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The Estimates of Diagonally Dominant Degree and Eigenvalue Inclusion Regions for the Schur Complement of Matrices

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Abstract

The theory of Schur complement plays an important role in many fields such as matrix theory, control theory and computational mathematics. In this paper, some new estimates of diagonally, α-diagonally and product α-diagonally dominant degree on the Schur complement of matrices are obtained, which improve some relative results. As an application, we present several new eigenvalue inclusion regions for the Schur complement of matrices. Finally, we give a numerical example to illustrate the advantages of our derived results.

Keywords

Schur Complement, Gerschgorin Theorem, Diagonally Dominant Degree, Eigenvalue

1. Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ complex matrices, $N=\{1,2,\cdots,n\}$ and $\boldsymbol{A}=(a_{ij})\in\mathbb{C}^{n\times n}\,(n\geq 2)$. We write

$$R_i(\boldsymbol{A})=\sum_{j\neq i}|a_{ij}|,\ C_i(\boldsymbol{A})=\sum_{j\neq i}|a_{ji}|,\ i\in N,$$

$$N_r(\boldsymbol{A})=\{i\,\|a_{ii}|>R_i(\boldsymbol{A}),i\in N\},\ N_c(\boldsymbol{A})=\{i\,\|a_{ii}|>C_i(\boldsymbol{A}),i\in N\}.$$

We know that $\boldsymbol{A}$ is called a strictly diagonally dominant matrix if

$$|a_{ii}|>R_i(\boldsymbol{A}),\ \forall\, i\in N.$$

$\boldsymbol{A}$ is called an Ostrowski matrix (see [1]) if

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$$|a_{ii}||a_{jj}| > R_i(\boldsymbol{A})R_j(\boldsymbol{A}),\ \forall i, j \in N, i \neq j.$$

SD_n and OS_n will be used to denote the sets of all $n\times n$ strictly diagonally dominant matrices and the sets all $n\times n$ Ostrowski matrices, respectively.

As shown in [2], for $1\le i\le n$ and $\alpha\in[0,1]$, we call $|a_{ii}|-R_i(\boldsymbol{A})$, $|a_{ii}|-\alpha R_i(\boldsymbol{A})-(1-\alpha)C_i(\boldsymbol{A})$ and $|a_{ii}|-[R_i(\boldsymbol{A})]^{\alpha}[C_i(\boldsymbol{A})]^{1-\alpha}$ the *i*-th diagonally, *α*-diagonally and product *α*-diagonally dominant degree of $\boldsymbol{A}$, respectively.

For $\beta\subseteq N$, denote by $|\beta|$ the cardinality of *β* and $\overline{\beta}=N/\beta$. If $\beta,\gamma\subseteq N$, then $\boldsymbol{A}(\beta,\gamma)$ is the submatrix of $\boldsymbol{A}$ with row indices in *β* and column indices in γ. In particular, $\boldsymbol{A}(\beta,\beta)$ is abbreviated to $\boldsymbol{A}(\beta)$. If $\boldsymbol{A}(\beta)$ is nonsingular,

$$\boldsymbol{A}/\beta=\boldsymbol{A}/\boldsymbol{A}(\beta)=\boldsymbol{A}(\overline{\beta})-\boldsymbol{A}(\overline{\beta},\beta)[\boldsymbol{A}(\beta)]^{-1}\boldsymbol{A}(\beta,\overline{\beta}),$$

is called the Schur complement of $\boldsymbol{A}$ with respect to $\boldsymbol{A}(\beta)$.

The comparison matrix of $\boldsymbol{A}$, $\mu(\boldsymbol{A})=(\alpha_{ij})$, is defined by

$$\alpha_{ij}=\begin{cases}|a_{ij}|, & \text{if } i=j,\\ -|a_{ij}|, & \text{if } i\neq j.\end{cases}$$

A matrix $\boldsymbol{A}=(a_{ij})\in\mathbb{C}^{n\times n}$ is called an *M*-matrix, if there exist a nonnegative matrix *B* and a real number $s>\rho(\boldsymbol{B})$, where $\rho(\boldsymbol{B})$ is the spectral radius of $\boldsymbol{B}$, such that $\boldsymbol{A}=s\boldsymbol{I}-\boldsymbol{B}$. It is known that *A* is an *h*-matrix if and only if $\mu(\boldsymbol{A})$ is an *m*-matrix, and if *A* is an *m*-matrix, then the Schur complement of *A* is also an *m*-matrix and $\det\boldsymbol{A}>0$ (see [3]). We denote by H_n and M_n the sets of *h*-matrices and *m*-matrices, respectively.

The Schur complement of matrix is an important part of matrix theory, which has been proved to be useful tools in many fields such as control theory, statistics and computational mathematics. A lot of work has been done on it (see [4]-[8]). We know that the Schur complements of strictly diagonally dominant matrices are strictly diagonally dominant matrices, and the Schur complements of Ostrowski matrices are Ostrowski matrices. These properties have been used for deriving matrix inequalities in matrix analysis and for the convergence of iterations in numerical analysis (see [9]-[12]). More importantly, studying the locations for the eigenvalues of the Schur complement is of great significance, as shown in [2] [6] [13]-[18].

The paper is organized as follows. In Section 2, we give some new estimates of diagonally dominant degree on the Schur complement of matrices. In Section 3, we present several new eigenvalue inclusion regions for the Schur complement of matrices. In Section 4, we give a numerical example to illustrate the advantages of our derived results.

2. The Diagonally Dominant Degree for the Schur Complement

In this section, we present several new estimates of diagonally, *α*-diagonally and product *α*-diagonally dominant degree on the Schur complement of matrices.

