

Analytic Theory of Finite Asymptotic Expansions in the Real Domain. Part II-A: The Factorizational Theory for Chebyshev Asymptotic Scales

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Received 30 April 2015; accepted 27 June 2015; published 30 June 2015

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Abstract

This paper, divided into three parts (Part II-A, Part II-B and Part II-C), contains the detailed factorizational theory of asymptotic expansions of type $(*) f(x) = a_1\phi_1(x) + \dots + a_n\phi_n(x) + o(\phi_n(x))$, $x \rightarrow x_0$, $n \geq 3$, where the asymptotic scale $\phi_1(x) \gg \phi_2(x) \gg \dots \gg \phi_n(x)$, $x \rightarrow x_0$, is assumed to be an extended complete Chebyshev system on a one-sided neighborhood of x_0 . It follows two previously published papers: the first, labelled as Part I, contains the complete (elementary but non-trivial) theory for $n = 2$; the second is a survey highlighting only the main results without proofs. All the material appearing in §2 of the survey is here reproduced in an expanded form, as it contains all the preliminary formulas necessary to understand and prove the results. The remaining part of the survey—especially the heuristical considerations and consequent conjectures in §3—may serve as a good introduction to the complete theory.

Keywords

Asymptotic Expansions, Formal Differentiation of Asymptotic Expansions, Factorizations of Ordinary Differential Operators, Chebyshev Asymptotic Scales

1. Introduction

Following the line of thought in [1], case $n = 2$, we develop in this paper a general analytic theory of asymptotic expansions of type

$$f(x) = a_1\phi_1(x) + \dots + a_n\phi_n(x) + o(\phi_n(x)), \quad x \rightarrow x_0, \quad n \geq 3, \tag{1.1}$$

where

$$\phi_1(x) \gg \phi_2(x) \gg \dots \gg \phi_n(x), \quad x \rightarrow x_0, \tag{1.2}$$

and the Hardy notation $\phi(x) \gg \psi(x), x \rightarrow x_0$ is alternative to $\psi(x) = o(\phi(x)), x \rightarrow x_0$. For the reader’s convenience, the paper has been divided into three parts: the present Part II-A contains all the general results obtainable through two approaches based on different special factorizations (called “canonical factorizations”) of the n th-order differential operator whose kernel is spanned by (ϕ_1, \dots, ϕ_n) . Our results are characterizations of (1.1) via integro-differential conditions useful for applications unlike the trivial characterization by means of the existence (as finite numbers) of the following n limits defining the coefficients a_i :

$$a_1 := \lim_{x \rightarrow x_0} f(x)/\phi_1(x); \quad a_i := \lim_{x \rightarrow x_0} \frac{[f(x) - a_1\phi_1(x) - \dots - a_{i-1}\phi_{i-1}(x)]}{\phi_i(x)}, \quad 2 \leq i \leq n, \tag{1.3}$$

the ϕ_i ’s being supposed non-vanishing on a deleted neighborhood of x_0 .

Our theory parallels: 1) the classical Taylor’s formula at a point $x_0 \in \mathbb{R}$; 2) the theory of polynomial expansions at ∞ systematized in [2]; 3) the theory of asymptotic expansions in real powers developed in [3] [4] from the standpoint of formal differentiability. A survey [5], previously published in this journal, contains the main results with no proofs and may be conveniently used to gain a quick view of the (rather long) detailed theory. The introductions in [1] [2] and [5] contain other comments; moreover, the heuristic considerations with related conjectures in §3 of [5] are quite helpful to properly grasp the presented results. Here, we only mention that our theory consists in studying (1.1) not by itself but matched to other expansions obtained by formal application of certain differential operators and we give a brief outline of the content.

