12}^2 w^2+e_{13}^2 w^3 \\ e_{21}^1 w^1+e_{22}^1 w^2+e_{23}^1 w^3 & e_{21}^2 w^1+e_{22}^2 w^2+e_{23}^2 w^3 \\ e_{31}^1 w^1+e_{32}^1 w^2+e_{33}^1 w^3 & e_{31}^2 w^1+e_{32}^2 w^2+e_{33}^2 w^3 \end{pmatrix}$$

$\boldsymbol{v}=\begin{pmatrix} x \\ y \end{pmatrix}$ and $W=\langle w_1, w_2, w_3\rangle$. Thus, using the basis vectors e_1 and e_2 in V, we have

$$\tilde{d}^0(e_1,e_2)=\begin{pmatrix} e_{11}^1 w^1+e_{12}^1 w^2+e_{13}^1 w^3 & e_{11}^2 w_1+e_{12}^2 w^2+e_{13}^2 w^3 \\ e_{21}^1 w^1+e_{22}^1 w^2+e_{23}^1 w^3 & e_{21}^2 w^1+e_{22}^2 w^2+e_{23}^2 w^3 \\ e_{31}^1 w^1+e_{32}^1 w^2+e_{33}^1 w^3 & e_{31}^2 w^1+e_{32}^2 w^2+e_{33}^2 w^3 \end{pmatrix}=\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \\ \gamma_5 & \gamma_6 \end{pmatrix}.$$

Hence, the mapping $\tilde{d}^0: W \to M_{3\times 2}$ is defined as;

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \mapsto \begin{pmatrix} e_{11}^1 w^1+e_{12}^1 w^2+e_{13}^1 w^3 & e_{11}^2 w^1+e_{12}^2 w^2+e_{13}^2 w^3 \\ e_{21}^1 w^1+e_{22}^1 w^2+e_{23}^1 w^3 & e_{21}^2 w^1+e_{22}^2 w^2+e_{23}^2 w^3 \\ e_{31}^1 w^1+e_{32}^1 w^2+e_{33}^1 w^3 & e_{31}^2 w^1+e_{32}^2 w^2+e_{33}^2 w^3 \end{pmatrix}$$

Corollary 2: *If* $\tilde{d}^0(e_1,e_2)=\begin{pmatrix} e_{11}^1 w^1+e_{12}^1 w^2+e_{13}^1 w^3 & e_{11}^2 w_1+e_{12}^2 w^2+e_{13}^2 w^3 \\ e_{21}^1 w^1+e_{22}^1 w^2+e_{23}^1 w^3 & e_{21}^2 w^1+e_{22}^2 w^2+e_{23}^2 w^3 \\ e_{31}^1 w^1+e_{32}^1 w^2+e_{33}^1 w^3 & e_{31}^2 w^1+e_{32}^2 w^2+e_{33}^2 w^3 \end{pmatrix}$

then the mapping $\tilde{d}^0: W \to M_{3\times 2}$ is defined as;

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \mapsto \begin{pmatrix} e_{11}^1 w^1+e_{12}^1 w^2+e_{13}^1 w^3 & e_{11}^2 w^1+e_{12}^2 w^2+e_{13}^2 w^3 \\ e_{21}^1 w^1+e_{22}^1 w^2+e_{23}^1 w^3 & e_{21}^2 w^1+e_{22}^2 w^2+e_{23}^2 w^3 \\ e_{31}^1 w^1+e_{32}^1 w^2+e_{33}^1 w^3 & e_{31}^2 w^1+e_{32}^2 w^2+e_{33}^2 w^3 \end{pmatrix}$$

The matrix $\tilde{d}^0(e_1,e_2)$ has been assigned to the matrix $\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \\ \gamma_5 & \gamma_6 \end{pmatrix}$ to simplify the composition of $\tilde{d}^1$ and $\tilde{d}^2$.

Proposition 2: $\tilde{d}^1 0 \tilde{d}^0 = 0$.