Lemma 1. [3] *If* $\boldsymbol{A}\in H_n$, *then* $[\mu(\boldsymbol{A})]^{-1}\ge|\boldsymbol{A}^{-1}|$.

Lemma 2. [3] *If* $\boldsymbol{A}\in SD_n$ *or* $\boldsymbol{A}\in OS_n$, *then* $\boldsymbol{A}\in H_n$, *i.e.,* $\mu(\boldsymbol{A})\in M_n$.

Lemma 3. [6] *If* $\boldsymbol{A}\in SD_n$ *or* $\boldsymbol{A}\in OS_n$ *and* $\beta\subseteq N$, *then the Schur complement of* $\boldsymbol{A}$ *is in* $SD_{|\overline{\beta}|}$ *or* $OS_{|\overline{\beta}|}$, *where* $\overline{\beta}=N-\beta$ *is the complement of β in N and* $|\overline{\beta}|$ *is the cardinality of* $\overline{\beta}$.

Lemma 4. [16] *Let* $a>b$, $c>b$, $b>0$ *and* $0\le\alpha\le1$. *Then*

$$a^{\alpha}c^{1-\alpha}\ge(a-b)^{\alpha}(c-b)^{1-\alpha}+b.$$

Theorem 1. *Let* $\boldsymbol{A}=(a_{ij})\in\mathbb{C}^{n\times n}$, $\beta=\{i_1,i_2,\cdots,i_k\}\subseteq N_r(\boldsymbol{A})\neq\phi$, $\overline{\beta}=\{j_1,j_2,\cdots,j_l\}$, $1\le k<n$ *and* $\boldsymbol{A}/\beta=(a'_{ts})$. *Then for all* $1\le t\le l$,

$$|a'_{tt}|-R_t(\boldsymbol{A}/\beta)\ge|a_{j_tj_t}|-R_{j_t}(\boldsymbol{A})+\delta_{j_t}\ge|a_{j_tj_t}|-R_{j_t}(\boldsymbol{A}),\tag{1}$$

and

$$|a'_{tt}|+R_t(\boldsymbol{A}/\beta)\le|a_{j_tj_t}|+R_{j_t}(\boldsymbol{A})-\delta_{j_t}\le|a_{j_tj_t}|+R_{j_t}(\boldsymbol{A}),\tag{2}$$

where

$$\delta_{j_t} = \sum_{v=1}^{k} \left|a_{j_t i_v}\right| \frac{\left|a_{i_v i_v}\right| - P_{i_v}(\boldsymbol{A})}{|a_{i_v i_v}|},\quad r = \max_{1\le v\le k} \frac{\left|a_{i_v i_v}\right| \left|\sum_{u=1}^{l} \left|a_{i_v j_u}\right|\right|}{\left(\left|a_{i_v i_v}\right| - \sum_{u=1,\neq v}^{k} \left|a_{i_v i_u}\right|\right) R_{i_v}},$$

$$P_{i_v}(\boldsymbol{A}) = rR_{i_v},\ 1 \le v \le k.$$

Proof. Since $\beta \subseteq N_r(\boldsymbol{A}) \neq \phi$, then $\boldsymbol{A}(\beta) \in H_k$ and $\mu(\boldsymbol{A}(\beta)) \in M_k$. From Lemma 1 and Lemma 2, we have

$$\left[\mu(\boldsymbol{A}(\beta))\right]^{-1} \ge \left|\left[\boldsymbol{A}(\beta)\right]^{-1}\right|.$$

Thus, for any $\varepsilon > 0$ and $1 \le t \le l$, we obtain

$$\begin{aligned}
&\left|a'_{tt}\right| - R_t(\boldsymbol{A}/\beta) \\
&= \left| a_{j_t j_t} - \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right) \left[\boldsymbol{A}(\beta)\right]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| - \sum_{s\neq t}^{l} \left| a_{j_t j_s} - \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right) \left[\boldsymbol{A}(\beta)\right]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
&\ge \left|a_{j_t j_t}\right| - \sum_{s\neq t}^{l} \left|a_{j_t j_s}\right| - \sum_{s=1}^{l} \left(\left|a_{j_t i_1}\right|, \cdots, \left|a_{j_t i_k}\right|\right) \left[\mu(\boldsymbol{A}(\beta))\right]^{-1} \begin{pmatrix} \left|a_{i_1 j_s}\right| \\ \vdots \\ \left|a_{i_k j_s}\right| \end{pmatrix} \\
&= \left|a_{j_t j_t}\right| - R_{j_t}(\boldsymbol{A}) + \sum_{v=1}^{k} \left|a_{j_t i_v}\right| + \left(\delta_{j_t} - \varepsilon\right) - \left(\delta_{j_t} - \varepsilon\right) - \sum_{s=1}^{l} \left(\left|a_{j_t i_1}\right|, \cdots, \left|a_{j_t i_k}\right|\right) \left[\mu(\boldsymbol{A}(\beta))\right]^{-1} \begin{pmatrix} \left|a_{i_1 j_s}\right| \\ \vdots \\ \left|a_{i_k j_s}\right| \end{pmatrix} \\
&= \left|a_{j_t j_t}\right| - R_{j_t}(\boldsymbol{A}) + \delta_{j_t} - \varepsilon + \frac{1}{\det\left[\mu(\boldsymbol{A}(\beta))\right]} \det \begin{pmatrix} \sum_{v=1}^{k} \left|a_{j_t i_v}\right| - \delta_{j_t} + \varepsilon & -\left|a_{j_t i_1}\right| & \cdots & -\left|a_{j_t i_k}\right| \\ -\sum_{s=1}^{l} \left|a_{i_1 j_s}\right| & & & \\ \vdots & & \mu(\boldsymbol{A}(\beta)) & \\ -\sum_{s=1}^{l} \left|a_{i_k j_s}\right| & & & \end{pmatrix}.
\end{aligned}$$