- In §2, we collect all the preliminary material concerning factorizations of a disconjugate operator and the nonvanishingness of various Wronskians involving certain bases of its kernel. The scale of comparison functions (ϕ_1, \dots, ϕ_n) is practically assumed to form an extended Chebyshev system on some left deleted neighborhood of x_0 and various properties are systematized around the concept of Chebyshev asymptotic scale.
- In §3, we exhibit the operators used in formal differentiation together with some of their elementary properties.
- In §4, we state characterizations of a set of asymptotic expansions obtained from (1.1) by formal applications of the differential operators implicitly defined by the “unique” canonical factorization, termed by us of type (I), which chronologically is the first to be introduced, studied and applied.
- In §5, we do the same job for the differential operators implicitly defined by a special canonical factorization termed by us of type (II) and constructed using the given asymptotic scale.
- In §6, all proofs are collected.
- In Part II-B, we specialize the results to the important class of functions satisfying a differential inequality linked to the scale, so obtaining many nice characterizations.
- In Part II-C, we exhibit two algorithms for constructing the canonical factorizations upon which our theory is built. These algorithms are simple to describe, admit of “natural” asymptotic interpretations (also showing the appropriateness of the used differential operators) and are of considerable help in building examples and counterexamples concerning formal differentiation of asymptotic expansions.

Occasionally, an asymptotic expansion

$$f(x) = a_1\phi_1(x) + \dots + a_i\phi_i(x) + o(\phi_i(x)), \quad x \rightarrow x_0, \quad 1 \leq i < n, \tag{1.4}$$

will be called “incomplete”—with respect to the given scale (ϕ_1, \dots, ϕ_n) , of course—whereas (1.1) will be called “complete”, and these locutions refer to the specified growth-order of the remainder and not to the terms effectively present in the expansion *i.e.* those with non-zero coefficients. In both approaches, characterizations of incomplete expansions via a differential operator whose kernel is spanned by the “complete” basis (ϕ_1, \dots, ϕ_n) are less simple than those of complete expansions presented in the survey [5] and will be illustrated in the present paper.

We stress that the guiding thread of our work is “formal differentiation of asymptotic expansions”, a theme going back to the early 20th century (Landau, Hardy, Boas and others), but usually referring to the use of standard derivatives. The factorizational approach clearly shows that formal differentiation is admissible only if one uses suitable differential operators strictly linked to the involved scale together with related factorizations. The case of standard derivatives is very special and is highlighted in Part II-B and Part II-C.

Notations

- $f \in AC^0(I) \equiv AC(I) \Leftrightarrow f$ is absolutely continuous on each compact subinterval of I ;
- $f \in AC^k(I) \Leftrightarrow f^{(k)} \in AC(I)$;
- For $f \in AC^k(I)$, we write $\lim_{x \rightarrow x_0} f^{(k+1)}(x)$, meaning that x runs through the points wherein $f^{(k+1)}$ exists as a finite number. Applying L’Hospital’s rule in such a context means using Ostrowski’s version [6] valid for absolutely continuous functions;
- $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ denotes the extended real line;
- If no ambiguity arises, we use the following shorthand notations or similar ones:

$$\int_x^{x_0} f_1 \int_{t_1}^{x_0} f_2 \cdots \int_{t_{n-2}}^{x_0} f_{n-1} \int_{t_{n-1}}^{x_0} f_n(t) dt := \int_x^{x_0} f_1(t_1) dt_1 \int_{t_1}^{x_0} f_2(t_2) dt_2 \cdots \int_{t_{n-2}}^{x_0} f_{n-1}(t_{n-1}) dt_{n-1} \int_{t_{n-1}}^{x_0} f_n(t_n) dt_n;$$

wherein each integral $\int^{\cdot} \equiv \int^{\rightarrow \cdot}$ or, alternatively, \int_T^{\cdot} may be a proper or improper integral. A notation such as “ $\int^{\rightarrow x_0} f$ convergent” means that $I(\epsilon) := \int_T^{x_0-\epsilon} f$ exists as a Lebesgue integral for some $T < x_0$ and each $\epsilon \in]0, x_0 - T]$ and that the $\lim_{\epsilon \rightarrow 0^+} I(\epsilon)$ exists in \mathbb{R} , so defining the improper integral $\int_T^{x_0} f$;