Proof. Since $d^2 \circ d^1 = 0, \tilde{d}^1 = i_1 o d^1 o i_1^{-1}$ and $\tilde{d}^0 = i_1 o d^0 o i_0^{-1}$, We have:

$$\tilde{d}^1 o \tilde{d}^0 = \left(i_1 o d^1 o i_1^{-1}\right) o \left(i_1 o d^0 o i_0^{-1}\right) = i_2 o d^1 o \left(i_1^{-1} o i_1\right) o d^0 o i_0^{-1} = i_2 o d^1 o d^0 o i_0^{-1} = i_2 0 i_0^{-1} = 0,$$

Which gives us our 2 - 3 matrix Chevalley Eilenberg homological hypothesis

$$\left(M_{e_1} M_{e_2} - M_{e_2} M_{e_1} - C_{12}^1 M_{e_1} - C_{12}^2 M_{e_2} = 0\right). \qquad \square$$

Remark 1: *By straightforward computation, we have*

$$\tilde{d}^1 o \tilde{d}^0(M)=\begin{pmatrix} \gamma_1 e_{11}^2 w^1-\gamma_2 e_{11}^1 w^1+\gamma_3 e_{21}^2 w^1-\gamma_4 e_{21}^1 w^1+\gamma_5 e_{31}^2 w^1-\gamma_6 e_{31}^1 w^1-\gamma_1 C_{12}^1 w^1-\gamma_2 C_{12}^2 w^1 \\ \gamma_1 e_{12}^2 w^2-\gamma_2 e_{12}^1 w^2+\gamma_3 e_{22}^2 w^2-\gamma_4 e_{22}^1 w^2+\gamma_5 e_{32}^2 w^2-\gamma_6 e_{32}^1 w^2-\gamma_3 C_{12}^1 w^2-\gamma_4 C_{12}^2 w^2 \\ \gamma_1 e_{13}^2 w^3-\gamma_2 e_{13}^1 w^3+\gamma_3 e_{23}^2 w^3-\gamma_4 e_{23}^1 w^3+\gamma_5 e_{33}^2 w^3-\gamma_6 e_{33}^1 w^3-\gamma_5 C_{12}^1 w^3-\gamma_6 C_{12}^2 w^3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2.6. Determination of the $ker\tilde{d}^1$ and $Im\tilde{d}^1$

$$ker\tilde{d}^1 = \left\{A \in M_{3\times 2}(\mathbb{R}) \middle| \tilde{d}^1 A = 0\right\}.$$

$\tilde{d}^1 A = 0$ iff

$$\begin{pmatrix} a_1e_{11}^2 - a_2e_{11}^1 + b_1e_{21}^2 - b_2e_{21}^1 + c_1e_{31}^2 - c_2e_{31}^1 - a_1C_{12}^1 - a_2C_{12}^2 \\ a_1e_{12}^2 - a_2e_{12}^1 + b_1e_{22}^2 - b_2e_{22}^1 + c_1e_{32}^2 - c_2e_{32}^1 - b_1C_{12}^1 - b_2C_{12}^2 \\ a_1e_{13}^2 - a_2e_{12}^1 + b_1e_{32}^2 - b_2e_{32}^1 + c_1e_{33}^2 - c_2e_{33}^1 - c_1C_{12}^1 - c_2C_{12}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

iff

$$a_1e_{11}^2 - a_2e_{11}^1 + b_1e_{21}^2 - b_2e_{21}^1 + c_1e_{31}^2 - c_2e_{31}^1 - a_1C_{12}^1 - a_2C_{12}^2 = 0 \tag{3}$$

$$a_1e_{12}^2 - a_2e_{12}^1 + b_1e_{22}^2 - b_2e_{22}^1 + c_1e_{32}^2 - c_2e_{32}^1 - b_1C_{12}^1 - b_2C_{12}^2 = 0 \tag{4}$$

$$a_1e_{13}^2 - a_2e_{12}^1 + b_1e_{32}^2 - b_2e_{32}^1 + c_1e_{33}^2 - c_2e_{33}^1 - c_1C_{12}^1 - c_2C_{12}^2 = 0. \tag{5}$$

Now, we compute the $Im\tilde{d}^1$ using the standard basis

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ then $\tilde{d}^1 A_1 = \begin{pmatrix} e_{11}^2 - C_{12}^2 \\ e_{12}^2 \\ e_{13}^2 \end{pmatrix}$.

If $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ then $\tilde{d}^1 A_2 = \begin{pmatrix} -e_{11}^1 - C_{12}^1 \\ -e_{12}^1 \\ -e_{13}^1 \end{pmatrix}$.

If $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $\tilde{d}^1 A_3 = \begin{pmatrix} e_{11}^2 \\ e_{22}^2 - C_{12}^2 \\ e_{23}^2 \end{pmatrix}$.

If $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $\tilde{d}^1 A_4 = \begin{pmatrix} -e_{21}^1 \\ -e_{22}^1 - C_{12}^1 \\ -e_{23}^1 \end{pmatrix}$.