For any $j_t \in \overline{\beta}$, denote

$$\boldsymbol{B}_t \equiv \begin{pmatrix} x & -\left|a_{j_t i_1}\right| & \cdots & -\left|a_{j_t i_k}\right| \\ -\sum_{v=1}^{l} \left|a_{i_1 j_v}\right| & & & \\ \vdots & & \mu(\boldsymbol{A}(\beta)) & \\ -\sum_{v=1}^{l} \left|a_{i_k j_v}\right| & & & \end{pmatrix}.$$

If

$$x > \sum_{v=1}^{k} \left|a_{j_t i_v}\right| \frac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|},$$

then there exists sufficiently small positive number ε_0 such that

$$x > \sum_{v=1}^{k} \left|a_{j_t i_v}\right| \left(\frac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} + \varepsilon_0 \right). \tag{3}$$

Construct a positive diagonal matrix $\boldsymbol{X} = diag\left(x_1, x_2, \cdots, x_{k+1}\right)$, where

$$x_t = \begin{cases} 1, & \text{if } t = 1 \\ \dfrac{P_{i_{t-1}}(\boldsymbol{A})}{\left|a_{i_{t-1} i_{t-1}}\right|} + \varepsilon_0, & \text{if } t = 2, 3, \cdots, k+1. \end{cases}$$

Let $\tilde{\boldsymbol{B}} = \boldsymbol{B}_t X = \left(\tilde{b}_{pq}\right)$. For $p = 1$, by (3), we have

$$\left|\tilde{b}_{pp}\right| - R_p\left(\tilde{\boldsymbol{B}}\right) = \left|\tilde{b}_{11}\right| - \sum_{j=2}^{k+1} \left|\tilde{b}_{1j}\right| = x - \sum_{v=1}^{k} \left|a_{j_t i_v}\right| \left(\frac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} + \varepsilon_0 \right) > 0.$$

And for $p = 2, 3, \cdots, k+1$, by $\dfrac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} \le r$, $1 \le v \le k$, we obtain

$$\begin{aligned} &\left|\tilde{b}_{pp}\right| - R_p\left(\tilde{\boldsymbol{B}}\right) \\ &= \left|a_{i_{p-1} i_{p-1}}\right| \left(\frac{P_{i_{p-1}}(\boldsymbol{A})}{\left|a_{i_{p-1} i_{p-1}}\right|} + \varepsilon_0 \right) - \sum_{v \ne p-1}^{k} \left|a_{i_{p-1} i_v}\right| \left(\frac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} + \varepsilon_0 \right) - \sum_{v=1}^{l} \left|a_{i_{p-1} j_v}\right| \\ &= P_{i_{p-1}}(\boldsymbol{A}) + \varepsilon_0 \left(\left|a_{i_{p-1} i_{p-1}}\right| - \sum_{v \ne p-1}^{k} \left|a_{i_{p-1} i_v}\right| \right) - \sum_{v \ne p-1}^{k} \left|a_{i_{p-1} i_v}\right| \frac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} - \sum_{v=1}^{l} \left|a_{i_{p-1} j_v}\right| \\ &= \varepsilon_0 \left(\left|a_{i_{p-1} i_{p-1}}\right| - \sum_{v \ne p-1}^{k} \left|a_{i_{p-1} i_v}\right| \right) + r \sum_{v \ne p-1}^{k} \left|a_{i_{p-1} i_v}\right| - \sum_{v \ne p-1}^{k} \left|a_{i_{p-1} i_v}\right| \frac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} > 0. \end{aligned}$$

Thus, $\tilde{\boldsymbol{B}} \in SD_{k+1}$, and so $\boldsymbol{B}_t \in H_{k+1}$. Note that $B_t = \mu\left(\boldsymbol{B}_t\right) \in M_{k+1}$, then

$$\det \boldsymbol{B}_t > 0. \tag{4}$$

Let x be $\sum_{v=1}^{k} \left|a_{j_t i_v}\right| - \delta_{j_t} + \varepsilon$ in $\boldsymbol{B}_t$. Then

$$\begin{aligned} &\sum_{v=1}^{k} \left|a_{j_t i_v}\right| - \delta_{j_t} + \varepsilon - \sum_{v=1}^{k} \left|a_{j_t i_v}\right| \frac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} \\ &= \sum_{v=1}^{k} \left|a_{j_t i_v}\right| - \sum_{v=1}^{k} \left|a_{j_t i_v}\right| \frac{\left|a_{i_v i_v}\right| - P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} - \sum_{v=1}^{k} \left|a_{j_t i_v}\right| \frac{P_{i_v}(\boldsymbol{A})}{\left|a_{i_v i_v}\right|} + \varepsilon > 0. \end{aligned}$$

Since $\det\left[\mu\left(\boldsymbol{A}(\beta)\right)\right] > 0$, by (4), we have

$$\left|a'_{tt}\right| - R_t\left(\boldsymbol{A}/\beta\right) > \left|a_{j_t j_t}\right| - R_{j_t}(\boldsymbol{A}) + \delta_{j_t} - \varepsilon.$$

Let $\varepsilon \to 0$. Then we obtain (1). Similarly, we can prove (2). □

Remark 1. *Note that*

$$P_{i_v}(\boldsymbol{A}) \le R_{i_v}(\boldsymbol{A}), \ 1 \le v \le k.$$

This shows that Theorem 1 *improves Theorem* 2 *of* [17] *and* [2], *respectively*.

