Advances in Pure Mathematics, 2015, 5, 828-834
Published Online December 2015 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2015.514077

Scientific Research Publishing

L^p Polyharmonic Dirichlet Problems in the Upper Half Plane

Kanda Pan

Department of Mathematics, Jinan University, Guangzhou, China
Email: kandapan@163.com

Received 11 November 2015; accepted 8 December 2015; published 11 December 2015

Abstract

In this article, a class of Dirichlet problem with L^p boundary data for polyharmonic function in the upper half plane is mainly investigated. By introducing a sequence of kernel functions called higher order Poisson kernels and a hierarchy of integral operators called higher order Pompeiu operators, we obtain a main result on integral representation solution as well as the uniqueness of the polyharmonic Dirichlet problem under a certain estimate.

Keywords

Dirichlet Problem, Polyharmonic Function, Higher Order Poisson Kernels, Higher Order Pompeiu Operators, Non-Tangential Maximal Function, Uniqueness

1. Introduction

Usually harmonic functions are defined by Laplace operator $\Delta = 4\partial_z\partial_{\bar{z}}$, where $\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial_x} + i\frac{\partial}{\partial_y}\right)$ is the Cauchy-Riemann operator and $\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial_x} - i\frac{\partial}{\partial_y}\right)$ is the adjoint operator of C-R operator. By iterating the Laplace operator, one can define the so-called polyharmonic functions by $\Delta^n\ (n \geq 2)$ [1]. In [2], Goursat obtained his decomposition formula, in [3], Vekua developed one method to construct an approximative solution of the biharmonic Dirichlet problem in a simply connected domain. In recent years, the study of explicit solution of BVPS (boundary value problems) has undergone a new phase of development [4]-[6]. There are Dirichlet, Neumann and Robin boundary value problems in regular domain (in the disc [4]; and in the upper half plane [5]) and in irregular domains (Lipschitz domains [6]). Although, there are many marked works about the BVPS, few of them give a certain estimate about the uniqueness of the solution. Thus, the purpose of this article is devoted to solving the unique solution of the following polyharmonic Dirichlet problems (for short, PHD) for L^p data in the upper half plane, H, *i.e*.

$$\begin{cases} \Delta^n u = 0 \text{ in H}, \\ \Delta^j u = f_j \text{ on } \mathbb{R}, \\ \mathrm{M}(u) \in L^P(\mathbb{R}) \end{cases} \tag{1.1}$$

with $\|\mathrm{M}(u)\|_{L^P(\mathbb{R})} \le C\sum_{j=0}^{n-1}\|f_j\|_{L^P(\mathbb{R})}$, where Δ is the Laplacian, and $\mathbb{R}$ is the real axis, $f_j \in L^P(\mathbb{R})$ for some suitable $p>1$, $n\in\mathbb{N}$, $0\le j<n$, $\mathrm{M}(u)$ is the non-tangential maximal function of *u*, which is defined by

$$\mathrm{M}(F)(x_0) = \sup_{X\in\Gamma_\gamma(x_0)} |F(X)|, \text{ for } x_0 \in \partial\mathrm{H},$$

where $\Gamma_\gamma(x_0)$ is the non-tangential approach region, viz.,

$$\Gamma_\gamma(x_0) = \{X \in \mathrm{H} : |X - x_0| < \gamma dist(X, \partial\mathrm{H})\}$$

where $\gamma > 1$.

It is clear that all the boundary data in BVPs (1.1) are non-tangential.

2. Preliminary and Some Lemmas

Definition 2.1. If a real valued function $f \in C^{2n}(D)$ satisfies the equation $(\partial_z \partial_{\bar z})^n f = 0$, in *D*, then *f* is called an n-harmonic function in *D*, concisely, a polyharmonic function.

We use the notation $Har_n(D)$ denoting the set of polyharmonic function of order *n* in *D*. Especially, $Har_n(D)$ is the set of all harmonic functions in *D*.