- The symbol $W(u_1(x), \dots, u_i(x))$ denotes the Wronskian determinant of the ordered i -tuple $(u_1(x), \dots, u_i(x))$, $(i-1)$ times differentiable at the specified point x ; $W(u(x)) := u(x)$; $W(u_1, \dots, u_i)$ denotes the Wronskian viewed as the operator $W : (u_1(x), \dots, u_i(x)) \mapsto W(u_1(x), \dots, u_i(x))$ on a specified interval;
- Two acronyms systematically used are T.A.S. = “Chebyshev asymptotic scale” as in Def. 2.1, and C.F. = “canonical factorization” defined in Proposition 2.1-(iv) and (v);
- Propositions are numbered consecutively in each section irrespective of their labelling as lemma, theorem and so on.

2. Canonical Factorizations of Disconjugate Operators and Chebyshev Asymptotic Scales

Our theory is built upon appropriate integral representations stemming from a special structure of the asymptotic scale (ϕ_1, \dots, ϕ_n) : practically this n -tuple forms a fundamental system of solutions of a disconjugate equation on a one-sided neighborhood of x_0 such that certain Wronskians do not vanish thereon, a property granted by a result by Levin [7] which justifies our definition of Chebyshev asymptotic scale. We preliminarily recall some facts about factorizations of differential operators.

In this section, $L_n, n \geq 2$ denotes a linear ordinary differential operator of type

$$L_n u := u^{(n)} + \alpha_{n-1}(x)u^{(n-1)} + \cdots + \alpha_0(x)u \quad \forall u \in AC^{n-1}(J), \tag{2.1}_1$$

$$\alpha_i \in L^1_{loc}(J), 0 \leq i \leq n-1, J \text{ a generic interval of } \mathbb{R}, \tag{2.1}_2$$

where $L^1_{loc}(J)$ denotes the class of functions Lebesgue-summable on every compact subinterval of J . The matters to be discussed depend on the property of disconjugacy and several characterizations involving factorizations are collected in the next proposition where special locutions are defined in the statement itself. For general properties about disconjugacy we refer to the book by Coppel [8] and the paper by Levin [7], and for facts concerning canonical factorizations we refer to the papers by Trench [9] and the author [10] [11].

Proposition 2.1 (Disconjugacy on an open interval via factorizations). *For an operator L_n of type (2.1)_{1,2}, $n \geq 2$, on an open interval $]a, b[$, bounded or not, the following properties are equivalent:*

- 1) L_n is disconjugate on $]a, b[$ in the sense that: every nontrivial solution of $L_n u = 0$ has at most $(n-1)$ zeros on $]a, b[$ counting multiplicities or, equivalently, has at most $(n-1)$ distinct zeros on $]a, b[$.
- 2) $L_n u = 0$ has a fundamental system of solutions on $]a, b[$, (u_1, \dots, u_n) , satisfying Pólya’s W -property

$$W(u_1(x), \dots, u_i(x)) > 0 \quad \forall x \in]a, b[, 1 \leq i \leq n; \tag{2.2}$$

or equivalently $L_n u = 0$ has solutions u_1, \dots, u_{n-1} satisfying (2.2) for $1 \leq i \leq n-1$.

3) L_n has a Pólya-Mammana factorization on $]a, b[$ i.e.

$$L_n u \equiv r_n \left[r_{n-1} \left(\dots \left(r_1 (r_0 u)' \right)' \dots \right)' \right] \quad \forall u \in AC^{n-1}]a, b[, \tag{2.3}$$

where the r_i 's are suitable functions such that

$$r_i(x) > 0 \quad \forall x \in]a, b[; r_i \in AC^{n-1-i}]a, b[, \quad 0 \leq i \leq n-1; r_n \in AC^0]a, b[. \tag{2.4}$$