Next, we present some new estimates of α-diagonally and product α-diagonally dominant degree of the Schur complement.

Theorem 2. *Let* $A=(a_{ij})\in\mathbb{C}^{n\times n}$, $\beta=\{i_1,i_2,\cdots,i_k\}\subseteq N_r(A)\cap N_c(A)\neq\phi$, $\overline{\beta}=\{j_1,j_2,\cdots,j_l\}$, $1\le k<n$ *and* $A/\beta=(a'_{ts})$. *Then for all* $1\le t\le l$, $0\le\alpha\le 1$,

$$|a'_{tt}|-(R_t(A/\beta))^{\alpha}(C_t(A/\beta))^{1-\alpha}\ge|a_{j_tj_t}|-(R_{j_t}(A)-\delta_t)^{\alpha}(C_{j_t}(A)-\delta_t^T)^{1-\alpha}, \quad (5)$$

and

$$|a'_{tt}|+(R_t(A/\beta))^{\alpha}(C_t(A/\beta))^{1-\alpha}\le|a_{j_tj_t}|+(R_{j_t}(A)-\delta_t)^{\alpha}(C_{j_t}(A)-\delta_t^T)^{1-\alpha}, \quad (6)$$

where for any $1\le v\le k$,

$$\delta_t=\sum_{v=1}^{k}|a_{j_ti_v}|\frac{|a_{i_vi_v}|-P_{i_v}(A)}{|a_{i_vi_v}|},\ \eta=\max_{1\le\omega\le k}\frac{|a_{i_\omega i_\omega}|\sum_{v=1}^{l}|a_{i_\omega j_v}|}{\left(|a_{i_\omega i_\omega}|-\sum_{v\neq\omega}^{k}|a_{i_\omega i_v}|\right)R_{i_\omega}(A)},$$

$$\delta_t^T=\sum_{v=1}^{k}|a_{i_vj_t}|\frac{|a_{i_vi_v}|-Q_{i_v}(A)}{|a_{i_vi_v}|},\ \xi=\max_{1\le\omega\le k}\frac{|a_{i_\omega i_\omega}|\sum_{v=1}^{l}|a_{j_vi_\omega}|}{\left(|a_{i_\omega i_\omega}|-\sum_{v\neq\omega}^{k}|a_{i_vi_\omega}|\right)C_{i_\omega}(A)},$$

$$P_{i_v}(A)=\eta R_{i_v}(A),\ Q_{i_v}(A)=\xi C_{i_v}(A).$$

Proof. By Lemma 1 and Lemma 2, we have $[\mu(A(\beta))]^{-1}\ge\left|[A(\beta)]^{-1}\right|$. Thus, for all $1\le t\le l$, $0\le\alpha\le1$, we have

$$\begin{aligned}
&|a'_{tt}|-(R_t(A/\beta))^{\alpha}(C_t(A/\beta))^{1-\alpha}\\
&=\left|a_{j_tj_t}-(a_{j_ti_1},\cdots,a_{j_ti_k})[A(\beta)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|-\left[\sum_{s\neq t}^{l}\left|a_{j_tj_s}+(a_{j_ti_1},\cdots,a_{j_ti_k})[A(\beta)]^{-1}\begin{pmatrix}a_{i_1j_s}\\ \vdots\\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\alpha}\\
&\times\left[\sum_{s\neq t}^{l}\left|a_{j_sj_t}+(a_{j_si_1},\cdots,a_{j_si_k})[A(\beta)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|\right]^{1-\alpha}\\
&\ge|a_{j_tj_t}|-(|a_{j_ti_1}|,\cdots,|a_{j_ti_k}|)[\mu(A(\beta))]^{-1}\begin{pmatrix}|a_{i_1j_t}|\\ \vdots\\ |a_{i_kj_t}|\end{pmatrix}-\left(\sum_{s\neq t}^{l}\left[|a_{j_tj_s}|+(|a_{j_ti_1}|,\cdots,|a_{j_ti_k}|)[\mu(A(\beta))]^{-1}\begin{pmatrix}|a_{i_1j_s}|\\ \vdots\\ |a_{i_kj_s}|\end{pmatrix}\right]\right)^{\alpha}\\
&\times\left(\sum_{s\neq t}^{l}\left[|a_{j_sj_t}|+(|a_{j_si_1}|,\cdots,|a_{j_si_k}|)[\mu(A(\beta))]^{-1}\begin{pmatrix}|a_{i_1j_t}|\\ \vdots\\ |a_{i_kj_t}|\end{pmatrix}\right]\right)^{1-\alpha}.
\end{aligned}$$

Let

$$\zeta=(|a_{j_ti_1}|,\cdots,|a_{j_ti_k}|)[\mu(A(\beta))]^{-1}\begin{pmatrix}|a_{i_1j_t}|\\ \vdots\\ |a_{i_kj_t}|\end{pmatrix}.$$

Similar as the proof of Theorem 1, we can prove

$$\sum_{s\neq t}^{l}\left[\left|a_{j_t j_s}\right|+\left(\left|a_{j_t i_1}\right|,\cdots,\left|a_{j_t i_k}\right|\right)\left[\mu\left(\boldsymbol{A}(\beta)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_s}\right|\\ \vdots\\ \left|a_{i_k j_s}\right|\end{pmatrix}\right]\leq R_{j_t}\left(\boldsymbol{A}\right)-\delta_t-\zeta.$$