If $A_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$ then $\tilde{d}^1 A_5 = \begin{pmatrix} e_{31}^2 \\ e_{32}^2 \\ e_{33}^2 - C_{12}^2 \end{pmatrix}$.

If $A_6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $\tilde{d}^1 A_6 = \begin{pmatrix} -e_{31}^1 \\ -e_{32}^1 \\ -e_{33}^1 - C_{12}^1 \end{pmatrix}$.

Thus, we have the image matrix as follows:

$$\tilde{M}_1 = \begin{pmatrix} e_{11}^2 - C_{12}^2 & -e_{11}^1 - C_{12}^1 & e_{21}^2 & -e_{21}^1 & e_{31}^2 & -e_{31}^1 \\ e_{12}^2 & -e_{12}^1 & e_{22}^2 - C_{12}^2 & -e_{22}^1 - C_{12}^1 & e_{32}^2 & -e_{32}^1 \\ e_{13}^2 & -e_{13}^1 & e_{23}^2 & -e_{23}^1 & e_{33}^2 - C_{12}^2 & -e_{33}^1 - C_{12}^1 \end{pmatrix}.$$

Next, we calculate the rank of the matrix $\tilde{M}_1$ which will help us to know the $ker\tilde{d}^1$ and $Im\tilde{d}^1$ by using the dimension rank theorem of the vector spaces [5] [6].

We now reduce the matrix $\tilde{M}_1$ to reduce row echelon form. We then replace the entries of the matrix $\tilde{M}_1$ by the follows constants:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{pmatrix}$$

where $\begin{cases} a_{11} = e_{11}^2 - C_{12}^2; \\ a_{12} = -e_{11}^1 - C_{12}^1; \\ a_{13} = e_{21}^2; \\ a_{14} = -e_{21}^1; \\ a_{15} = -e_{31}^2; \\ a_{16} = -e_{31}^1. \end{cases}$ $\begin{cases} a_{21} = e_{12}^2; \\ a_{22} = -e_{12}^1; \\ a_{23} = e_{22}^2 - C_{12}^2; \\ a_{24} = -e_{22}^1 - C_{12}^1; \\ a_{25} = e_{32}^2; \\ a_{26} = -e_{31}^1. \end{cases}$ $\begin{cases} a_{31} = e_{13}^2; \\ a_{31} = -e_{13}^1; \\ a_{31} = e_{23}^2; \\ a_{31} = -e_{23}^1; \\ a_{31} = e_{33}^2 - C_{12}^2; \\ a_{31} = e_{12}^2 - C_{12}^2. \end{cases}$ Let $e_{11}^2 - C_{12}^2 \neq 0$ and by dividing each of the entries of row 1 by $e_{11}^2 - C_{12}^2$ and carrying out the following row operation $e_{12}^2 R_1 - R_2$ and $e_{13}^2 R_1 - R_3$, we obtain

$$A = \begin{pmatrix} 1 & \dfrac{a_{12}}{a_{11}} & \dfrac{a_{13}}{a_{11}} & \dfrac{a_{14}}{a_{11}} & \dfrac{a_{15}}{a_{11}} & \dfrac{a_{16}}{a_{11}} \\ 0 & a_{21}\left(\dfrac{a_{12}}{a_{11}}\right) - a_{22} & a_{21}\left(\dfrac{a_{13}}{a_{11}}\right) - a_{23} & a_{21}\left(\dfrac{a_{14}}{a_{11}}\right) - a_{24} & a_{21}\left(\dfrac{a_{15}}{a_{11}}\right) - a_{25} & a_{21}\left(\dfrac{a_{16}}{a_{11}}\right) - a_{26} \\ 0 & a_{31}\left(\dfrac{a_{12}}{a_{11}}\right) - a_{32} & a_{31}\left(\dfrac{a_{13}}{a_{11}}\right) - a_{33} & a_{31}\left(\dfrac{a_{14}}{a_{11}}\right) - a_{34} & a_{31}\left(\dfrac{a_{15}}{a_{11}}\right) - a_{35} & a_{31}\left(\dfrac{a_{16}}{a_{11}}\right) - a_{36} \end{pmatrix}$$