4) L_n has a “canonical factorization (C.F. for short) of type (I) at the endpoint a ” i.e. a factorization of type (2.3)-(2.4) with the additional conditions

$$\int_{\rightarrow a} (1/r_i) = +\infty, \quad 1 \leq i \leq n-1, \tag{2.5}_a$$

and a similar “C.F. of type (I) at the endpoint b ”, i.e. with the r_i 's satisfying

$$\int^{\rightarrow b} (1/r_i) = +\infty, \quad 1 \leq i \leq n-1. \tag{2.5}_b$$

5) For each $c, a < c < b$, L_n has a “C.F. on the interval $]a, c[$ which is of type (II) at the endpoint a ” i.e. a factorization (2.3)-(2.4) valid on the interval $]a, c[$ and with the r_i 's satisfying

$$\int_{\rightarrow a} (1/r_i) < +\infty, \quad 1 \leq i \leq n-1, \tag{2.6}_a$$

and L_n has a “C.F. on the interval $]c, b[$ which is of type (II) at the endpoint b ” i.e. a factorization (2.3)-(2.4) valid on the interval $]c, b[$ and with the r_i 's satisfying

$$\int^{\rightarrow b} (1/r_i) < +\infty, \quad 1 \leq i \leq n-1. \tag{2.6}_b$$

Remarks. 1) In the definition of a C.F. conditions (2.5) or (2.6) are required to hold for the index i running from 1 to $(n-1)$: there are no conditions on r_0 and r_n . Factorizations in properties 3)-4) are global i.e. valid on the whole given interval $]a, b[$, whereas property 5) claims the existence of local C.F.'s of type (II). The existence of a global C.F. of type (II) at a or at b is a special circumstance ([10], Th. 3.11, p. 163).

2) A global C.F. of type (I) at a specified endpoint does always exist for a disconjugate operator on an open interval and is “essentially” unique in the sense that the functions r_i are determined up to multiplicative constants with product 1, Trench [9]. The situation is quite different for C.F.'s of type (II). For example the operator $L_n \equiv u^{(n)}$ has no global C.F. on $(-\infty, +\infty)$ of type (II) at any of the endpoints for it admits of only “one” (up to constant factors) Pólya-Mammana factorization on $(-\infty, +\infty)$ namely

$$u^{(n)} \equiv \left(\dots (u)' \dots \right)', \tag{2.7}$$

which is a special contingency characterized in ([10], Th. 3.3) and in ([11], Th. 7.1). But the operator $u^{(n)}$ thought of as acting on the space $AC^{n-1}]0, +\infty)$, or even on the space $C^\infty]0, +\infty)$, has infinitely many “essentially” different C.F.'s of type (II), for instance the following ones

$$u^{(n)} \equiv \frac{1}{(x-c)^{n-1}} \left[(x-c)^2 \left(\dots \left((x-c)^2 \left(\frac{u}{(x-c)^{n-1}} \right)' \right)' \dots \right)' \right]', \tag{2.8}$$

which are C.F.'s of type (II) at both the endpoints “0” and “ $+\infty$ ” whatever the choice of the constant $c < 0$. For $c = 0$, we get a factorization on $]0, +\infty)$ which is a C.F. of type (I) at “0” and of type (II) at “ $+\infty$ ”; for $c > 0$ we have nonglobal factorizations which are of type (II) at $+\infty$.

C.F.'s are naturally linked to bases of $\ker L_n$ forming asymptotic scales at one or both endpoints and the

following results, due to Levin ([7], §2), highlight important properties of the Wronskians constructed with an asymptotic scale.

Proposition 2.2 (Wronskians of asymptotic scales and their hierarchies).

(I) (Results involving a differential operator). Let L_n be an operator of type (2.1)_{1,2} disconjugate on an open interval $]a, b[$. Then:

1) Its kernel has some basis (ϕ_1, \dots, ϕ_n) satisfying

$$\phi_i(x) > 0 \text{ on some interval }]b - \epsilon, b[, 1 \leq i \leq n; \phi_1(x) \gg \phi_2(x) \gg \dots \gg \phi_n(x), x \rightarrow b^-. \quad (2.9)$$

2) For each such basis

$$W(\phi_n(x), \phi_{n-1}(x), \dots, \phi_i(x)) > 0 \text{ on the whole interval }]a, b[, 1 \leq i \leq n, \quad (2.10)$$

noticing the reversed order of the ϕ_i 's in the Wronskians.