Similarly, we have

$$\sum_{s\neq t}^{l}\left[\left|a_{j_s j_t}\right|+\left(\left|a_{j_s i_1}\right|,\cdots,\left|a_{j_s i_k}\right|\right)\left[\mu\left(\boldsymbol{A}(\beta)\right)\right]^{-1}\begin{pmatrix}\left|a_{i_1 j_t}\right|\\ \vdots\\ \left|a_{i_k j_t}\right|\end{pmatrix}\right]\leq C_{j_t}\left(\boldsymbol{A}\right)-\delta_t^T-\zeta.$$

By Lemma 4, we have

$$\begin{aligned}&\left|a'_{tt}\right|-\left(R_t\left(\boldsymbol{A}/\beta\right)\right)^{\alpha}\left(C\left(\boldsymbol{A}/\beta\right)\right)^{1-\alpha}\\&\geq\left|a_{j_t j_t}\right|-\zeta-\left(R_{j_t}\left(\boldsymbol{A}\right)-\delta_t-\zeta\right)^{\alpha}\left(C_{j_t}\left(\boldsymbol{A}\right)-\delta_t^T-\zeta\right)^{1-\alpha}\\&\geq\left|a_{j_t j_t}\right|-\zeta-\left[\left(R_{j_t}\left(\boldsymbol{A}\right)-\delta_t\right)^{\alpha}\left(C_{j_t}\left(\boldsymbol{A}\right)-\delta_t^T\right)^{1-\alpha}-\zeta\right]\\&=\left|a_{j_t j_t}\right|-\left(R_{j_t}\left(\boldsymbol{A}\right)-\delta_t\right)^{\alpha}\left(C_{j_t}\left(\boldsymbol{A}\right)-\delta_t^T\right)^{1-\alpha}.\end{aligned}$$

Hence, (5) holds. Similarly, we can prove (6).

Remark 2. *Note that*

$$P_{i_v}\left(\boldsymbol{A}\right)\leq R_{i_v}\left(\boldsymbol{A}\right),\quad Q_{i_v}\left(\boldsymbol{A}\right)\leq C_{i_v}\left(\boldsymbol{A}\right).$$

This shows that Theorem 3 *improves Theorem* 4 *of* [2].

Similar as the proof of Theorem 2, we can prove the following theorem immediately, which improves Theorem 2 of [2].

Theorem 3. *Let* $\boldsymbol{A}=\left(a_{ij}\right)\in\mathbb{C}^{n\times n}$, $\beta=\{i_1,i_2,\cdots,i_k\}\subseteq N_r\left(A\right)\cap N_c\left(A\right)\neq\phi$, $\overline{\beta}=\{j_1,j_2,\cdots,j_l\}$, $1\leq k<n$ *and* $\boldsymbol{A}/\beta=\left(a'_{ts}\right)$. *Then for all* $1\leq t\leq l$, $0\leq\alpha\leq1$,

$$\begin{aligned}&\left|a'_{tt}\right|-\alpha R_t\left(\boldsymbol{A}/\beta\right)-\left(1-\alpha\right)C_t\left(\boldsymbol{A}/\beta\right)\\&\geq\left|a_{j_t j_t}\right|-\alpha R_{j_t}\left(\boldsymbol{A}\right)-\left(1-\alpha\right)C_{j_t}\left(\boldsymbol{A}\right)+\alpha\delta_t+\left(1-\alpha\right)\delta_t^T\\&\geq\left|a_{j_t j_t}\right|-\alpha R_{j_t}\left(\boldsymbol{A}\right)-\left(1-\alpha\right)C_{j_t}\left(\boldsymbol{A}\right),\end{aligned}$$

and

$$\begin{aligned}&\left|a'_{tt}\right|+\alpha R_t\left(\boldsymbol{A}/\beta\right)+\left(1-\alpha\right)C_t\left(\boldsymbol{A}/\beta\right)\\&\leq\left|a_{j_t j_t}\right|+\alpha R_{j_t}\left(\boldsymbol{A}\right)+\left(1-\alpha\right)C_{j_t}\left(\boldsymbol{A}\right)-\alpha\delta_t-\left(1-\alpha\right)\delta_t^T\\&\leq\left|a_{j_t j_t}\right|+\alpha R_{j_t}\left(\boldsymbol{A}\right)+\left(1-\alpha\right)C_{j_t}\left(\boldsymbol{A}\right).\end{aligned}$$

3. Eigenvalue Inclusion Regions of the Schur Complement

In this section, based on these derived results in Section 2, we present new eigenvalue inclusion regions for the Schur complement of matrices.

