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The Space of Bounded $p(\cdot)$-Variation in Wiener's Sense with Variable Exponent

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Abstract

In this paper, we proof some properties of the space of bounded $p(\cdot)$-variation in Wiener's sense. We show that a functions is of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent if and only if it is the composition of a bounded nondecreasing functions and hölderian maps of the $\frac{1}{p(\cdot)}$ variable exponent. We show that the composition operator H, associated with $h:\mathbb{R}\to\mathbb{R}$, maps the spaces $WBV_{p(\cdot)}([a,b])$ into itself if and only if h is locally Lipschitz. Also, we prove that if the composition operator generated by $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ maps this space into itself and is uniformly bounded, then the regularization of h is affine in the second variable.

Keywords

Generalized Variation, $p(\cdot)$-Variation in Wiener's Sense, Variable Exponent, Composition Operator, Matkowski's Condition

1. Introduction

Since Camile Jordan in 1881 first gave the notion of variation of a function in the paper [1] devoted to the convergence of Fourier series, a number of generalizations and extensions have been given in many directions. Such extensions find many applications in different areas of mathematics. Consequently, the study of notions of generalized bounded variation forms an important direction in the field of mathematical analysis. Two well-known generalizations are the functions of bounded p-variation and the functions of bounded φ-variation, due to N. Wiener [2] and L. C. Young [3] respectively. In 1924, N. Wiener [2] generalized the Jordan notion and introduced the notion of p-variation (variation in the sense of Wiener). L. Young [3] introduced the notion of φ-variation of a function. The p-variation of a function f is the supremum of the sums of the *pth* powers of

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absolute increments of f over no overlapping intervals. Wiener mainly focused on the case $p = 2$, the 2-variation. For p-variations with $p \neq 2$, the first major work was done by Young [3], partly with Love [4]. After a long hiatus following Young's work, *pth*-variations were reconsidered in a probabilistic context by R. Dudley [5] [6]. Many basic properties of the variation in the sense of Wiener and a number of important applications of the concept can be found in [7] [8]. Also the paper by V. V. Chistyakov and O. E. Galkin [9] is very important in the context of p-variation. They study properties of maps of bounded p-variation $(p > 1)$ in the sense of Wiener are defined on a subset of the real line and take values in metric or normed spaces.

In recent years, there has been an increasing interest in the study of various mathematical problems with variable exponents. With the emergency of nonlinear problems in applied sciences, standard Lebesgue and Sobolev spaces demonstrated their limitations in applications. The class of nonlinear problems with exponent growth is a new research field and it reflects a new kind of physical phenomena. In 2000, the field began to expand even further. Motivated by problems in the study of electrorheological fluids, Diening [10] raised the question of when the Hardy-Littlewood maximal operator and other classical operators in harmonic analysis were bounded on the variable Lebesgue spaces. These and related problems are the subject of active research to this day. These problems were interesting in applications (see [11]-[14]) and gave rise to a revival of the interest in Lebesgue and Sobolev spaces with variable exponent, the origins of which could be traced back to the work of Orlicz in the 1930's [15]. In the 1950's, this study was carried on by Nakano [16] [17] who made the first systematic study of spaces with variable exponent. Later, Polish and Czechoslovak mathematicians investigated the modular function spaces (see for the example Musielak [18] [19], Kovacik and Rakosnik [20]). We refer to books [14] for the detailed information on the theoretical approach to the Lebesgue and Sobolev spaces with variable exponents. In [21], Castillo, Merentes and Rafeiro studied a new space of functions of generalized bounded variation. There, the authors introduced the notion of bounded variation in the Wiener sense with the exponent $p(\cdot)$-variable.

The main purpose of this paper is threefold: First, we provide a further develop of the results of the article [21]. We give a detailed description of the new class formed by the functions of bounded variation in the sense of Wiener with the exponent $p(\cdot)$-variable. Second, in the spirit of some results of Federer ([22] sec. 2.5.16), Sierpinski [23], and Chistyakov and Galkin [9], we provide a characterization of the functions with variable bounded variation in the sense of Wiener. We prove a structural theorem for mappings of bounded variation in the sense of Wiener with the exponent $p(\cdot)$-variable. Finally, we analyze a necessary and sufficient conditions for the acting of composition operator (Nemystskij) on the space $WBV_{p(\cdot)}[a,b]$.

This paper is organized as follows: Section 2 contains definitions, notations, and necessary background about the class of functions of bounded $p(\cdot)$-variation in Wiener's sense; Section 3 contains some properties of this space; Section 4 contains a main theorem, which is a characterization of the functions of bounded $p(\cdot)$-variation in Wiener's sense of the composition of two functions with certain properties; Section 5 contains another main theorem, in which we prove a result in the case when h is locally Lipschitz if and only if the composition operator maps the space $WBV_{p(\cdot)}[a,b]$ into itself; Finally, in section 6 we give the last main theorem, namely, we show that any uniformly bounded composition operator that maps the space $WBV_{p(\cdot)}[a,b]$ into itself necessarily satisfies the so called Matkowski's weak condition.

2. Preliminaries

Throughout this paper, we use the following notation: we will denote by $\omega_{p(x_{ts})}(f,[a,b]) = \sup\left\{\left|f(t)-f(s)\right|^{p(x_{ts})} : t,s \in [a,b]\right\}$ the diameter of the image $f([a,b])$ (or the oscillation of f on $[a,b]$) and by x_{ts} a number between $[t,s]$.

The concept of functions of bounded variation has been well-known since C. Jordan in 1881 (see [1]) gave the complete characterization of functions of bounded variation as a difference of two increasing functions. This class of functions exhibit so many interesting properties that it makes them a suitable class of functions in a variety of contexts with wide applications in pure and applied mathematics (see [8] [24]).

Definition 1 *Let* $f:[a,b] \to \mathbb{R}$ *be a function. For each partition* $\pi : a = t_0 < t_1 < \cdots < t_n = b$ *of* $[a,b]$, *we define*

$$V(f;[a,b]) := \sup_{\pi} \sum_{i=1}^{n} \left|f(t_i) - f(t_{i-1})\right|, \tag{2.1}$$

where the supremum is taken over all partitions π of the interval $[a,b]$. If $V(f;[a,b])<\infty$, we say that *f* has bounded variation. The collection of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

The notion of bounded variation due to Jordan was generalized in 1924 by Wiener (see [2]) who introduced the definition of *p*-variation as follows.

Definition 2 *Given a real number* $p\geq 1$, *a partition* $\pi: a=t_0<t_1<\cdots<t_n=b$ *of* $[a,b]$, *and a function* $f:[a,b]\to\mathbb{R}$. *The nonnegative real number*

$$V_p^W(f)=V_p^W(f;[a,b]):=\sup_{\pi}\sum_{j=1}^{n}\left(\left|f(t_j)-f(t_{j-1})\right|\right)^p \tag{2.2}$$

is called the Wiener variation (or *p*-variation in Wiener's sense) of *f* on $[a,b]$ where the supremum is taken over all partitions of π. In case that $V_p^W(f;[a,b])<\infty$, we say that *f* has bounded Wiener variation (or bounded *p*-variation in Wiener's sense) on $[a,b]$. The symbol $WBV_p([a,b])$ will denote the space of functions of bounded *p*-variation in Wiener's sense on $[a,b]$.

In 2013 R. Castillo, N. Merentes and H. Rafeiro [21] introduce the notation of bounded variation space in the Wiener sense with variable exponent on $[a,b]$ and study some of its basic properties.