Let $\alpha = e_{12}^2\left(\dfrac{-e_{11}^1 - C_{12}^1}{e_{11}^2 - C_{12}^2}\right) - \left(e_{22}^2 - C_{12}^2\right)$ be such that $\alpha \neq 0$ and by carrying the following row operations $\dfrac{1}{\alpha}R_2$, $\dfrac{-e_{11}^1 - C_{12}^1}{e_{11}^2 - C_{12}^2}R_2 - R_1$ and $\left(e_{13}^2\left(\dfrac{\left(-e_{11}^1 - C_{12}^1\right)}{\left(e_{11}^2 - C_{12}^2\right)}\right) - e_{23}^2\right)R_2 - R_3$, and setting $\left(e_{13}^2\left(\dfrac{\left(-e_{11}^1 - C_{12}^1\right)}{\left(e_{11}^2 - C_{12}^2\right)}\right) - \left(-e_{13}^1\right)\right) = t$, thus we obtain the following matrix.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{1}{\alpha}\dfrac{a_{12}}{a_{11}}\left(a_{21}\left(\dfrac{a_{13}}{a_{11}}\right) - a_{23}\right) - \dfrac{a_{13}}{a_{11}} & \dfrac{1}{\alpha}\left(a_{21}\left(\dfrac{a_{13}}{a_{11}}\right) - a_{23}\right) & \dfrac{1}{\alpha}t\left(a_{21}\left(\dfrac{a_{13}}{a_{11}}\right) - a_{23}\right) - \left(a_{31}\left(\dfrac{a_{13}}{a_{11}}\right) - a_{33}\right) \\ \dfrac{1}{\alpha}\dfrac{a_{12}}{a_{11}}\left(a_{21}\left(\dfrac{a_{14}}{a_{11}}\right) - a_{24}\right) - \dfrac{a_{14}}{a_{11}} & \dfrac{1}{\alpha}\left(a_{21}\left(\dfrac{a_{14}}{a_{11}}\right) - a_{24}\right) & \dfrac{1}{\alpha}t\left(a_{21}\left(\dfrac{a_{14}}{a_{11}}\right) - a_{24}\right) - \left(a_{31}\left(\dfrac{a_{14}}{a_{11}}\right) - a_{34}\right) \\ \dfrac{1}{\alpha}\dfrac{a_{12}}{a_{11}}\left(a_{21}\left(\dfrac{a_{15}}{a_{11}}\right) - a_{25}\right) - \dfrac{a_{15}}{a_{11}} & \dfrac{1}{\alpha}\left(a_{21}\left(\dfrac{a_{15}}{a_{11}}\right) - a_{25}\right) & \dfrac{1}{\alpha}t\left(a_{21}\left(\dfrac{a_{15}}{a_{11}}\right) - a_{25}\right) - \left(a_{31}\left(\dfrac{a_{15}}{a_{11}}\right) - a_{35}\right) \\ \dfrac{1}{\alpha}\dfrac{a_{12}}{a_{11}}\left(a_{21}\left(\dfrac{a_{16}}{a_{11}}\right) - a_{26}\right) - \dfrac{a_{16}}{a_{11}} & \dfrac{1}{\alpha}\left(a_{21}\left(\dfrac{a_{16}}{a_{11}}\right) - a_{26}\right) & \dfrac{1}{\alpha}t\left(a_{21}\left(\dfrac{a_{16}}{a_{11}}\right) - a_{26}\right) - \left(a_{31}\left(\dfrac{a_{16}}{a_{11}}\right) - a_{36}\right) \end{pmatrix}$$

Let $\beta = \dfrac{1}{\alpha}t\left(e_{12}^2\left(\dfrac{e_{11}^2}{\left(e_{11}^2 - C_{12}^2\right)}\right) - \left(e_{22}^2 - C_{12}^2\right)\right) - \left(e_{13}^2\left(\dfrac{e_{11}^2}{\left(e_{11}^2 - C_{12}^2\right)}\right) - \left(e_{23}^2\right)\right)$ be such that $\beta \neq 0$, and by carry the following row operations

$\dfrac{1}{\beta}R_3$, $\dfrac{1}{\beta}\left[\dfrac{1}{\alpha}\dfrac{a_{12}}{a_{11}}\left(a_{21}\left(\dfrac{a_{13}}{a_{11}}\right) - a_{23}\right) - \dfrac{a_{13}}{a_{11}}\right]R_3 - R_1$ and $\dfrac{1}{\beta}\dfrac{1}{\alpha}\left(a_{21}\left(\dfrac{a_{13}}{a_{11}}\right) - a_{23}\right)R_3 - R_2$. By setting