Lemma 2.2. [7] Let *D* be a simply connected (bounded or unbounded) domain in the complex plane with smooth boundary ∂D. If $f \in Har_n(D)$, then for any $z_0 \in D$, there exist functions $f_j \in H^j_{1,z_0}(D)$, $j = 0,1,\cdots,n-1$ such that

$$f(z) = 2\Re\left\{\sum_{j=0}^{n-1} (\bar z - \bar z_0)^j f_j(z)\right\}, z \in D, \tag{2.1}$$

where $\Re$ denotes the real part. The above decomposition expression of *f* is unique in the sense of the equivalence relation $\sim$, more precisely, $\sim_j$ for f_j.

Corollary 2.3. If the sequence of functions $\{f_n\}$ defined in *D* satisfy

(1) $f_1 \in Har_1(D)$;

(2) $(\partial_z \partial_{\bar z}) f_n = f_{n-1}$ in *D* for $n > 1$.

Then $f_n \in Har_n(D)$ for $n > 1$, and

$$\partial_z f_{n,j} = j^{-1} f_{n-1,j-1}, \quad 1 \le j \le n-1 \tag{2.2}$$

where $f_{n,j}$ is the analytic *jth* decomposition component of the n-harmonic function f_n. It must be noted that (2.2) holds in the sense of the equivalence relation $\sim$.

Definition 2.4. A sequence of real-valued functions of two variables $\{G_n(z,t)\}_{n=1}^{\infty}$ defined on $\mathbf{H}\times\mathbb{R}$ is called a sequence of higher order Poisson kernels, more precisely, $G_n(z,t)$ is called the *nth* order Poisson kernel, if they satisfy the following conditions.

(1) For all $n\in\mathbb{N}, G_n(z,t) \in C(\mathbf{H}\times\mathbb{R})$; $G_n(.,t) \in C^{2n}(\mathbf{H})$ with any fixed $t\in\mathbb{R}$; and $G_n(z,.) \in L^p, p>1$, with any fixed $z \in \mathbf{H}$, and the non-tangential boundary value

$$\lim_{\substack{z\to s \\ z\in\mathbf{H}, s\in\mathbb{R}}} G_n(z,t) = G_n(s,t)$$

exists for all *t* and $s \ne t$; $G_n(.,t)$ can be continuously extended to $\bar{\mathbf{H}} \setminus \{t\}$ for any fixed $t\in\mathbb{R}$;

(2) $G_1(z,t)=\frac{1}{2i}\left(\frac{1}{t-z}-\frac{1}{t-\overline{z}}\right)$ and $G_n(i,t)=0, n\ge 2$ and $t\in\mathbb{R}$, and for any $n\in\mathbb{N}$

$$|G_n(z,t)|\le\frac{M}{|t-z'|}$$

uniformly on $D_c\times\{t\in\mathbb{R}:|t|>T\}$ whenever $z'\in D_c$, where D_c is any compact set in $\overline{\mathbf{H}}$, *M*, *T* are positive constants depending only on D_c and *n*;

(3) $(\partial_z\partial_{\overline{z}})G_1(z,t)=0$ and $\Delta G_n(z,t)=G_{n-1}(z,t)$ for $n>1$;

(4) $\lim\limits_{z\to s,z\in\mathbf{H}}\frac{1}{\pi}\int_{-\infty}^{+\infty}G_1(z,t)\gamma(t)\mathrm{d}t=\gamma(s)$, a.e., for any $\gamma\in L^p(\mathbb{R}), P\ge 1$;

(5) $\lim\limits_{z\to s,z\in\mathbf{H}}\frac{1}{\pi}\int_{-\infty}^{+\infty}G_n(z,t)\gamma(t)\mathrm{d}t=0$, for any $\gamma\in L^P(\mathbb{R}), P\ge 1, n\ge 2$,

where all limits are non-tangential.