Definition 3 *Given a function* $p:[a,b]\to(1,\infty)$, *a partition* $\pi: a=t_0<t_1<\cdots<t_n=b$ *of the interval* $[a,b]$, *and a function* $f:[a,b]\to\mathbb{R}$. *The nonnegative real number*

$$V_{p(\cdot)}^W(f)=V_{p(\cdot)}^W(f,[a,b]):=\sup_{\pi^*}\sum_{j=1}^{n}\left(\left|f(t_j)-f(t_{j-1})\right|\right)^{p(x_{j-1})} \tag{2.3}$$

is called Wiener variation with variable exponent (or $p(\cdot)$-variation in Wiener's sense) of *f* on $[a,b]$ where π^* is a tagged partition of the interval $[a,b]$, *i.e.*, a partition of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-1}$ subject to the conditions that for each *i*, $t_i\leq x_i\leq t_{i+1}$.

In case that $V_{p(\cdot)}^W(f;[a,b])<\infty$, we say that *f* has bounded Wiener variation with variable exponent (or bounded $p(\cdot)$-variation in Wiener's sense) on $[a,b]$. The symbol $WBV_{p(\cdot)}([a,b])$ will denote the space of functions of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent on $[a,b]$.

Remark 1 *Given a function* $p:[a,b]\to[1,\infty)$

1) If $p(x)=1$ for all *x* in $[a,b]$, then $WBV_{p(\cdot)}([a,b])=BV([a,b])$.

2) If $p(x)=p$ for all *x* in $[a,b]$ and $1<p<\infty$, then $WBV_{p(\cdot)}([a,b])=WBV_p([a,b])$.

3. Properties of the Space

Definition 4 (*Norm in* $WBV_{p(\cdot)}([a,b])$)

$$\|f\|_{p(\cdot)}^W:=|f(a)|+\mu_{p(\cdot)}(f),\quad f\in WBV_{p(\cdot)}([a,b]) \tag{3.1}$$

where $\mu_{p(\cdot)}(f):=\inf_{\lambda>0}\left\{\lambda>0: V_{p(\cdot)}^W\left(\frac{f}{\lambda}\right)\leq 1\right\}$.

In [21] is shown that the space $WBV_{p(\cdot)}([a,b])$ endowed with the norm $\|\cdot\|_{p(\cdot)}^W$ is a Banach space.

Theorem 2 *Let* $p:[a,b]\to(1,\infty)$ *be a function, then* $\left(WBV_{p(\cdot)}([a,b]),\|\cdot\|_{p(\cdot)}^W\right)$ *is a Banach space.*

Lemma 1 *Let* $f:[a,b]\to\mathbb{R}$ *be a function such that* $f\in WBV_{p(\cdot)}([a,b])$ *then f has the left-hand and right-hand limits in all point on* $[a,b]$.

Proof. Without loss of generality we can show that *f* has a left limit on $x_0\in[a,b]$. Assume that the $\lim_{x\to x_0^-} f(x)$ do not exist. Then

Case 1: If $\lim_{x\to x_0^-} f(x)\to\pm\infty$, then

$\left|f(x)-f(x_0)\right|\leq V_{p(\cdot)}^W(f;[a,b])^{1/p(x)}$ so $\lim_{x\to x_0^-}\left|f(x)-f(x_0)\right|\leq V_{p(\cdot)}^W(f;[a,b])^{1/p(x)}$. Since $f(x_0)=L<\infty$ and $\lim_{x\to x_0^-} f(x)\to\pm\infty$, then $V_{p(\cdot)}^W(f;[a,b])=\infty$, which is a contradiction.

Case 2: $\lim_{x\to x_0^-} f(x)$ do not converge a any point. That means that the function *f* oscillates. Let $\{t_n\}_{n\geq 1}$ be a sequence such that $t_n \to x_0$ when $n \to \infty$

$$\left| f(t_n) - f(t_m) \right| > r \quad \text{for all } n,m \geq N$$

$nr < \sum_{i=1}^{n} \left| f(t_i) - f(t_{i-1}) \right| \leq V_{p(\cdot)}^{W}(f,[a,b])$ therefore $V_{p(\cdot)}^{W}(f,[a,b]) = \infty$, which is a contradiction as well. □

Remark 3 *Without loss of generality we can take* $[a,b]=[0,1]$. *If* $f \in Lip[0,1]$ *then* $\left| f(t_j) - f(t_{j-1}) \right| \leq k \left| t_j - t_{j-1} \right|$, *further, as* $p(\cdot)$ *is bounded*

$$V_{p(\cdot)}^{W}(f,[0,1]) = \sup_{\pi^*} \sum_{j=1}^{n} \left| f(t_j) - f(t_{j-1}) \right|^{p(x_{j-1})} \leq \sup_{\pi^*} \sum_{j=1}^{n} k^{p(x_{j-1})} \left| t_j - t_{j-1} \right|^{p(x_{j-1})}$$

$$\leq k^{p(x_{01})} \sup_{\pi^*} \sum_{j=1}^{n} \left| t_j - t_{j-1} \right|^{p(x_{j-1})}$$

since $p(x) \geq 1$ and $\left| t_j - t_{j-1} \right| \leq 1$ we have

$$V_{p(\cdot)}^{W}(f,[0,1]) \leq k^{p(x_{01})} \sup_{\pi^*} \sum_{j=1}^{n} \left| t_j - t_{j-1} \right|^{p(x_{j-1})} \leq k^{p(x_{01})} < \infty.$$

So $f \in WBV_{p(\cdot)}[a,b]$, *i.e.*, $Lip[a,b] \subset WBV_{p(\cdot)}[a,b]$.

The following properties of elements of $WBV_{p(\cdot)}[a,b]$ allow us to get characterizations of them.

Lemma 2 (General properties of the *p*(·)-variation) *Let* $f:[a,b] \to \mathbb{R}$ *be an arbitrary map. We have*

(P1) minimality: if $t,s \in [a,b]$, then

$$\left| f(t) - f(s) \right|^{p(x_{ts})} \leq \omega_{p(x_{ts})}(f,[a,b]) \leq V_{p(\cdot)}^{W}(f,[a,b]).$$

(P2) monotonicity: if $a,t,s,b \in [a,b]$ and $a \leq t \leq s \leq b$, then $V_{p(\cdot)}^{W}(f,[a,t]) \leq V_{p(\cdot)}^{W}(f,[a,s])$, $V_{p(\cdot)}^{W}(f,[s,b]) \leq V_{p(\cdot)}^{W}(f,[t,b])$ and $V_{p(\cdot)}^{W}(f,[t,s]) \leq V_{p(\cdot)}^{W}(f,[a,b])$.

(P3) semi-additivity: if $t \in [a,b]$, then

$$2^{1-p^+} V_{p(\cdot)}^{W}(f,[a,b]) \leq V_{p(\cdot)}^{W}(f,[a,t]) + V_{p(\cdot)}^{W}(f,[t,b]) \leq V_{p(\cdot)}^{W}(f,[a,b]).$$

(P4) change of a variable: if $[c,d] \subset \mathbb{R}$ and $\varphi:[c,d] \to [a,b]$ is a (not necessarily strictly) monotone function, then $V_{p(\cdot)}^{W}(f,\varphi([c,d])) = V_{p(\cdot)}^{W}(f \circ \varphi,[c,d])$.

(P5) regularity: $V_{p(\cdot)}^{W}(f,[a,b]) = \sup\left\{ V_{p(\cdot)}^{W}(f,[s,t]);\, s,t \in [a,b], a \leq b \right\}$.