$m=\frac{1}{\beta}\left[\frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{13}}{a_{11}}\right)-a_{23}\right)-\frac{a_{13}}{a_{11}}\right]$ and $n=\frac{1}{\beta}\frac{1}{\alpha}\left(a_{21}\left(\frac{a_{13}}{a_{11}}\right)-a_{23}\right)$. Also, if we let

$$x=\left[\frac{1}{\alpha}t\left(a_{21}\left(\frac{a_{14}}{a_{11}}\right)-a_{24}\right)-\left(a_{31}\left(\frac{a_{14}}{a_{11}}\right)-a_{34}\right)\right],$$

$$y=\left[\frac{1}{\alpha}t\left(a_{21}\left(\frac{a_{15}}{a_{11}}\right)-a_{25}\right)-\left(a_{31}\left(\frac{a_{15}}{a_{11}}\right)-a_{35}\right)\right]$$

$$z=\left[\frac{1}{\alpha}t\left(a_{21}\left(\frac{a_{16}}{a_{11}}\right)-a_{26}\right)-\left(a_{31}\left(\frac{a_{16}}{a_{11}}\right)-a_{36}\right)\right],$$

we obtain the following matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ m\frac{1}{\beta}x-\frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{14}}{a_{11}}\right)-a_{24}\right)-\frac{a_{14}}{a_{11}} & n\frac{1}{\beta}x-\frac{1}{\alpha}\left(a_{21}\left(\frac{a_{14}}{a_{11}}\right)-a_{24}\right) & \frac{1}{\beta}x \\ m\frac{1}{\beta}y-\frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{15}}{a_{11}}\right)-a_{25}\right)-\frac{a_{15}}{a_{11}} & n\frac{1}{\beta}y-\frac{1}{\alpha}\left(a_{21}\left(\frac{a_{15}}{a_{11}}\right)-a_{25}\right) & \frac{1}{\beta}y \\ m\frac{1}{\beta}z-\frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{16}}{a_{11}}\right)-a_{26}\right)-\frac{a_{16}}{a_{11}} & n\frac{1}{\beta}z-\frac{1}{\alpha}\left(a_{21}\left(\frac{a_{16}}{a_{11}}\right)-a_{26}\right) & \frac{1}{\beta}z \end{pmatrix}.$$

Hence we obtain the reduce row echelon form of $\tilde{M}_1$ of rank 3 [5] [6].

We wish to consider now the cases of the matrix $\tilde{M}_1$ of rank 1 and rank 2 since the case of rank Zero is trivial.

Rank 1: By setting each of the entries on row 2 and 3 of matrix *A* to zero, we obtain the rank of $\tilde{M}_1$ to be 1.

Rank 2: By setting each of the entries on row 3 of matrix *B* to zero, we obtain the rank of $\tilde{M}_1$ to be 2.

Proposition 3: *if*

$$\left(e_{11}^2-C_{12}^2\right)=0,\left(-e_{11}^1-C_{12}^1\right)=0, e_{11}^2=0,-e_{21}^1=0,e_{31}^2=0,-e_{31}^1=0,$$

$$e_{12}^2=0,-e_{12}^1=0,\left(e_{22}^2-C_{12}^2\right)=0,\left(-e_{22}^1-C_{12}^1\right)=0,e_{32}^2=0,-e_{32}^1=0,$$

$$e_{13}^2=0,-e_{13}^1=0,e_{23}^2=0,-e_{23}^1=0,e_{33}^2-C_{12}^2=0,-e_{33}^1-C_{12}^1=0,$$

then $\begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$ and the $rank\tilde{M}_1=0$. Further

$ker\tilde{d}^1\cong\mathbb{R}^6$ and $Im\tilde{d}^1\cong 0$.

Proposition 4: *From matrix A, if* $e_{11}^2-C_{12}^2\neq 0$, $\left(e_{11}^2-C_{12}^2\right)\left(\frac{\left(-e_{11}^1-C_{12}^1\right)}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{12}^1\right)=0,$

$$e_{12}^2\left(\frac{e_{11}^2}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(e_{22}^2-C_{12}^2\right)=0,\quad e_{12}^2\left(\frac{\left(-e_{21}^1\right)}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{22}^1-C_{12}^1\right)=0,\quad e_{12}^2\left(\frac{e_{31}^2}{\left(e_{11}^2-C_{12}^2\right)}\right)-e_{32}^2=0,$$

$$e_{12}^2\left(\frac{\left(-e_{31}^1\right)}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{32}^1\right)=0, \quad then$$

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{a_{14}}{a_{11}} & \frac{a_{15}}{a_{11}} & \frac{a_{16}}{a_{11}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ and $rank\tilde{M}_1=1$. Further, $ker\tilde{d}^1\cong\mathbb{R}^5$ and $Im\tilde{d}^1\cong\mathbb{R}$.