Theorem 4. *Let* $\boldsymbol{A}=\left(a_{ij}\right)\in\mathbb{C}^{n\times n}$, $\beta=\{i_1,i_2,\cdots,i_k\}\subseteq N_r\left(\boldsymbol{A}\right)\neq\phi$, $\overline{\beta}=\{j_1,j_2,\cdots,j_l\}$, $1\leq k<n$ *and* $\boldsymbol{A}/\beta=\left(a'_{ts}\right)$ *and* λ *be eigenvalue of* $\boldsymbol{A}/\beta$. *Then there exists* $1\leq t\leq l$ *such that*

$$\left|\lambda-a_{j_t j_t}\right|\leq R_{j_t}\left(\boldsymbol{A}\right)-\delta_{j_t}\leq R_{j_t}\left(\boldsymbol{A}\right). \tag{7}$$

Proof. By Gerschgorin Circle Theorem, we know that there exists $1\leq t\leq l$ such that $\left|\lambda-a'_{tt}\right|\leq R_t\left(\boldsymbol{A}/\beta\right)$. Thus, by Lemma 1 and Lemma 2, we have

$$
\begin{aligned}
0 &\ge \left|\lambda - a'_{tt}\right| - R_t\left(A/\beta\right) \\
&= \left|\lambda - a_{j_t j_t} + \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| - \sum_{s=1,\ne t}^{l}\left|a_{j_t j_s} - \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right| \\
&\ge \left|\lambda - a_{j_t j_t}\right| - \sum_{s=1,\ne t}^{l}\left|a_{j_t j_s}\right| - \sum_{s=1}^{l}\left(\left|a_{j_t i_1}\right|, \cdots, \left|a_{j_t i_k}\right|\right)\left[\mu\left(A(\beta)\right)\right]^{-1}\begin{pmatrix} \left|a_{i_1 j_s}\right| \\ \vdots \\ \left|a_{i_k j_s}\right|\end{pmatrix} \\
&= \left|\lambda - a_{j_t j_t}\right| - R_{j_t}(A) + \sum_{v=1}^{k}\left|a_{j_t i_v}\right| + \delta_{j_t} - \delta_{j_t} - \sum_{s=1}^{l}\left(\left|a_{j_t i_1}\right|, \cdots, \left|a_{j_t i_k}\right|\right)\left[\mu\left(A(\beta)\right)\right]^{-1}\begin{pmatrix} \left|a_{i_1 j_s}\right| \\ \vdots \\ \left|a_{i_k j_s}\right|\end{pmatrix} \\
&\ge \left|\lambda - a_{j_t j_t}\right| - R_{j_t}(A) + \delta_{j_t},
\end{aligned}
$$

i.e.,

$$\left|\lambda - a_{j_t j_t}\right| \le R_{j_t}(A) - \delta_{j_t} \le R_{j_t}(A).$$

Thus, (7) holds.

Lemma 5. [2] *Let* $A = \left(a_{ij}\right) \in \mathbb{C}^{n\times n}$ *and* $0 \le \alpha \le 1$. *Then for any eigenvalue* μ *of* **A**, *there exists* $1 \le t \le n$ *such that*

$$\left|\mu - a_{tt}\right| \le \left(R_t(A)\right)^{\alpha}\left(C_t(A)\right)^{1-\alpha}.$$

Theorem 5. *Let* $A = \left(a_{ij}\right) \in \mathbb{C}^{n\times n}$, $\beta = \left\{i_1, i_2, \cdots, i_k\right\} \subseteq N_r(A) \cap N_c(A) \ne \phi$, $\bar{\beta} = \left\{j_1, j_2, \cdots, j_l\right\}$, $1 \le k < n$, $A/\beta = \left(a'_{ts}\right)$ *and* λ *be eigenvalue of* A/β. *Then for any* $0 \le \alpha \le 1$, *there exists* $1 \le t \le l$ *such that*

$$\left|\lambda - a_{j_t j_t}\right| \le \left(R_{j_t}(A) - \delta_t\right)^{\alpha}\left(C_{j_t}(A) - \delta_t^{T}\right)^{1-\alpha} \le \left(R_{j_t}(A)\right)^{\alpha}\left(C_{j_t}(A)\right)^{1-\alpha}. \tag{8}$$

Proof. By Lemma 5, we know that there exists $1 \le t \le l$ such that

$$\left|\lambda - a'_{tt}\right| \le \left(R_t\left(A/\beta\right)\right)^{\alpha}\left(C_t\left(A/\beta\right)\right)^{1-\alpha}.$$

Therefore,

$$
\begin{aligned}
0 &\ge \left|\lambda - a'_{tt}\right| - \left(R_t\left(A/\beta\right)\right)^{\alpha}\left(C_t\left(A/\beta\right)\right)^{1-\alpha} \\
&= \left|\lambda - a_{j_t j_t} - \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| - \left[\sum_{s=1,\ne t}^{l}\left|a_{j_t j_s} + \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right|\right]^{\alpha} \\
&\quad \times \left[\sum_{s=1,\ne t}^{l}\left|a_{j_s j_t} + \left(a_{j_s i_1}, \cdots, a_{j_s i_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right|\right]^{1-\alpha} \\
&\ge \left|\lambda - a_{j_t j_t}\right| - \left|\left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| - \left(\sum_{s=1,\ne t}^{l}\left[\left|a_{j_t j_s}\right| + \left|\left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right|\right]\right)^{\alpha} \\
&\quad \times \left(\sum_{s=1,\ne t}^{l}\left[\left|a_{j_s j_t}\right| + \left|\left(a_{j_s i_1}, \cdots, a_{j_s i_k}\right)\left[A(\beta)\right]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right|\right]\right)^{1-\alpha}.
\end{aligned}
$$

Similar as the proof of Theorem 2, we can prove

$$\left|\left(a_{j_t i_1},\cdots,a_{j_t i_k}\right)\left[\boldsymbol{A}(\beta)\right]^{-1}\begin{pmatrix}a_{i_1 j_t}\\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right|$$
$$+\left(\sum_{s=1,\neq t}^{l}\left[\left|a_{j_t j_s}\right|+\left|\left(a_{j_t i_1},\cdots,a_{j_t i_k}\right)\left[\boldsymbol{A}(\beta)\right]^{-1}\begin{pmatrix}a_{i_1 j_s}\\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right|\right]\right)^{\alpha}$$
$$\times\left(\sum_{s=1,\neq t}^{l}\left[\left|a_{j_s j_t}\right|+\left|\left(a_{j_s i_1},\cdots,a_{j_s i_k}\right)\left[\boldsymbol{A}(\beta)\right]^{-1}\begin{pmatrix}a_{i_1 j_t}\\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right|\right]\right)^{1-\alpha}$$
$$\le\left(R_{j_t}(\boldsymbol{A})-\delta_t\right)^{\alpha}\left(C_{j_t}(\boldsymbol{A})-\delta_t^T\right)^{1-\alpha}.$$