Proof. (P1) Let $a,t,s,b \in [a,b]$, $a \leq t \leq s \leq b$

$$\left| f(t) - f(s) \right|^{p(x_{ts})} \leq \sup_{\pi^*} \left\{ \left| f(t) - f(s) \right|^{p(x_{ts})} : t,s \in [a,b] \right\} = \omega_{p(x_{ts})}(f,[a,b])$$

$$\leq \sup_{\pi^*} \sum_{j=1}^{n} \left(\left| f(t_j) - f(t_{j-1}) \right| \right)^{p(x_{j-1})} = V_{p(\cdot)}^{W}(f,[a,b]).$$

(P2) Let $a,t,s,b \in [a,b]$, $a \leq t \leq s \leq b$ and the partition $\pi: a = t_0 < t_1 < \cdots < t_{m1} = t < \cdots < t_{m2} = s < \cdots < t_n = b$ so

$$V_{p(\cdot)}^{W}(f,[a,t]) = \sup_{\pi^*} \sum_{j=1}^{m1} \left(\left| f(t_j) - f(t_{j-1}) \right| \right)^{p(x_{j-1})}$$

$$\leq \sup_{\pi^*} \sum_{j=1}^{m1} \left(\left| f(t_j) - f(t_{j-1}) \right| \right)^{p(x_{j-1})} + \sup_{\pi^*} \sum_{j=m1+1}^{m2} \left(\left| f(t_j) - f(t_{j-1}) \right| \right)^{p(x_{j-1})}$$

$$= \sup_{\pi^*} \sum_{j=1}^{m2} \left(\left| f(t_j) - f(t_{j-1}) \right| \right)^{p(x_{j-1})} = V_{p(\cdot)}^{W}(f,[a,s])$$

the other cases are similarly.

(P3) Let $T = \{t_j\}_{j=0}^{m} \in \pi$ and denote $p^+(\cdot) = \sup_{x \in [a,b]} p(x)$ and $S = T \cup \{t\}$. We consider the following two cases:

1) if $t \leq t_0$ or $t_m \leq t$, then

$$V_{p(\cdot)}(f,T) \le V_{p(\cdot)}(f,T\cup\{t\}).$$

2) if $t_{k-1} \le t \le t_k$ for some $1 \le k \le m$, then

$$V_{p(\cdot)}(f,T) \le 2^{p^+-1} V_{p(\cdot)}(f,T\cup\{t\}).$$

For the case (a) we have

$$V_{p(\cdot)}(f,T) \le V_{p(\cdot)}(f,T\cup\{t\}) \le V_{p(\cdot)}(f,[a,t]) + V_{p(\cdot)}(f,[t,b]).$$

For the case (b) we get

$$\begin{aligned}
V^W_{p(\cdot)}(f,T) &= \sup_{\pi^*}\left[\sum_{j=1}^{k-1}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})} + \left|f(t_k)-f(t_{k-1})\right|^{p(x_{k-1})} + \sum_{j=k+1}^{m}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}\right] \\
&\le \sup_{\pi^*}\left[\sum_{j=1}^{k-1}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})} + \left(\left|f(t)-f(t_{k-1})\right| + \left|f(t_k)-f(t)\right|\right)^{p(x_{k-1})} + \sum_{j=k+1}^{m}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}\right] \\
&\le \sup_{\pi^*}\left[\sum_{j=1}^{k-1}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})} + 2^{p(x_{k-1})-1}\left(\left|f(t)-f(t_{k-1})\right|^{p(x_{k-1})} + \left|f(t_k)-f(t)\right|^{p(x_{k-1})}\right)\right. \\
&\quad \left. + \sum_{j=k+1}^{m}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}\right] \\
&\le \sup_{\pi^*}\left[\sum_{j=1}^{k-1} 2^{p(x_{k-1})-1}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})} + 2^{p(x_{k-1})-1}\left|f(t)-f(t_{k-1})\right|^{p(x_{k-1})}\right. \\
&\quad \left. + \sum_{j=k+1}^{m} 2^{p(x_{k-1})-1}\left|f(t_k)-f(t)\right|^{p(x_{k-1})} + 2^{p(x_{k-1})-1}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}\right] \\
&\le 2^{p^+-1} V^W_{p(\cdot)}(f,T\cup\{t\}),
\end{aligned}$$

also

$$\begin{aligned}
V^W_{p(\cdot)}(f,T\cup\{t\}) &= V^W_{p(\cdot)}(f,T) + \left|f(t)-f(t_{k-1})\right|^{p(x_{j-1})} + \left|f(t_k)-f(t)\right|^{p(x_{j-1})} \\
&\quad - \left|f(t_k)-f(t_{k-1})\right|^{p(x_{j-1})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
V^W_{p(\cdot)}(f,T) &\le 2^{p^+-1} V^W_{p(\cdot)}(f,S) \le 2^{p^+-1} V_{p(\cdot)}(f,[a,b]) \\
&\le 2^{p^+-1}\left(V_{p(\cdot)}(f,[a,t]) + V_{p(\cdot)}(f,[t,b])\right).
\end{aligned}$$

Taking the supremum over all $T \in \pi^*$, we arrive at the left hand side inequality in (P3).

Now we prove the right hand side inequality. Let $V^W_{p(\cdot)}(f,[a,t]) < \infty$ and $V^W_{p(\cdot)}(f,[t,b]) < \infty$. Then for every $\varepsilon > 0$ there are partitions $\pi_1 \in \pi^*$ and $\pi_2 \in \pi^*$ of the interval $[a,t]$ and $[t,b]$ respectively, such that

$$V_{p(\cdot)}(f,[a,t]) \le V_{p(\cdot)}(f,\pi_1) + \frac{\varepsilon}{2} \quad \text{and} \quad V_{p(\cdot)}(f,[t,b]) \le V_{p(\cdot)}(f,\pi_2) + \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned}
V^W_{p(\cdot)}(f,[a,t]) + V^W_{p(\cdot)}(f,[t,b]) &\le V^W_{p(\cdot)}(f,\pi_1) + V^W_{p(\cdot)}(f,\pi_2) + \varepsilon \\
&\le V^W_{p(\cdot)}(f,\pi_1\cup\{t\}) + V^W_{p(\cdot)}(f,\pi_2\cup\{t\}) + \varepsilon \\
&= V^W_{p(\cdot)}(f,\pi_1\cup\pi_2\cup\{t\}) + \varepsilon \\
&\le V^W_{p(\cdot)}(f,[a,b]) + \varepsilon,
\end{aligned}$$

and take into account the arbitrariness of $\varepsilon > 0$.