Definition 2.5. Let *D* be a simply connected (bounded or unbounded) domain in the plane with smooth boundary ∂D, and $H(D)$ denote the set of all analytic functions in *D*. If *f* is a continuous function defined on $D\times\partial D$ satisfying $f(.,t)\in H(D)$ for any fixed $t\in\partial D$, and $f(z,.)\in L^p(\partial D)$, $P\ge 1$, for any fixed $z\in D$, then *f* is called $H\times L^P$ on $D\times\partial D$ and this is noted by $f\in(H\times L^P)(D\times\partial D)$.

Lemma 2.6. [8] If $G_n(z,t)_{n=1}^{\infty}$ is a sequence of higher order Poisson kernels defined on $\mathbf{H}\times\mathbb{R}$, *i.e.*, $G_n(z,t)_{n=1}^{\infty}$ fulfills the aforementioned properties 1 - 5 in Definition 2.4, then, for $n>1$, there exist functions $G_{n,0}(z,t), G_{n,1}(z,t),\cdots,G_{n,n-1}(z,t)$ defined on $\mathbf{H}\times\mathbb{R}$ such that

$$G_n(z,t)=2\Re\left\{\sum_{j=0}^{n-1}(\overline{z}+i)^j G_{n,j}(z,t)\right\}, z\in\mathbf{H}, t\in\mathbb{R}, \tag{2.3}$$

with

$$\partial z G_{n,j}(z,t)=j^{-1}G_{n-1,j-1}(z,t) \tag{2.4}$$

for $1\le j\le n-1$

$$\partial_z^k G_{n,j}(i,t)=0 \tag{2.5}$$

for $1\le k\le j-1$ with respect to $t\in\mathbb{R}$ and

$$G_{n,0}(z,t)=-\sum_{j=1}^{n-1}(z+i)^j G_{n,j}(z,t) \tag{2.6}$$

Moreover,

$$G_1(z,t)=\frac{1}{2i}\left(\frac{1}{t-z}-\frac{1}{t-\overline{z}}\right) \tag{2.7}$$

is the classical Poisson kernel for the upper half plane. All of the above $G_{n,j}\in(H\times L^p)(\mathbf{H}\times\mathbb{R})$, the non-tangential boundary value

$$\lim_{\substack{z\to s\\ z\in\mathbf{H},s\in\mathbb{R}}} G_{n,j}(z,t)=G_{n,j}(s,t) \tag{2.8}$$

exists on $\mathbb{R}$, except $t\in\mathbb{R}$ and $G_{n,j}(s,.)\in L^p(\mathbb{R})$ for any fixed $s\in\mathbb{R}$. We can further show that $G_{n,j}(.,t)$ can be continuously extended to $\overline{\mathbf{H}}\setminus\{t\}$ for any fixed $t\in\mathbb{R}$, and

$$|G_{n,j}(z,t)|\le M\frac{1}{|t-z'|} \tag{2.9}$$

uniformly on $D_c\times\{t\in\mathbb{R}:|t|>T\}$ whenever $z'\in D_c$ which is any compact set in $\overline{\mathbf{H}}$, where *M*, *T* are positive constants depending only on D_c.

Moreover,

$$\lim_{\substack{z\to s\\ z\in\bar{\mathbf{H}},s\in\mathbb{R}}}\left|G_{n,j}(z,s)\right|=+\infty \text{ and } \lim_{\substack{z\to s\\ z\in\bar{\mathbf{H}},s\in\mathbb{R}}}\left|(z-s)G_{n,j}(z,s)\right|=0 \tag{2.10}$$

for any $s\in\mathbb{R}$ and $n\geq 2$.

Remark 2.7. Lemma 2.6 provides a algorithm to obtain all explicit expressions of higher order Poisson kernels appeared in [8].

3. Homogeneous PHD Problem in the Upper Half Plane

In order to solve the homogeneous PHD problems (1.1) and get the uniqueness of its solution, we need the following lemmas.