Proof. Since the $rank\tilde{M}_1=1$, we have that $dim\left(Im\tilde{d}^1\right)=1$, thus $Im\tilde{d}^1\cong\mathbb{R}$. We now show that $ker\tilde{d}^1\cong\mathbb{R}^5$. By the dimension rank theorem, we have that $dim\left(ker\tilde{d}^1\right)+dim\left(Im\tilde{d}^1\right)=6$ which is $dim\left(ker\tilde{d}^1\right)=6-1=5$. □

Proposition 5: *From matrix B, if* $e_{12}^2\left(\frac{\left(-e_{11}^1-C_{12}^1\right)}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{12}^1\right)\neq 0$,

$$\frac{1}{\alpha}t\left(e_{12}^2\left(\frac{e_{11}^2}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(e_{22}^2-C_{12}^2\right)\right)-\left(e_{13}^2\left(\frac{e_{11}^2}{\left(e_{11}^2-C_{12}^2\right)}\right)-e_{23}^2\right)=0,$$

$$\frac{1}{\alpha}t\left(e_{12}^2\left(\frac{-e_{21}^1}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{22}^1-C_{12}^1\right)\right)-\left(e_{13}^2\left(\frac{-e_{21}^1}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{23}^1\right)\right)=0,$$

$$\frac{1}{\alpha}t\left(e_{12}^2\left(\frac{e_{31}^2}{\left(e_{11}^2-C_{12}^2\right)}\right)-e_{32}^2\right)-\left(e_{13}^2\left(\frac{e_{31}^2}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(e_{33}^2-C_{12}^2\right)\right)=0,$$

$$\frac{1}{\alpha}t\left(e_{12}^2\left(\frac{-e_{31}^1}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{32}^1\right)\right)-\left(e_{13}^2\left(\frac{-e_{31}^1}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{33}^1-C_{12}^1\right)\right)=0, \quad \text{then}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{13}}{a_{11}}\right)-a_{23}\right)-\frac{a_{13}}{a_{11}} & \frac{1}{\alpha}\left(a_{21}\left(\frac{a_{13}}{a_{11}}\right)-a_{23}\right) & 0 \\ \frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{14}}{a_{11}}\right)-a_{24}\right)-\frac{a_{14}}{a_{11}} & \frac{1}{\alpha}\left(a_{21}\left(\frac{a_{14}}{a_{11}}\right)-a_{24}\right) & 0 \\ \frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{15}}{a_{11}}\right)-a_{25}\right)-\frac{a_{15}}{a_{11}} & \frac{1}{\alpha}\left(a_{21}\left(\frac{a_{15}}{a_{11}}\right)-a_{25}\right) & 0 \\ \frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{16}}{a_{11}}\right)-a_{26}\right)-\frac{a_{16}}{a_{11}} & \frac{1}{\alpha}\left(a_{21}\left(\frac{a_{16}}{a_{11}}\right)-a_{26}\right) & 0 \end{pmatrix}$$

and $rank\tilde{M}_1=2$. Further, $ker\tilde{d}^1\cong\mathbb{R}^4$ and $Im\tilde{d}^1\cong\mathbb{R}^2$.

Proof. Since the $rank\tilde{M}_1=2$, we have that $dim\left(Im\tilde{d}^1\right)=2$, thus $Im\tilde{d}^1\cong\mathbb{R}^2$. We now show that $ker\tilde{d}^1\cong\mathbb{R}^4$. By the dimension rank theorem, we have that $dim\left(ker\tilde{d}^1\right)+dim\left(Im\tilde{d}^1\right)=6$, that is $dim\left(ker\tilde{d}^1\right)=6-2=4$.