(P4) Let $[c,d] \subset \mathbb{R}$, $\varphi:[c,d] \to [a,b]$ a (not necessarily strictly) monotone function, π_0 a tagged partition of the interval $[c,d]$, $T_1 = \{\tau_j\}_{j=0}^m \in \pi_0$ and $T = \{t_j\}_{j=0}^m$ with $t_j = \varphi(\tau_j)$, then

$$\begin{aligned} V_{p(\cdot)}^W(f\circ\varphi, T_1) &= \sup_{T_1}\sum_{j=1}^m \left| f(\varphi(\tau_j)) - f(\varphi(\tau_{j-1}))\right|^{p(x_{j-1})} \\ &= \sup_{T}\sum_{j=1}^m \left| f(t_j) - f(t_{j-1})\right|^{p(x_{j-1})} \\ &= V_{p(\cdot)}^W(f,T) \le V_{p(\cdot)}^W(f,\varphi([c,d])). \end{aligned}$$

On the other hand, if a partition $T = \{t_j\}_{j=0}^m$ of $\varphi([c,d])$ is such that $t_{j-1} < t_j$ for $j = 1,\cdots,m$ then there exist $\tau_j \in [c,d]$ such that $t_j = \varphi(\tau_j)$ and, again by the monotonicity of φ

$$V_{p(\cdot)}^W(f,T) = V_{p(\cdot)}^W(f\circ\varphi, T_1) \le V_{p(\cdot)}^W(f\circ\varphi,[c,d]).$$

(P5) By monotonicity of $V_{p(\cdot)}^W$ we get

$$V_{p(\cdot)}^W(f,[a,b]) \ge \sup\left\{V_{p(\cdot)}^W(f,[s,t]);\, s,t\in[a,b], a\le b\right\}.$$

On the other hand, for any number $\alpha < V_{p(\cdot)}^W(f,[a,b])$ such that there is a partition $T = \{t_i\}_{i=0}^m \in \pi^*$ with $V_{p(\cdot)}^W(f,T) \ge \alpha$. We define $\hat{\pi}$ a partition of the interval $[t_0,t_m]$ then $T \in \hat{\pi}$ and $V_{p(\cdot)}^W(f,\hat{\pi}) \ge V_{p(\cdot)}^W(f,T) \ge \alpha$, *i.e.*,

$$V_{p(\cdot)}^W(f,[a,b]) \ge \sup\left\{V_{p(\cdot)}^W(f,[s,t]);\, s,t\in[a,b], a\le b\right\}. \qquad \square$$

4. Characterization

W. Sierpiński in 1933 (See [23]) showed that a function $f:[a,b]\to\mathbb{R}$ is regular function if and only if it is the composition of increasing function and continuous function. This is a notable result which links regular functions with continuous functions. In 1969 (see [22]), H. Federer demostrated that function is of bounded variation if and only if it is the composition of a Lipschitz function with a monotone function. In the year 1998 (see [9]) V. V. Chistyakov and O. E. Galkin proved similar result for bounded *p*-variation with $p > 1$, they show that a function is of bounded *p*-variation if and only if it is the composition of a bounded nondecreasing function with a Hölder function. In this section we show that a function is of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent if and only if it is the composition of a bounded nondecreasing function with a Hölderian function with variable exponent equal to $\dfrac{1}{p(\cdot)}$.

We say that $g(x) \in H^{\gamma(x)}$, the Hölder space of variable exponent, where $\gamma(x)$ is a positive function, $0 < \gamma(x) \le 1$, if

$$\left|g(t_i) - g(t_{i-1})\right| \le C\left|t_i - t_{i-1}\right|^{\gamma(x_{i-1})}$$

for all $x_{i-1} \in [a,b]$. The least number *C* satisfying the above inequality is called the Hölder constant of *g*.

Theorem 4 *The map $f:[a,b]\to\mathbb{R}$ is of bounded $p(\cdot)$-variation if and only if there exists a bounded nondecreasing function $\varphi:[a,b]\to\mathbb{R}$ a Hölderian map $g:\varphi([a,b])\to\mathbb{R}$ of exponent $\gamma = 1/p(\cdot)$ and $H(g)\le 1$ such that $f = g\circ\varphi$ on $[a,b]$.*

The proof of this theorem is contained in the following two lemmas.

Lemma 4.1 *If $\varphi:[a,b]\to\mathbb{R}$ is bounded monotone, $g:\varphi([a,b])\to\mathbb{R}$ is Hölderian of exponent $\gamma(\cdot) = 1/p(\cdot)$ and $f = g\circ\varphi$, then $f \in V_{p(\cdot)}^W([a,b];\mathbb{R})$*

Proof. Assume that φ is nondecreasing. Since

$$\varphi([a,b]) = [\varphi(a),\varphi(b)]$$

by virtue of change of a variable (P4) we have

$$V^W_{p(\cdot)}\left(f,[a,b]\right)=V^W_{p(\cdot)}\left(g\circ\varphi,[a,b]\right)=V^W_{p(\cdot)}\left(g,\varphi([a,b])\right)=V^W_{p(\cdot)}\left(g,\left[\varphi(a),\varphi(b)\right]\right).$$

If $T=\{t_i\}_{i=0}^m$ is a partition of $\left[\varphi(a),\varphi(b)\right]$

$$V^W_{p(\cdot)}\left(g,T\right)=\sup_T\sum_{i=1}^m\left|g(t_i)-g(t_{i-1})\right|^{p(x_{i-1})}\le\sup_T\sum_{i=1}^m C^{p(x_{i-1})}\left|t_i-t_{i-1}\right|^{p(x_{i-1})\gamma(x_{i-1})}$$
$$=\sum_{i=1}^m C^{p(x_{i-1})}\left|t_i-t_{i-1}\right|\le C^{p(x_\varphi)}\left(\varphi(b)-\varphi(a)\right),$$

where $\varphi(a)\le x_\varphi\le\varphi(b)$. Therefore, by boundedness of φ yield

$$V^W_{p(\cdot)}\left(f,[a,b]\right)\le C^{p(x_\varphi)}\left(\varphi(b)-\varphi(a)\right)<\infty\quad\text{for all } a,b\in[a,b].$$

If φ is nonincreasing the proof is similarly. □

Lemma 4.2 *Let* $f:[a,b]\to\mathbb{R}$ *be a map of bounded* $p(\cdot)$*-variation. Then, there exist a bounded nondecreasing nonnegative function* $\varphi:[a,b]\to\mathbb{R}$ *and a Hölderian map* $g:\varphi([a,b])\to\mathbb{R}$ *of exponent* $\gamma(\cdot)=1/p(\cdot)$ *and the Hölder constant* $H(g)\le 1$ *such that*

1) $f=g\circ\varphi$ on $[a,b]$.
2) $g\left(\varphi([a,b])\right)=f([a,b])$ in $\mathbb{R}$.
3) $V^W_{p(\cdot)}\left(g,\varphi([a,b])\right)=V^W_{p(\cdot)}\left(f,[a,b]\right)$.

Proof. We define the function φ by $\varphi(t)=V^W_{p(\cdot)}\left(f,[a,t]\right)$; by (P2) φ it is well define nonnegative bounded and nondecreasing. If $\tau\in\varphi([a,b])$ denote by $\varphi^{-1}(\tau)=\{t\in[a,b]:\varphi((t)=\tau\}$ the inverse image of the one-point set $\{\tau\}$ under the function φ. Define the map $g:\varphi([a,b])\to\mathbb{R}$ as follows if $\tau\in\varphi([a,b])=\left[\varphi(a),\varphi(b)\right]$

$$g(\tau)=f(t)\quad\text{for any point } t\in\varphi^{-1}(\tau).\tag{4.1}$$

By (P1) and (P3),

$$\left|f(t)-f(s)\right|^{p(x_{ts})}\le V^W_{p(\cdot)}\left(f,[t,s]\right)\le V^W_{p(\cdot)}\left(f,[a,s]\right)-V^W_{p(\cdot)}\left(f,[a,t]\right)=\varphi(s)-\varphi(t).$$

The representation of f in (1) follows from (5), for if $t\in[a,b]$, then $\tau:=\varphi(t)\in\left[\varphi(a),\varphi(b)\right]$ and $t\in\varphi^{-1}(\tau)$, so that (5) yields $f(t)=g(\tau)=g(\varphi(t))=(g\circ\varphi)(t)$.