Lemma 3.1. [8] Let D be a simply connected unbounded domain in the plane with smooth boundless boundary ∂D. If $f\in\left(H\times L^1\right)(D\times\partial D)$ and there exists $g\in L^p(\partial D),P\geq 1$, such that

$$\left|f(z,t)\right|\leq M\frac{g(t)}{\left|t-z_0\right|} \tag{3.1}$$

uniformly on $D_c\times\left\{t\in\partial D:|t|>T\right\}$ whenever $z_0\in D_c$ which is any compact set in *D*, where *M*, *T*, are positive constants depending only on D_c. Then

$$\frac{\partial}{\partial_z}\left(\int_{\partial D}f(z,t)\,\mathrm{d}t\right)=\int_{\partial D}\frac{\partial f}{\partial_z}(z,t)\,\mathrm{d}t.$$

Lemma 3.2. [8] Let $\left\{G_n(z,t)\right\}_{n=1}^{\infty}$ be the sequence of higher order Poisson kernels defined on $\mathbf{H}\times\mathbb{R}$, then for any $n>1$ and $\gamma\in L^p(\mathbb{R}),P\geq 1$,

$$\frac{\partial^2}{\partial_z\partial_{\bar{z}}}\left(\int_{-\infty}^{+\infty}G_n(z,t)\gamma(t)\,\mathrm{d}t\right)=\int_{-\infty}^{+\infty}G_{n-1}(z,t)\gamma(t)\,\mathrm{d}t. \tag{3.3}$$

Lemma 3.3. [9] Let $f\in L^P(\mathbb{R})$, $1\leq p\leq\infty$, and $u(z)$ be the Poisson integral of *f* (in our notations, $u(z)=\frac{1}{\pi}\int_{-\infty}^{+\infty}G_1(z,t)f(t)\,\mathrm{d}t$, $z\in\mathbf{H}$), then

$$M[u](x_0)=\sup_{z\in\Gamma_\alpha(x_0)}\left|u(z)\right|\leq C_\alpha\mathfrak{M}f(x_0) \tag{3.4}$$

where $\Gamma_\alpha(x_0)$ is the cone in $\mathbf{H}$ with the vertex at $(x_0,0)$ and the aperture α, $x_0\in\mathbb{R}$, $\alpha>0$; C_α is a positive constant depending only on α, $M[u]$ is the non-tangential maximal function, and $\mathfrak{M}f$ is the standard Hardy-Littlewood maximal function defined by

$$\mathfrak{M}f(x_0)=\sup_{\rho>0}\frac{1}{2\rho}\int_{x_0-\rho}^{x_0+\rho}\left|f(x)\right|\mathrm{d}x \tag{3.5}$$

Lemma 3.4. (Hardy-Littlewood maximal theorem, see [10]) Let $f\in L^P(\mathbb{R})$, $1\leq P\leq\infty$, then $\mathfrak{M}f$ is finite almost everywhere on $\mathbb{R}$. Moreover,

(1) If $f\in L^1(\mathbb{R})$, then $\mathfrak{M}f$ is in $L^{1,\infty}(\mathbb{R})$, more precisely

$$\left|\left\{x\in\mathbb{R}:\mathfrak{M}f(x)>\lambda\right\}\right|\leq\frac{2}{\lambda}\|f\|_{L^1}(\mathbb{R}). \tag{3.6}$$

(2) If $f\in L^P(\mathbb{R})$, $1<P\leq\infty$, then

$$\|\mathfrak{M}f\|_{L^P(\mathbb{R})}\leq A_p\|f\|_{L^P(\mathbb{R})}, \tag{3.7}$$

where A_p is a constant depending only on p.

Corollary 3.5. $\|M[u]\|_{L^P(\mathbb{R})}\leq C_{p,\alpha}\|f\|_{L^P(\mathbb{R})}$ for any $f\in L^P(\mathbb{R})$ with $1<p\leq\infty$, where $C_{p,\alpha}$ is a constant depending only on p,α. Moreover,

$$\left|\left\{x\in\mathbb{R}:M[u](x)>\lambda\right\}\right|\leq\frac{2C_\alpha}{\lambda}\|f\|_{L^1(\mathbb{R})} \tag{3.8}$$

for any $f \in L^1(\mathbb{R})$, and for any $f \in L^P(\mathbb{R})$, $1 \le P \le \infty$, $M[u]$ is finite almost everywhere on $\mathbb{R}$, C_α is a positive constant depending only on α.