Proposition 6: *if*

$$e_{11}^2-C_{12}^2\neq 0, \quad \alpha=e_{12}^2\left(\frac{\left(-e_{11}^1-C_{12}^1\right)}{\left(e_{11}^2-C_{12}^2\right)}\right)-\left(-e_{11}^1-C_{12}^1\right)\neq 0,$$

$\beta = \frac{1}{\alpha} t \left(e_{12}^2 \left(\frac{e_{11}^2}{\left(e_{11}^2 - C_{12}^2\right)} \right) - \left(e_{22}^2 - C_{12}^2\right) \right) - \left(e_{13}^2 \left(\frac{a_{13}}{\left(e_{11}^2 - C_{12}^2\right)} \right) - e_{23}^2 \right) \neq 0,$ then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ m\frac{1}{\beta}x - \frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{14}}{a_{11}}\right) - a_{24}\right) - \frac{a_{14}}{a_{11}} & n\frac{1}{\beta}x - \frac{1}{\alpha}\left(a_{21}\left(\frac{a_{14}}{a_{11}}\right) - a_{24}\right) & \frac{1}{\beta}x \\ m\frac{1}{\beta}y - \frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{15}}{a_{11}}\right) - a_{25}\right) - \frac{a_{15}}{a_{11}} & n\frac{1}{\beta}y - \frac{1}{\alpha}\left(a_{21}\left(\frac{a_{15}}{a_{11}}\right) - a_{25}\right) & \frac{1}{\beta}y \\ m\frac{1}{\beta}z - \frac{1}{\alpha}\frac{a_{12}}{a_{11}}\left(a_{21}\left(\frac{a_{16}}{a_{11}}\right) - a_{26}\right) - \frac{a_{16}}{a_{11}} & n\frac{1}{\beta}z - \frac{1}{\alpha}\left(a_{21}\left(\frac{a_{16}}{a_{11}}\right) - a_{26}\right) & \frac{1}{\beta}z \end{pmatrix}$$

and the $rank\tilde{M}_1 = 0$. Further, $ker\tilde{d}^1 \cong \mathbb{R}^3$ and $Im\tilde{d}^1 \cong \mathbb{R}^3$.

Proof. Since the $rank\tilde{M}_1 = 3$, we have that $dim\left(Im\tilde{d}^1\right) = 3$, thus $Im\tilde{d}^1 \cong \mathbb{R}^3$. We now show that $ker\tilde{d}^1 \cong \mathbb{R}^3$. By the dimension rank theorem, we have that $dim\left(ker\tilde{d}^1\right) + dim\left(Im\tilde{d}^1\right) = 6,$ that is $dim\left(ker\tilde{d}^1\right) = 6 - 3 = 3.$ □

Now, we compute our quotient spaces of the 2 - 3 matrix Chevalley Eilenberg cohomology which are $H^0(V) = Ker\tilde{d}^0 / Im\tilde{d}^{-1}$, $H^1(V) = Ker\tilde{d}^1 / Im\tilde{d}^0$ and $H^2(V) = Ker\tilde{d}^2 / Im\tilde{d}^1$.

For $H^0(V)$, we have the following quotient space:

Case 1: $Im\tilde{d}^{-1} \cong 0$ and $Ker\tilde{d}^0 \cong 0$

$$Ker\tilde{d}^0 / Im\tilde{d}^{-1} \cong 0.$$

For $H^1(V)$, we have the following quotient spaces:

Case 1: $Im\tilde{d}^0 \cong \mathbb{R}^0$ and $Ker\tilde{d}^1 \cong \mathbb{R}^6$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^6.$$

Case 2: $Im\tilde{d}^0 \cong \mathbb{R}$ and $Ker\tilde{d}^1 \cong \mathbb{R}^6$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^5.$$

Case 3: $Im\tilde{d}^0 \cong \mathbb{R}^2$ and $Ker\tilde{d}^1 \cong \mathbb{R}^6$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^4.$$

Case 4: $Im\tilde{d}^0 \cong \mathbb{R}^3$ and $Ker\tilde{d}^1 \cong \mathbb{R}^6$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^3.$$

Case 5: $Im\tilde{d}^0 \cong \mathbb{R}^4$ and $Ker\tilde{d}^1 \cong \mathbb{R}^6$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^2.$$

Case 6: $Im\tilde{d}^0 \cong \mathbb{R}^5$ and $Ker\tilde{d}^1 \cong \mathbb{R}^6$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}.$$

Case 7: $Im\tilde{d}^0 \cong \mathbb{R}^6$ and $Ker\tilde{d}^1 \cong \mathbb{R}^6$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong 0.$$

Case 1: $Im\tilde{d}^0 \cong 0$ and $Ker\tilde{d}^1 \cong \mathbb{R}^5$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^5.$$

Case 2: $Im\tilde{d}^0 \cong \mathbb{R}$ and $Ker\tilde{d}^1 \cong \mathbb{R}^5$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^4.$$

Case 3: $Im\tilde{d}^0 \cong \mathbb{R}^2$ and $Ker\tilde{d}^1 \cong \mathbb{R}^5$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^3 .$$