The assertions in (2) and (3) follows from (1) and (P4). Now we will show that g is Hölderian. We have

$$V^W_{p(\cdot)}\left(g,\left[\varphi(a),\varphi(b)\right]\right)=V^W_{p(\cdot)}\left(g\circ\varphi,[a,t]\right)=V^W_{p(\cdot)}\left(f,[a,t]\right)=\varphi(t)=\tau.$$

Hence, if $\alpha,\beta\in\left[\varphi(a),\varphi(b)\right]$, $\alpha\le\beta$, then by (P1) and (P3) we get

$$\left|g(\beta)-g(\alpha)\right|^{p(x_{\beta\alpha})}\le V^W_{p(\cdot)}\left(g,[\alpha,\beta]\right)\le V^W_{p(\cdot)}\left(g,[\alpha,\beta]\right)-V^W_{p(\cdot)}\left(g,[a,\alpha]\right)=\beta-\alpha,$$

then

$$\left|g(\beta)-g(\alpha)\right|\le\left|\beta-\alpha\right|^{1/p(x_{\beta\alpha})}=\left|\beta-\alpha\right|^{\gamma(x_{\beta\alpha})}.$$ □

In the next section we will be dealing with the composition operator (Nemitskij).

5. Composition Operator between the Space $WBV_{p(\cdot)}\left(f,[a,b]\right)$

In any field of nonlinear analysis composition operators (Nemytskij), the superposition operators generated by appropriate functions, play a crucial role in the theory of differential, integral and functional equations. Their analytic properties depend on the postulated properties of the defining function and on the function space in which they are considered. A rich source of related questions are the monograph by J. Appell and P. P. Zabrejko [25] and J. Appell, J. Banas, N. Merentes [8].

Given a function $h:\mathbb{R}\to\mathbb{R}$, the composition operator H, associated to a function f (autonomous case) maps each function $f:[a,b]\to\mathbb{R}$ into the composition function $Hf:[a,b]\to\mathbb{R}$, given by

$$Hf(t) := h(f(t)), \quad (t \in [a,b]). \tag{5.1}$$

More generally, given $f:[a,b]\times\mathbb{R}\to\mathbb{R}$, we consider the operator *H*, defined by

$$Hf(t) := h(t, f(t)), \quad (t \in [a,b]). \tag{5.2}$$

This operator is also called *superposition operator* or *susbtitution operator* or *Nemytskij operator*. In what follows, will refer (5.1) as the *autonomus case* and to (5.2) as the *non-autonomus case*.

One of our main goals is to prove a result in the case when *h* is locally Lipschitz if and only if the composition operator maps the space of functions of bounded $p(\cdot)$-variation into itself.

Theorem 5 *Let H be a composition operator associated to* $h:\mathbb{R}\to\mathbb{R}$. *H maps the space* $WBV_{p(\cdot)}(f)$ *into itself if and only if h is locally Lipschitz.*

Proof. We may suppose without loss generality that $[a,b]=[0,1]$. First, let $u:\mathbb{R}\to\mathbb{R}$ be locally Lipschitz on $\mathbb{R}$, and let $u\in WBV_{p(\cdot)}([0,1])$. Then $V^W_{p(\cdot)}(\lambda u;[0,1])<\infty$ for some $\lambda>0$. Considering the local Lipschitz condition

$$|h(u)-h(v)| \le k(r)|u-v| \quad (u,v\in\mathbb{R}, |u|,|v|\le\mathbb{R}) \tag{5.3}$$

for $r:=\|f\|_\infty$, for any partition $\pi: 0=t_0<t_1<\cdots<t_n=1$ we obtain the estimate

$$\sum_{j=1}^{n}\left(\frac{\lambda}{k(\|f\|_\infty)}\left|h(u(t_j))-h(u(t_{j-1}))\right|\right)^{p(t_{j-1})}$$
$$\le \sum_{j=1}^{n}\left(\frac{\lambda}{k(\|f\|_\infty)}k(\|f\|_\infty)\left|u(t_j)-u(t_{j-1})\right|\right)^{p(t_{j-1})}$$
$$= \sum_{j=1}^{n}\left(\lambda\left|u(t_j)-u(t_{j-1})\right|\right)^{p(t_{j-1})} = V^W_{p(\cdot)}(\lambda u,[0,1]).$$

This shows that for $\mu := \dfrac{\lambda}{k(\|f\|_\infty)}$, $V^W_{p(\cdot)}(\mu Hu,[0,1])<\infty$, and hence $Hu\in WBV_{p(\cdot)}([0,1])$ as claimed.

For the converse implication, suppose that *h* does not satisfy a local Lipschitz condition (5.3), in this way for any increasing sequence of positive real numbers $\{k_j\}_{j\ge1}$ that converges to infinite, that we will be defined later, we can choose sequences $\{u_j\}_{j\ge1}$, $\{v_j\}_{j\ge1}$, with $\delta_j : v_j-u_j<\dfrac{1}{k_j}$ and

$$|h(v_j)-h(u_j)| > k_j|v_j-u_j| \quad (j\in\mathbb{N}, u_j<v_j). \tag{5.4}$$

Considering subsequences if necessary, we can assume that the sequence $\{u_j\}_{j\ge1}$ is monotone. We supposed without loss of generality the sequence $\{u_j\}_{j\ge1}$ is increasing. Since $[0,1]$ is compact, from de inequality (5.2) we have that there exists subsequences of $\{u_j\}_{j\ge1}$ and $\{v_j\}_{j\ge1}$ that we will denote in the same way, and that converge to $u_\infty\in[0,1]$. Since the sequence $\{u_j\}_{j\ge1}$ is a Cauchy sequence, we can assume that $u_\infty\in[-r,r]$ such that $|u_j-u_\infty|\le\dfrac{1}{2k_j}$ for all *k*, and so $|u_j-u_{j+1}|\le\dfrac{1}{k_j}$. Choose $n_j:=\dfrac{1}{k_j\delta_j}$.

Pick the sequence defined recursively $\{t_k\}_{k\ge1}$ by

$$t_1 := 0, \; t_{k+1} := t_k + |u_{k+1}-u_k| + 2n_k\delta_k.$$

This sequence is strictly increasing and

$$t_j \to t_\infty := \sum_{j=1}^{\infty}(t_{j+1}-t_j) = \sum_{j=1}^{\infty}|u_{j+1}-u_j| + 2\sum_{j=1}^{\infty}n_j\delta_j \le 3\sum_{j=1}^{\infty}\frac{1}{k_j}.$$

So to ensure that $t_\infty\in[0,1]$, it is sufficient to suppose that $\sum_{j=1}^{\infty}\frac{1}{k_j}\le\frac{1}{3}$. We define the continuous zig-zag functions $f:[0,1]\to\mathbb{R}$, as

$$f(t):=\begin{cases} u_j, & t=t_j+2i(b_j-a_j),\ i=0,\cdots,[n_j].\\ v_j, & t=t_j+(2i+1)(b_j-a_j),\ i=0,\cdots,[n_j]-1.\\ u_\infty, & t_\infty\le t\le 1.\\ \text{affine}, & \text{otherwise}.\end{cases}$$

Put $t_{j,i}:=t_j+i(v_j-u_j),\ j\in\mathbb{N}, i=0,\cdots,2[n_j]$ and write each interval $I_j=[t_j,t_{j+1}],\ j\in\mathbb{N}$, as the union of the family of non-overlapping ones

$$I_{j,i}:=[t_{j,i},t_{j,i+1}],\ I_{j,2[n_j]}:=[t_{j,2[n_j]},t_{j+1}]\ (i=0,\cdots,[n_j]-1).$$