Theorem 1. *Let* $\{G_n(z,t)\}_{n=1}^{\infty}$ *be the sequence of higher order Poisson kernels defined on* $\mathbf{H} \times \mathbb{R}$, *then for any* $n \ge 1$,

$$u(z) = \sum_{j=1}^{n} \frac{4^{1-j}}{\pi} \int_{-\infty}^{+\infty} G_j(z,t) f_{j-1}(t) \mathrm{d}t \tag{3.9}$$

is the unique solution of PHD problem (1.1)

Proof. Since the higher order Poisson kernels possess the inductive property as stated in Definition 2.4. Act on the two sides of (3.9) with the polyharmonic operator Δ^l, $1 \le l \le n-1$. We have

$$\Delta^l u(z) = \sum_{j=l+1}^{n} \frac{4^{1-j+l}}{\pi} \int_{-\infty}^{+\infty} G_{j-l}(z,t) f_{j-1}(t) \mathrm{d}t \tag{3.10}$$

since the Laplace operator is $\Delta = 4\frac{\partial^2}{\partial_z \partial_{\bar{z}}}$. Thus, for $\Delta^l u_h = 0$ on $\mathbb{R}$,

$$\Delta^l u(s) = f_l(s), \ s \in \mathbb{R}, 0 \le l \le n-1, \tag{3.11}$$

follow from Lemma 2.6 and the nice property of G, *i.e.*,

$$\lim_{\substack{z \to s \\ z \in \mathrm{H}, s \in \mathbb{R}}} \frac{1}{\pi} \int_{-\infty}^{+\infty} G_1(z,t) \gamma(t) \mathrm{d}t = \gamma(s) \tag{3.12}$$

for any $\gamma \in L^p(\mathbb{R}), p \ge 1$.

Similarly, letting the polyharmonic operator Δ^n act on the two sides of (3.9), we have $\Delta^n u(z) = 0$ for any $z \in \mathbf{H}$. Thus (3.9) is a solution of the PHD problem (1.1).

Next we turn to the estimate and uniqueness of the solution. By Definition 2.4 and Corollary 3.5, we have

$$\begin{aligned} \|Mu\|_{L^P(\mathbb{R})} &= \left\| M\left[\sum_{j=1}^{n} \frac{4^{1-j}}{\pi} \int_{-\infty}^{+\infty} G_j(z,t) f_{j-1}(t) \mathrm{d}t \right] \right\|_{L^P(\mathbb{R})} \\ &\le \sum_{j=1}^{n} \left\| M\left[\frac{4^{1-j}}{\pi} \int_{-\infty}^{+\infty} G_j(z,t) f_{j-1}(t) \mathrm{d}t \right] \right\|_{L^P(\mathbb{R})} \le C \sum_{j=0}^{n-1} \|f_j\|_{L^P(\mathbb{R})} \end{aligned}$$

As discussed above, the uniqueness of solution follows.

4. Inhomogeneous PHD Problem in the Upper Plane

Due to the limited knowledge of the author, at this section, we only consider the bounded domain D for inhomogeneous PHD problem in the upper half-plane, *i.e*.

$$\begin{cases} \Delta^n u = g \ \text{ in } D, \\ \Delta^j u = f_j \ \text{ on } \partial D, \end{cases} \tag{4.1}$$

where $g \in L^1_{loc}(\mathbb{C})$, such that, for some $\delta > 0$, $|g(z)| = O\left(|z|^{-m-n-\delta}\right)$, as $z \to \infty$ and $f_j \in L^p(\partial D)$ for some suitable $p > 1$. In order to solve the inhomogeneous PHD problem (4.1), we need the higher order Pompeiu operators which are higher order analogues of the classical Pompeiu operators.