Case 4: $Im\tilde{d}^0 \cong \mathbb{R}^3$ and $Ker\tilde{d}^1 \cong \mathbb{R}^5$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^2 .$$

Case 5: $Im\tilde{d}^0 \cong \mathbb{R}^4$ and $Ker\tilde{d}^1 \cong \mathbb{R}^5$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R} .$$

Case 6: $Im\tilde{d}^0 \cong \mathbb{R}^5$ and $Ker\tilde{d}^1 \cong \mathbb{R}^5$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong 0 .$$

Case 1: $Im\tilde{d}^0 \cong 0$ and $Ker\tilde{d}^1 \cong \mathbb{R}^4$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^4 .$$

Case 2: $Im\tilde{d}^0 \cong \mathbb{R}$ and $Ker\tilde{d}^1 \cong \mathbb{R}^4$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^3 .$$

Case 3: $Im\tilde{d}^0 \cong \mathbb{R}^2$ and $Ker\tilde{d}^1 \cong \mathbb{R}^4$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^2 .$$

Case 4: $Im\tilde{d}^0 \cong \mathbb{R}^3$ and $Ker\tilde{d}^1 \cong \mathbb{R}^4$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R} .$$

Case 5: $Im\tilde{d}^0 \cong \mathbb{R}^4$ and $Ker\tilde{d}^1 \cong \mathbb{R}^4$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong 0 .$$

Case 1: $Im\tilde{d}^0 \cong 0$ and $Ker\tilde{d}^1 \cong \mathbb{R}^3$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^3 .$$

Case 2: $Im\tilde{d}^0 \cong \mathbb{R}$ and $Ker\tilde{d}^1 \cong \mathbb{R}^3$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^2 .$$

Case 3: $Im\tilde{d}^0 \cong \mathbb{R}^2$ and $Ker\tilde{d}^1 \cong \mathbb{R}^3$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R} .$$

Case 4: $Im\tilde{d}^0 \cong \mathbb{R}^3$ and $Ker\tilde{d}^1 \cong \mathbb{R}^3$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong 0 .$$

Case 1: $Im\tilde{d}^0 \cong 0$ and $Ker\tilde{d}^1 \cong \mathbb{R}^2$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R}^2 .$$

Case 2: $Im\tilde{d}^0 \cong \mathbb{R}$ and $Ker\tilde{d}^1 \cong \mathbb{R}^2$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R} .$$

Case 3: $Im\tilde{d}^0 \cong \mathbb{R}^2$ and $Ker\tilde{d}^1 \cong \mathbb{R}^2$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong 0 .$$

Case 1: $Im\tilde{d}^0 \cong 0$ and $Ker\tilde{d}^1 \cong \mathbb{R}$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong \mathbb{R} .$$

Case 2: $Im\tilde{d}^0 \cong \mathbb{R}$ and $Ker\tilde{d}^1 \cong \mathbb{R}$

$$Ker\tilde{d}^1 / Im\tilde{d}^0 \cong 0.$$

For $H^2(V)$**, we have the following quotient spaces**:
Case 1: $Im\tilde{d}^1 \cong 0$ and $Ker\tilde{d}^2 \cong \mathbb{R}^3$

$$Ker\tilde{d}^2 / Im\tilde{d}^1 \cong \mathbb{R}^3.$$

Case 2: $Im\tilde{d}^1 \cong \mathbb{R}$ and $Ker\tilde{d}^2 \cong \mathbb{R}^3$

$$Ker\tilde{d}^2 / Im\tilde{d}^1 \cong \mathbb{R}^2.$$

Case 3: $Im\tilde{d}^1 \cong \mathbb{R}^2$ and $Ker\tilde{d}^2 \cong \mathbb{R}^3$

$$Ker\tilde{d}^2 / Im\tilde{d}^1 \cong \mathbb{R}.$$

Case 4: $Im\tilde{d}^1 \cong \mathbb{R}^3$ and $Ker\tilde{d}^2 \cong \mathbb{R}^3$

$$Ker\tilde{d}^2 / Im\tilde{d}^1 \cong 0.$$

We suggest that further research in this direction is to carry out the deformation on the Cohomological spaces H^0, H^1 and H^2 which are 32 in number and apply a specific example with $\mathfrak{sl}_2(\mathbb{C})$. We will also carry out an extensive study on the solution of our system of linear equations on the 2 - 3 matrix Chavelley Eilenberg fundamental condition.

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