The function *f* is defined on $I_{j,i},\ i=0,\cdots,2[n_j]$ as follows:

$$f(t)=t-(t_j+2i(v_j-u_j))+u_j\ \ (t\in I_{j,2i})$$

$$f(t)=t-t_j+(2i+1)(v_j-u_j)+v_j\ \ (t\in I_{j,2i+1})$$

$$f(t)=t-t_{j+1}+u_j\ \ \left(t\in I_{j,2[n_j]}\right)$$

Let $0\le s<t\le 1$, then the possibilities for the location of *s* and *t* on $[0,1]$ are as follows

Case 1. If $s,t\in I_j,(j\in\mathbb{N})$ and are in the same interval $I_{j,i},\ i=0,\cdots,2[n_j]$.

$$V^W_{p(\cdot)}(f,[s,t])\le |f(t)-f(s)|^{p(x_{ts})}=|t-s|^{p(x_{ts})}\le 1.$$

Case 2. If $s,t\in I_j,(j\in\mathbb{N})$ and are in two different intervals $I_{j,i},\ i=0,\cdots,2[n_j]$. $s\in I_{j,i},\ t\in I_{j,k},\ i<k<2[n_j]$. We get

$$\begin{aligned}V^W_{p(\cdot)}(f,[s,t])&\le 2^{p^+-1}\left(|f(t_{j,i+1})-f(s)|^{p\left(x_{(j,i+1)s}\right)}+|f(t)-f(t_{j,i+1})|^{p\left(x_{t(j,i+1)}\right)}\right)\\&\le 2^{p^+-1}\left(|u_j-v_j|^{p\left(x_{(j,i+1)s}\right)}+|u_j-v_j|^{p\left(x_{t(j,i+1)}\right)}\right)\\&\le 2^{p^+}\max\left\{|u_j-v_j|^{p\left(x_{(j,i+1)s}\right)},|u_j-v_j|^{p\left(x_{t(j,i+1)}\right)}\right\}\\&\le 2^{p^+}\max\left\{|\delta_j|^{p\left(x_{(j,i+1)s}\right)},|\delta_j|^{p\left(x_{t(j,i+1)}\right)}\right\}<\infty.\end{aligned}$$

Case 3. If $s\in I_j,\ t\in I_k,\ k,j\in\mathbb{N}, j<k$.

$$\begin{aligned}V^W_{p(\cdot)}(f,[s,t])&\le 2^{p^+-1}\left(\left|f\left(t_{j,2[n_j]+1}\right)-f(s)\right|^{p\left(x_{(j,2[n_j]+1)s}\right)}+|f(t)-f(t_{j,i+1})|^{p\left(x_{t(j,i+1)}\right)}\right)\\&\le 2^{p^+-1}\left(|u_j-v_j|^{p\left(x_{(j,2[n_j]+1)s}\right)}+|u_j-v_j|^{p\left(x_{t(j,i+1)}\right)}\right)\\&\le 2^{p^+}\max\left\{|\delta_j|^{p\left(x_{(j,2[n_j]+1)s}\right)},|\delta_j|^{p\left(x_{t(j,i+1)}\right)}\right\}<\infty.\end{aligned}$$

Case 4. If $s\in I_j, j\in\mathbb{N}, t=t_\infty$.

$$\begin{aligned}V_{p(\cdot)}^{W}\left(f,[s,t]\right) &\le 2^{p^+-1}\left(\left|f\left(t_{j,i+1}\right)-f\left(s\right)\right|^{p\left(x_{(j,i+1)s}\right)}+\left|f\left(t_\infty\right)-f\left(t_{j,i+1}\right)\right|^{p\left(x_{\infty(j,i+1)}\right)}\right)\\ &\le 2^{p^+-1}\left(\left|u_j-v_j\right|^{p\left(x_{(j,i+1)s}\right)}+\left|u_\infty-u_j\right|^{p\left(x_{\infty(j,i+1)}\right)}\right)\\ &\le 2^{p^+}\max\left\{\left|\delta_j\right|^{p\left(x_{(j,i+1)s}\right)},\left|\delta_j\right|^{p\left(x_{\infty(j,i+1)}\right)}\right\}<\infty.\end{aligned}$$

Case 5. If $s<t_\infty<t\le 1$.

$$\begin{aligned}V_{p(\cdot)}^{W}\left(f,[s,t]\right) &\le 2^{p^+-1}\left(\left|f\left(t_\infty\right)-f\left(s\right)\right|^{p\left(x_{\infty s}\right)}+\left|f\left(t_\infty\right)-f\left(t\right)\right|^{p\left(x_{\infty t}\right)}\right)\\ &\le 2^{p^+-1}\left(\left|u_\infty-v_j\right|^{p\left(x_{\infty s}\right)}+\left|u_\infty-u_j\right|^{p\left(x_{\infty t}\right)}\right)\\ &\le 2^{p^+}\max\left\{\left|\delta_j\right|^{p\left(x_{\infty s}\right)},\left|\delta_j\right|^{p\left(x_{\infty t}\right)}\right\}<\infty.\end{aligned}$$

Case 6: If $t_\infty\le s<t\le 1$.
In this circumstance $f\left(s\right)=f\left(t\right)=f_\infty$ and the situation is trivial.
So $f\in WBV_{p(\cdot)}[0,1]$, for each partition of the interval $[0,1]$ of the form

$$\begin{aligned}\pi: 0=t_1&<t_1+\left(v_1-u_1\right)<\cdots<t_1+2\left[n_1\right]\left(v_1-u_2\right)\\ &<t_2<t_2+\left(v_2-u_2\right)<\cdots<t_k<\cdots<t_k+2\left[n_k\right]\left(v_k-u_k\right)<1\end{aligned}$$

and using the inequality (5.4) and definition of $n_j,\ j\in\mathbb{N}$, we have

$$\begin{aligned}V_{p(\cdot)}^{W}\left(\left(h\circ f\right),[0,1]\right) &= \sup_{\pi^*}\sum_{j=1}^{n}\left(h\circ f\left(t_j\right)-h\circ f\left(t_{j-1}\right)\right)^{p\left(x_{j-1}\right)}\\ &= \sup_{\pi^*}\sum_{j=1}^{n}\left(h\left(f\left(t_j\right)\right)-h\left(f\left(t_{j-1}\right)\right)\right)^{p\left(x_{j-1}\right)}\\ &\ge \sup_{\pi^*}\sum_{j=1}^{n}\left(2\left[n_j\right]\left(h\left(b_j\right)-h\left(a_j\right)\right)\right)^{p\left(x_{j-1}\right)}\\ &\ge \sup_{\pi^*}\sum_{j=1}^{n}\left(1\right)^{p\left(x_{j-1}\right)}.\end{aligned}$$

Hence series $\sum_{j=1}^{n}\left(1\right)^{p\left(x_{j-1}\right)}$ diverges, $h\circ f\notin WBV_{p(\cdot)}[0,1]$, which is a contradiction. □

6. Uniformly Continuous Composition Operator

In this section, we give the other main result of this paper, namely, we show that any uniformly bounded composition operator that maps the space the $WBV_{p(\cdot)}[a,b]$ into itself necessarily satisfies the so called Matkowski's weak condition.

First of all we will give the definition of left regularization of a function.