Thus, we have

$$0\ge\left|\lambda-a'_{tt}\right|-\left(R_t\left(\boldsymbol{A}/\beta\right)\right)^{\alpha}\left(C_t\left(\boldsymbol{A}/\beta\right)\right)^{1-\alpha}$$
$$\ge\left|\lambda-a_{j_t j_t}\right|-\left(R_{j_t}(\boldsymbol{A})-\delta_t\right)\left(C_{j_t}(\boldsymbol{A})-\delta_t^T\right)^{1-\alpha}.$$

Further, we obtain (8).

4. A Numerical Example

In this section, we present a numerical example to illustrate the advantages of our derived results.

Example 1. Let

$$\boldsymbol{A}=\begin{pmatrix}20&2&5&1&4\\2&15&2&4&1\\2&3&17&2&1\\4&3&4&8&1\\5&1&3&3&12\end{pmatrix},\quad \beta=\{1,3\}.$$

By calculation with Matlab 7.1, we have that

$$R_1(\boldsymbol{A})=12;\ R_2(\boldsymbol{A})=9;\ R_3(\boldsymbol{A})=8;\ R_4(\boldsymbol{A})=12;\ R_5(\boldsymbol{A})=12;$$

$$C_1(\boldsymbol{A})=13;\ C_2(\boldsymbol{A})=9;\ C_3(\boldsymbol{A})=14;\ C_4(\boldsymbol{A})=10;\ C_5(\boldsymbol{A})=7;$$

$$\delta_2=2.1800;\delta_4=4.3600;\delta_5=4.2500;\delta_2^T=1.4550;\delta_4^T=0.8404;\delta_5^T=1.7813.$$

Since $\beta\in N_r(\boldsymbol{A})$, by Theorem 4, the eigenvalue inclusion set of $\boldsymbol{A}/\beta$ is

$$\Gamma_1=\{\lambda\,|\,|\lambda-15|\le 6.8200\}\cup\{\lambda\,|\,|\lambda-8|\le 7.6400\}\cup\{\lambda\,|\,|\lambda-12|\le 7.7500\}.$$

From Theorem 4 of [2], the eigenvalue inclusion set of $\boldsymbol{A}/\beta$ is

$$\Gamma_1'=\{\lambda\,|\,|\lambda-15|\le 7.1412\}\cup\{\lambda\,|\,|\lambda-8|\le 8.2824\}\cup\{\lambda\,|\,|\lambda-12|\le 8.4118\}.$$

We use **Figure 1** to illustrate $\Gamma_1\subset\Gamma_1'$. And the eigenvalues of $\boldsymbol{A}/\beta$ are denoted by "+" in **Figure 1**. The blue dotted line and green dashed line denote the corresponding discs Γ_1 and Γ_1' respectively.

Meanwhile, since $\beta\in N_r(\boldsymbol{A})\cap N_c(\boldsymbol{A})$, by taking $\alpha=0.5$ in Theorem 5, the eigenvalue inclusion set of $\boldsymbol{A}/\beta$ is

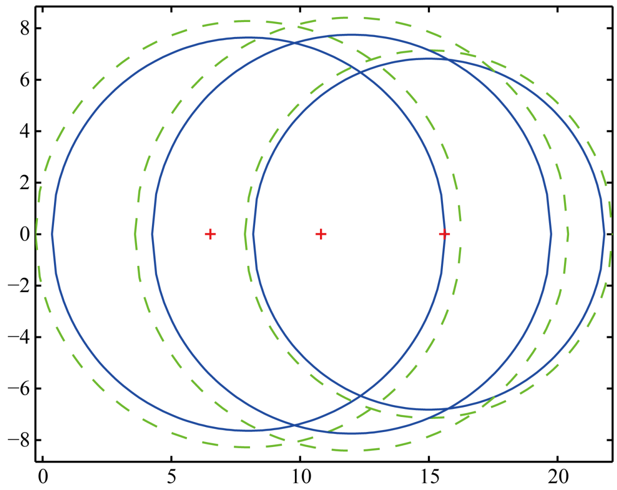

Figure 1. The blue dotted line and green dashed line denote the corresponding discs Γ_1 and Γ_1', respectively.