Definition 4.1. [11] Let kernels

$$K_{m,n}(z) = \begin{cases} \dfrac{(-m)!(-1)^m}{(n-1)!\pi} z^{m-1}\bar{z}^{n-1}, & m \le 0; \\ \dfrac{(-n)!(-1)^n}{(m-1)!\pi} z^{m-1}\bar{z}^{n-1}, & n \le 0; \\ \dfrac{1}{(m-1)!(n-1)!\pi} z^{m-1}\bar{z}^{n-1} \cdot \left[\log|z|^2 - \sum_{k=1}^{m-1} \dfrac{1}{k} - \sum_{l=1}^{n-1} \dfrac{1}{l} \right], & m,n \ge 1 \end{cases} \tag{4.2}$$

where m and n are integer, with $m+n\geq 0$ but $(m,n)\neq(0,0)$. Then, we formally define operators $T_{m,n,D}$, acting on suitable complex valued function w defined in D, a domain in the plane, according to

$$T_{m,n,D}w(z)=\iint_D K_{m,n}(z-\zeta)w(\zeta)\mathrm{d}\xi\mathrm{d}\eta \tag{4.3}$$

The following properties of $T_{m,n,D}$ are needed in the sequel. They are partial results from [11].

Lemma 4.2. Assume $m+n\geq 1$, and let w be a complex valued function in $L^1_{loc}(\mathbb{C})$ such that for some $\delta>0$,

$$|w(z)|=O\left(|z|^{-m-n-\delta}\right),\quad \text{as } z\to\infty.$$

Then, the integral $T_{m,n}w(z)$ converges absolutely for almost all z in $\mathbb{C}$ and, provides that p satisfies conditions,

$$\begin{cases}(a)\ 1\leq p<2, \text{ when } m+n=1,\\ (b)\ 1\leq p\leq\infty, \text{ when } m+n=2, mn\leq 0,\\ (c)\ 1\leq p<\infty, \text{ when } m+n=2, mn>0,\\ (d)\ 1\leq p\leq\infty, \text{ when } m+n\geq 3,\end{cases}$$

$$T_{m,n}w\in L^p_{loc}(\mathbb{C}).$$

Proof. See Corollary 4.6 in [11].

Lemma 4.3. Assume $m+n\geq 1$, and let w be a measurazble complex valued function in $\mathbb{C}$ such that for some $\delta>0$,

$$|w(z)|=O\left(|z|^{-m-n-\delta}\right),\quad \text{as } z\to\infty. \tag{4.4}$$

(a) If $m+n\geq 2$ and $w\in L^1_{loc}(\mathbb{C})$, then in the sense of Sobolev derivatives in the entire plane $\mathbb{C}$,

$$\partial_z T_{m,n}w=T_{m-1,n}w, \tag{4.5}$$

$$\partial_{\bar z} T_{m,n}w=T_{m,n-1}w \tag{4.6}$$

(b) If $m+n=1$ and $w\in L^p_{loc}(\mathbb{C})$ for some $p>1$, then (4.5) and (4.6) again hold in the sense of Sobolev derivatives in $\mathbb{C}$; moreover, the formulas

$$\partial_z T_{1,0}w=\partial_{\bar z}T_{0,1}w=w \tag{4.7}$$

are valid in $\mathbb{C}$ even in the case of $p=1$.

Proof. See Corollary 5.4 in [11].

Theorem 2. *The problem of* (4.1) *is solvable and its unique solution is*

$$u(z)=T_{n,n,\mathbb{C}}g(z)+\sum_{j=1}^{n}\frac{4^{1-j}}{\pi}\int_{-\infty}^{+\infty}G_j(z,t)\left[f_{j-1}(t)-T_{n+1-j,n+1-j,\mathbb{C}}g(t)\right]\mathrm{d}t \tag{4.8}$$

where $z\in\mathbf{H}$ and $T_{l,l,\mathbb{C}}(1\leq l\leq n)$ are the higher order Pompeiu operators, $G_j(z,t)$ and $(1\leq j\leq n)$ are the former n higher order Poisson kernel functions.