Definition 5 *Let* $f\in WBV_{p(\cdot)}\left([a,b]\right)$, *its left regularization* $f^-:(a,b]\to\mathbb{R}$ *of mapping f is the function given as*

$$f^-\left(t\right):=\begin{cases}\lim\limits_{s\to t^-} f\left(s\right) & t\in(a,b];\\ f\left(a\right) & t=a.\end{cases}$$

We will denote by $WBV_{p(\cdot)}^{-}\left([a,b]\right)\left([a,b]\right)$ the subset in $WBV_{p(\cdot)}\left([a,b]\right)$ which consists of those functions that are left continuous on $(a,b]$.

Lemma 6.1 *If* $f\in WBV_{p(\cdot)}\left([a,b]\right)$, *then* $f^-\in WBV_{p(\cdot)}\left([a,b]\right)$.

Thus, if a function *f* has Wiener variation with variable exponent, then its left regularization is a left continuous function.

Theorem 6 *Suppose that the composition operator H generated by* $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ *maps* $WBV_{p(\cdot)}([a,b])$ *into itself and satisfies the following inequality*

$$\|Hf_1-Hf_2\|_{p(\cdot)}^W \le \gamma\left(\|f_1-f_2\|_{p(\cdot)}^W\right) \quad \left(f_1,f_2\in WBV_{p(\cdot)}([a,b])\right) \tag{6.1}$$

for some function $\gamma:[0,\infty)\to[0,\infty)$. Then, there exist functions $\alpha,\beta\in WBV_{p(\cdot)}([a,b])$ such that

$$h^-(t,x)=\alpha(t)x+\beta(t),\ t\in[a,b],\ x\in\mathbb{R} \tag{6.2}$$

where $h^-(\cdot,x):(a,b]\to\mathbb{R}$ is the left regularization of $h(\cdot,x)$ for all $x\in\mathbb{R}$.

Proof. By hypothesis, for $x\in\mathbb{R}$ fixed the constant function $f(t)=x,\ t\in[a,b]$ belongs to $WBV_{p(\cdot)}([a,b])$. Since H maps $WBV_{p(\cdot)}([a,b])$ into itself, we have $(Hf)(t)=h(t,f(t))\in WBV_{p(\cdot)}([a,b])$. By Lemma 6.1 the left regularization $h^-(\cdot,x)\in WBV_{p(\cdot)}^-([a,b])$ for every $x\in\mathbb{R}$.

From the inequality (6.1) and definition of the norm $\|\cdot\|_{p(\cdot)}^W$ we obtain for $f_1,f_2\in WBV_{p(\cdot)}([a,b])$,

$$\mu_{p(\cdot)}(H(f_1)-H(f_2))\le\|H(f_1)-H(f_2)\|_{p(\cdot)}^W\le\gamma\left(\|f_1-f_2\|_{p(\cdot)}^W\right). \tag{6.3}$$

From the inequality (6.3) and Lemma 6.1, if $\gamma\left(\|f_1-f_2\|_{p(\cdot)}^W\right)>0$ then

$$V_{p(\cdot)}^W\left(\frac{H(f_1)-H(f_2)}{\gamma\left(\|f_1-f_2\|_{p(\cdot)}^W\right)}\right)\le 1. \tag{6.4}$$

Let $a\le s<t\le b$, and let $\pi_m:=\{t_0,t_1,\cdots,t_{2m}\}\in\pi$ be the equidistant partition defined by

$$t_0=s,\ t_j-t_{j-1}=\frac{t-s}{2m}\quad (j=1,2,\cdots,2m).$$

Given $u,v\in\mathbb{R}$ with $u\ne v$, define $f_1,f_2:[a,b]\to\mathbb{R}$ by

$$f_1(x):=\begin{cases} v, & \text{if } x=t_j \text{ for some even } j,\\ \dfrac{u+v}{2}, & \text{if } x=t_j \text{ for some odd } j,\\ \text{linear}, & \text{otherwise}\end{cases} \tag{6.5}$$

and

$$f_2(x):=\begin{cases} \dfrac{u+v}{2}, & \text{if } x=t_j \text{ for some even } j,\\ u, & \text{if } x=t_j \text{ for some odd } j,\\ \text{linear}, & \text{otherwise.}\end{cases} \tag{6.6}$$

Then, the difference f_1-f_2 satisfies

$$|f_1(x)-f_2(x)|\equiv\frac{|u-v|}{2}\quad (a\le x\le b).$$

Consequently, by the inequality (6.1)

$$\|Hf_1-Hf_2\|_{p(\cdot)}^W\le\gamma\left(\|f_1-f_2\|_{p(\cdot)}^W\right)\le\gamma\left(\frac{|u-v|}{2}\right).$$

From the inequality (6.4) and the definition of $p(\cdot)$-variation in Wiener's sense, we have

$$\sum_{j=1}^m\left(\frac{\left|(h^-\circ f_1)(t_{2j})-(h^-\circ f_2)(t_{2j})-(h^-\circ f_1)(t_{2j-1})+(h^-\circ f_2)(t_{2j-1})\right|}{\gamma\left(2^{-1}|u-v|\right)}\right)^{p(x_{j-1})}\le 1.$$

However, by definition of the definition of the functions f_1 and f_2,

$$\begin{aligned}&\left|\left(h^{-}\circ f_1\right)\left(t_{2j}\right)-\left(h^{-}\circ f_2\right)\left(t_{2j}\right)-\left(h^{-}\circ f_1\right)\left(t_{2j-1}\right)+\left(h^{-}\circ f_2\right)\left(t_{2j-1}\right)\right|\\&=\left|h^{-}\left(v\right)-h^{-}\left(\frac{u+v}{2}\right)-h^{-}\left(\frac{u+v}{2}\right)+h^{-}\left(u\right)\right|\\&=\left|h^{-}\left(v\right)-2h^{-}\left(\frac{u+v}{2}\right)+h^{-}\left(u\right)\right|.\end{aligned}$$

Then

$$\sum_{j=1}^{m}\left(\frac{\left|h^{-}\left(v\right)-2h^{-}\left(\frac{u+v}{2}\right)+h^{-}\left(u\right)\right|}{\gamma\left(2^{-1}\left|u-v\right|\right)}\right)^{p\left(x_{j-1}\right)}\leq 1.$$

Since $1\leq p\left(x_{j-1}\right)<\infty$ for all $j=1,2,\cdots,2m$ and passing to the limit as $m\to\infty$, necessarily $h^{-}\left(v\right)-2h^{-}\left(\frac{u+v}{2}\right)+h^{-}\left(u\right)=0.$

So, we conclude that $h^{-}\left(s,\cdot\right)$ satisfies the Jensen equation in $\mathbb{R}$ (see [26], p. 315). The continuity of h^{-} with respect of the second variable implies that for every $t\in[a,b]$ there exist $\alpha,\beta:[a,b]\to\mathbb{R}$ such that

$$h^{-}\left(t,x\right)=\alpha\left(t\right)x+\beta\left(t\right)\ \ t\in[a,b],x\in\mathbb{R}.$$

Because $\beta\left(t\right)=h^{-}\left(t,0\right),t\in[a,b]$, $\alpha\left(t\right)=h^{-}\left(t,1\right)-\beta\left(t\right)$ and $h^{-}\left(\cdot,x\right)\in WBV_{p(\cdot)}\left([a,b]\right)$, for each $x\in\mathbb{R}$, we obtain that $\alpha,\beta\in WBV_{p(\cdot)}\left([a,b]\right)$. □

J. Matkowski [27] introduced the notion of a uniformly bounded operator and proved that any uniformly bounded composition operator acting between general Lipschitz function normed spaces must be of the form (11).