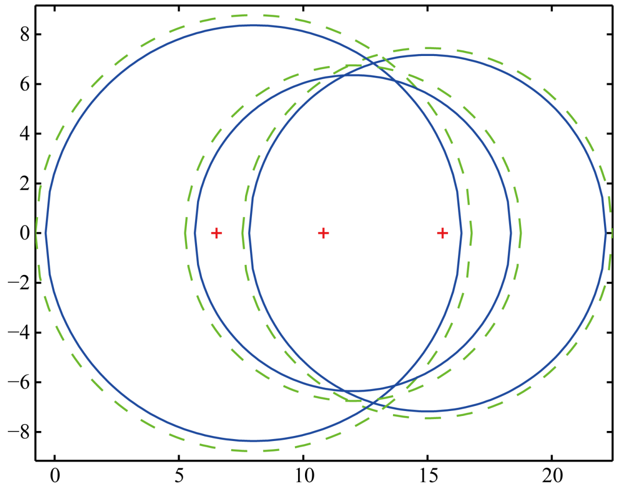

Figure 2. The blue dotted line and green dashed line denote the corresponding discs Γ_2 and Γ_2', respectively.

$$\Gamma_2 = \{\lambda \| \lambda - 15| \le 7.1733\} \cup \{\lambda \| \lambda - 8| \le 8.3654\} \cup \{\lambda \| \lambda - 12| \le 6.3596\}.$$

From Theorem 5 of [2], the eigenvalue inclusion set of A/β is

$$\Gamma_2' = \{\lambda \| \lambda - 15| \le 7.4492\} \cup \{\lambda \| \lambda - 8| \le 8.7751\} \cup \{\lambda \| \lambda - 12| \le 6.7544\}.$$

We use **Figure 2** to illustrate $\Gamma_2 \subset \Gamma_2'$. And the eigenvalues of A/β are denoted by "+" in **Figure 2**. The blue dotted line and green dashed line denote the corresponding discs Γ_2 and Γ_2' respectively. It is clear that $\Gamma_1 \not\subseteq \Gamma_2$ and $\Gamma_2 \not\subseteq \Gamma_1$.

References

[1] Cvetković, Lj. (2009) A New Subclass of *h*-Matrices. *Applied Mathematics and Computation*, **208**, 206-210. http://dx.doi.org/10.1016/j.amc.2008.11.037

[2] Liu, J.Z. and Huang, Z.J. (2010) The Dominant Degree and Disc Theorem for the Schur Complement. *Applied Mathematics and Computation*, **215**, 4055-4066. http://dx.doi.org/10.1016/j.amc.2009.12.063

[3] Horn, R.A. and Johnson, C.R. (1991) Topics in Matrix Analysis. Cambridge University Press, New York.

http://dx.doi.org/10.1017/CBO9780511840371

[4] Carlson, D. and Markham, T. (1979) Schur Complements on Diagonally Dominant Matrices. *Czechoslovak Mathematical Journal*, **29**, 246-251.

[5] Ikramov, K.D. (1989) Invariance of the Brauer Diagonal Dominance in Gaussian Elimination. *Moscow University Computational Mathematics and Cybernetics*, **2**, 91-94.

[6] Li, B. and Tsatsomeros, M. (1997) Doubly Diagonally Dominant Matrices. *Linear Algebra and Its Applications*, **261**, 221-235. http://dx.doi.org/10.1016/S0024-3795(96)00406-5

[7] Smith, R. (1992) Some Interlacing Properties of the Schur Complement of a Hermitian Matrix. *Linear Algebra and Its Applications*, **177**, 137-144. http://dx.doi.org/10.1016/0024-3795(92)90321-Z

[8] Zhang, F.Z. (2005) The Schur Complement and Its Applications. Springer-Verlag, New York. http://dx.doi.org/10.1007/b105056

[9] Demmel, J.W. (1997) Applied Numerical Linear Algebra. SIAM, Philadephia.

[10] Golub, G.H. and Van Loan, C.F. (1996) Matrix Computations. 3rd Edition, Johns Hopkins University Press, Baltimore.

[11] Kress, R. (1998) Numerical Analysis. Springer, New York. http://dx.doi.org/10.1007/978-1-4612-0599-9

[12] Xiang, S.H. and Zhang, S.L. (2006) A Convergence Analysis of Block Accelerated Over-Relaxation Iterative Methods for Weak Block *H*-Matrices to Partition π. *Linear Algebra and Its Applications*, **418**, 20-32. http://dx.doi.org/10.1016/j.laa.2006.01.013

[13] Liu, J.Z., Li, J.C., Huang, Z.H. and Kong, X. (2008) Some Properties on Schur Complement and Diagonal Schur Complement of Some Diagonally Dominant Matrices. *Linear Algebra and Its Applications*, **428**, 1009-1030. http://dx.doi.org/10.1016/j.laa.2007.09.008

[14] Liu, J.Z. and Huang, Y.Q. (2004) The Schur Complements of Generalized Doubly Diagonally Dominant Matrices. *Linear Algebra and Its Applications*, **378**, 231-244. http://dx.doi.org/10.1016/j.laa.2003.09.012

[15] Liu, J.Z. and Huang, Y.Q. (2004) Some Properties on Schur Complements of *H*-Matrices and Diagonally Dominant Matrices. *Linear Algebra and Its Applications*, **389**, 365-380. http://dx.doi.org/10.1016/j.laa.2004.04.012

[16] Liu, J.Z. and Huang, Z.J. (2010) The Schur Complements of γ-Diagonally and Product γ-Diagonally Dominant Matrix and their Disc Separation. *Linear Algebra and Its Applications*, **432**, 1090-1104. http://dx.doi.org/10.1016/j.laa.2009.10.021

[17] Liu, J.Z. and Zhang, F.Z. (2005) Disc Separation of the Schur Complements of Diagonally Dominant Matrices and Determinantal Bounds. *SIAM Journal on Matrix Analysis and Applications*, **27**, 665-674. http://dx.doi.org/10.1137/040620369

[18] Li, Y.T. Ouyang, S.P. Cao, S.J. and Wang, R.W. (2010) On Diagonal-Schur Complements of Block Diagonally Dominant Matrices. *Applied Mathematics and Computation*, **216**, 1383-1392. http://dx.doi.org/10.1016/j.amc.2010.02.038