Proof. Through Lemma 4.2 and Lemma 4.3, we get

$$\partial_z^k\partial_{\bar z}^l T_{m,n,\mathbb{C}}g(z)=T_{m-k,n-l,\mathbb{C}}g(z),\ 0\leq k+l\leq m+n \tag{4.9}$$

in the Sobolev sense. Moreover,

$$T_{m-k,n-l,\mathbb{C}}g(z)\in L^p_{loc}(\mathbb{C}) \quad \text{as } 0\leq k+l\leq m+n. \tag{4.10}$$

Noting (4.9) we know that $u(z)=T_{n,n,\mathbb{C}}g(z)$ is a week solution of the inhomogeneous equation

$$\left(\partial_z^n\partial_{\bar z}^n\right)u(z)=g(z),\ z\in\mathbb{C}, g(z)\in L^1_{loc}(\mathbb{C}),$$

and for some

$$\delta > 0, \left|g(z)\right| = O\left(\left|z\right|^{-m-n-\delta}\right), \text{ as } z \to \infty. \tag{4.11}$$

By the aforementioned, the problem (4.1) is equivalent to the PHD problem of simplified form

$$\begin{cases} \Delta^n\left(u - T_{n,n,\mathbb{C}}g\right) = 0, g \in L^1_{loc}(\mathbb{C}), \left|g(z)\right| = O\left(\left|z\right|^{-m-n-\delta}\right), \text{ for some } \delta > 0, \text{ as } z \to \infty \\ \left(\partial_z \partial_{\bar{z}}\right)^j \left[u - T_{n,n,\mathbb{C}}g\right] = f_j - T_{n-j,n-j,\mathbb{C}}g, \quad f_j \in L^p(\partial D), 0 \le j < n, p > 1. \end{cases}$$

So, through Theorem 1 as well as the estimate of the solution, we complete the proof of Theorem 2.

References

[1] Aronszajn, N., Cresse, T. and Lipkin, L. (1983) Polyharmonic Functions, Oxford Math. Clarendon, Oxford.

[2] Goursat, E. (1898) Sur I'équation $\Delta\Delta u = 0$. *Bulletin de la Société Mathématique de France*, **26**, 236-237.

[3] Vekua, I.N. (1976) On One Method of Solving the First Biharmonic Boundary Value Problem and the Dirichlet Problem. *American Mathematical Society Translations*, **104**, 104-111.

[4] Begehr, H., Du, J. and Wang, Y. (2008) A Dirichlet Problem for Polyharmonic Functions. *Annali di Matematica Pura ed Applicata*, **187**, 435-457. http://dx.doi.org/10.1007/s10231-007-0050-5

[5] Begehr, H. and Gaertner, E. (2007) A Dirichlet Problem for the Inhomogeneous Polyharmonic Equations in the Upper Half Plane. *Georgian Mathematical Journal*, **14**, 33-52.

[6] Verchota, G.C. (2005) The Biharmonic Neumann Problem in Lipschitz Domain. *Acta Mathematica*, **194**, 217-279. http://dx.doi.org/10.1007/BF02393222

[7] Du, Z. (2008) Boundary Value Problems for Higher Order Complex Differential Equations. Doctoral Dissertation, Freie Universität Berlin, Berlin.

[8] Du, Z., Qian, T. and Wang, J.X. (2012) Polyharmonic Dirichlet Problem in Regular Domain: The Upper Half Plane. *Journal of Differential Equations*, **252**, 1789-1812. http://dx.doi.org/10.1016/j.jde.2011.08.024

[9] Stein, E.M. and Weiss, G. (1971) Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton, New Jersey.

[10] Garnett, J. (2007) Bounded Analytic Functions. Springer, New York.

[11] Begehr, H. and Hile, G.N. (1997) A Hierarchy of Integral Operators. *Rocky Mountain Journal of Mathematics*, **27**, 669-706. http://dx.doi.org/10.1216/rmjm/1181071888