Definition 6 ([27], *Def*. 1]) *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be two metric* (*or normed*) *spaces. We say that a mapping* $H:\mathcal{X}\to\mathcal{Y}$ *is uniformly bounded if, for any* $t>0$ *there exists a nonnegative real number* $\gamma\left(t\right)$ *such that for any nonempty set* $B\subset\mathcal{X}$ *we have*

$$diamB\leq t\Rightarrow diamH\left(B\right)\leq\gamma\left(t\right).$$

Remark 6.2 *Every uniformly continuous operator or Lipschitzian operator is uniformly bounded.*

Theorem 7 *Let* $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ *and H the composition operator associated with h. Suppose that H maps* $WBV_{p(\cdot)}\left([a,b]\right)$ *into itself and is uniformly continuous, then, there exist functions* $\alpha,\beta\in WBV_{p(\cdot)}\left([a,b]\right)$ *such that*

$$h^{-}\left(t,x\right)=\alpha\left(t\right)x+\beta\left(t\right),\ \ t\in[a,b],x\in\mathbb{R}.$$

where $h^{-}\left(\cdot,x\right):(a,b]\to\mathbb{R}$ is the left regularization of $h\left(\cdot,x\right)$ for all $x\in\mathbb{R}$.

Proof. Take any $t\geq 0$ and $f,g\in WBV_{p(\cdot)}\left([a,b]\right)$ such that

$$\left\|f-g\right\|_{p(\cdot)}^{W}\leq diamH\left(\left\{f,g\right\}\right)$$

Since $diam\left\{f,g\right\}\leq t$ by the uniform boundedness of *H*, we have

$$diamH\left(\left\{f,g\right\}\right)\leq\gamma\left(t\right),$$

that is,

$$\left\|H\left(f\right)-H\left(g\right)\right\|_{p(\cdot)}^{W}=diamH\left(\left\{f,g\right\}\right)\leq\gamma\left(\left\|f-g\right\|_{p(\cdot)}^{W}\right),$$

therefore, by the Theorem 6 we get

$$h^{-}\left(t,x\right)=\alpha\left(t\right)x+\beta\left(t\right),\ \ \ t\in[a,b],x\in\mathbb{R}.$$

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References

[1] Jordan, C. (1881) Sur la série de Fourier. *Comptes Rendus de l'Académie des Sciences*, *Paris*, **92**, 228-230.

[2] Wiener, N. (1924) The Quadratic Variation of a Function and Its Fourier Coefficients. *Journal of Mathematical Physics*, **3**, 73-94.

[3] Young, L.C. (1937) Sur une généralisation de la notion de variation de pussance piéme bornée au sens de M. Wiener, et sur la convergence des séries de Fourier. *Comptes Rendus de l'Académie des Sciences*, *Paris*, **204**, 470-472.

[4] Love, E.R. and Young, L.C. (1937) Sur une classe de fonctionelles linéaires. *Fundamenta Mathematicae*, **28**, 243-257.

[5] Dudley, R.M. (1994) The Order of the Remainder in Derivatives of Composition and Inverse Operators for *p*-Variation Norms. *Annals of Statistics*, **22**, 1-20. http://dx.doi.org/10.1214/aos/1176325354

[6] Dudley, R.M. (1997) Empirical Processes and *p*-Variation. In: Pollard, D., Torgersen, E. and Yang, G.L., Eds., *Festschrift for Lucien Le Cam*, Springer, New York, 219-233. http://dx.doi.org/10.1007/978-1-4612-1880-7_13

[7] Dudley, R.M. and Norvaiša, R. (1999) Differentiability of Six Operators on Nonsmooth Functions and *p*-Variation. *Lecture Notes in Mathematics*, **1703**. Springer, Berlin.

[8] Appell, J., Banas, J. and Merentes, N. (2014) Bounded Variation and Around. De Gruyter, Boston.

[9] Chistyakov, V.V. and Galkin, O.E. (1998) On Maps of Bounded *p*-Variation with $p > 1$. *Positivity*, **2**, 19-45. http://dx.doi.org/10.1023/A:1009700119505

[10] Diening, L. (2004) Maximal Function on Generalized Lebesgue Spaces $L^{p(\cdot)}$. *Mathematical Inequalities & Applications*, **7**, 245-253. http://dx.doi.org/10.7153/mia-07-27

[11] Azroul, E., Barbara, A. and Redwane, H. (2014) Existence and Nonexistence of a Solution for a Nonlinear $p(x)$-Elliptic Problem with Right-Hand Side Measure. *International Journal of Analysis*, **2014**, Article ID: 320527. http://dx.doi.org/10.1155/2014/320527

[12] Fan, X.L., Zhao, Y.Z. and Zhao, D. (2001) Compact Imbedding Theorems with Symmetry of Strauss-Lions Type for the Space $W^{1,p(x)}(\Omega)$. *Journal of Mathematical Analysis and Applications*, **255**, 333-348. http://dx.doi.org/10.1006/jmaa.2000.7266

[13] Yin, L., Liang, Y., Zhang, Q. and Zhao, C.S. (2015) Existence of Solutions for a Variable Exponent System without PS Conditions. *Journal of Differential Equations*, **2015**, 1-23.

[14] Rădulescu, V.D. and Repovš, D.D. (2015) Partial Differential Equations with Variable Exponent: Variational Methods and Qualitative Analysis. CRC Press, Taylor & Francis Group, Boca Raton.

[15] Orlicz, W. (1931) Über konjugierte exponentenfolgen. *Studia Mathematica*, **3**, 200-211.

[16] Nakano, H. (1950) Modulared Semi-Ordered Linear Spaces. Maruzen Co., Ltd., Tokyo.

[17] Nakano, H. (1951) Topology and Topological Linear Spaces. Maruzen Co., Ltd., Tokyo.

[18] Musielak, J. (1983) Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics. Vol. 1034, Springer-Verlag, Berlin.

[19] Musielak, J. and Orlicz, W. (1959) On Modular Spaces. *Studia Mathematica*, **18**, 49-65.

[20] Kováčik, O. and Rákosník, J. (1991) On Spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Mathematical Journal*, **41**, 592-618.

[21] Castillo, R., Merentes, N. and Rafeiro, H. (2014) Bounded Variation Spaces with *p*-Variable. *Mediterranean Journal of Mathematics*, **11**, 1069-1079. http://dx.doi.org/10.1007/s00009-013-0342-5

[22] Federer, H. (1969) Geometric Measure Theory. Springer-Verlag, Heidelberg.

[23] Sierpiński, W. (1933) Sur une propriété des fonctions qui n'ont que des discontinuités de première espèce. *Bulletin de la Section Scientifique de l'Académie Roumaine*, **16**, 1-4.

[24] Merentes, N. and Rivas, S. (1996) El Operador de Composición en Espacios de Funciones con Algún Tipo de Variación Acotada. IX Escuela Venezolana de Matemáticas, Facultad de Ciencias-ULA, Mérida.

[25] Appell, J. and Zabreĭko, P.P. (1990) Nonlinear Superposition Operators. Cambridge University Press, Cambrige. http://dx.doi.org/10.1017/CBO9780511897450

[26] Kuczma, M. (1885) An Introduction to the Theory of Functional Equations and Inequalities. Polish Scientific Editors and Silesian University, Warszawa-Krakow-Katowice.

[27] Matkowski, J. (2011) Uniformly Bounded Composition Operators between General Lipschitz Function Normed Spaces. *Topological Methods in Nonlinear Analysis*, **38**, 395-405.