3) For any strictly decreasing set of indexes $\{i_1, \dots, i_k\}$ i.e. such that

$$n \geq i_1 > i_2 > \dots > i_k \geq 1, 1 \leq k \leq n-1, \quad (2.11)$$

we have

$$W(\phi_{i_1}(x), \dots, \phi_{i_k}(x)) > 0 \text{ on a left deleted neighborhood of } b, \quad (2.12)$$

and in particular we have the inequalities

$$\text{sign } W(\phi_1(x), \dots, \phi_i(x)) = (-1)^{i(i-1)/2} \text{ on a left deleted neighborhood of } b, 1 \leq i \leq n. \quad (2.13)$$

4) For each k , $1 \leq k \leq n-1$, and for any two distinct and strictly increasing sets of indexes i_1, \dots, i_k and j_1, \dots, j_k such that $i_h \leq j_h, 1 \leq h \leq k$, we have

$$W(\phi_{i_1}(x), \dots, \phi_{i_k}(x)) \gg W(\phi_{j_1}(x), \dots, \phi_{j_k}(x)), x \rightarrow b^-. \quad (2.14)$$

Notice the ordering of the ϕ_i 's and the ϕ_j 's in (2.14): if each ϕ_i has a growth-order at b^- greater than that of the corresponding ϕ_j then the same is true for the Wronskians. In the claim 3), we have a different ordering of the ϕ_i 's as this grants the positivity of the Wronskians in (2.12).

(II) (Results involving scales with less regularity). Let (ϕ_1, \dots, ϕ_n) be functions of class $C^{n-1}]a, b[$ satisfying conditions (2.9) and condition

$$W(\phi_n(x), \phi_{n-1}(x), \dots, \phi_1(x)) \text{ either } \geq 0 \text{ or } \leq 0 \text{ on }]a, b[, 1 \leq i \leq n; \quad (2.15)$$

and let there exist an integer $r, 1 \leq r \leq n$, such that

$$W_r(\phi_n(x), \phi_{n-1}(x), \dots, \phi_i(x)) \neq 0 \text{ on }]a, b[, \forall i, 1 \leq i \leq n; \quad (2.16)$$

where the symbol $W_r(\phi_n(x), \phi_{n-1}(x), \dots, \phi_i(x))$ denotes the Wronskian determinant wherein the column involving ϕ_r has been suppressed. Then the following inequalities hold true

$$W(\phi_n(x), \phi_{n-1}(x), \dots, \phi_1(x)) \geq 0 \text{ on }]a, b[; \quad (2.17)$$

$$W(\phi_n(x), \phi_{n-1}(x), \dots, \phi_i(x)) > 0 \text{ on }]a, b[, 1 \leq i \leq n-1; \quad (2.18)$$

together with the above-stated properties in 3) and 4). Notice that in (2.17)-(2.18) the signs of the Wronskians are well defined even if they remain undefined in the assumptions (2.15)-(2.16).

To visualize (2.14), we list a few asymptotic scales at b^- constructed with the Wronskians:

$$\begin{cases} W(\phi_1, \phi_2) \gg W(\phi_1, \phi_3) \gg \dots \gg W(\phi_1, \phi_n), \\ W(\phi_2, \phi_3) \gg W(\phi_2, \phi_4) \gg \dots \gg W(\phi_2, \phi_n), \\ \vdots \\ W(\phi_{n-2}, \phi_{n-1}) \gg W(\phi_{n-2}, \phi_n), \end{cases} \quad x \rightarrow b^-; \quad (2.19)$$

$$W(\phi_1, \phi_2, \phi_3) \gg W(\phi_1, \phi_2, \phi_4) \gg \dots \gg W(\phi_1, \phi_2, \phi_n), x \rightarrow b^-. \quad (2.20)$$

It is quite important to note the order of the ϕ_i 's forming the asymptotic scale in (2.9); if we maintain the same ordering in the analogous statement for $x \rightarrow a^+$, i.e. $\phi_1(x) \gg \phi_2(x) \gg \dots \gg \phi_n(x)$, $x \rightarrow a^+$, then the Wronskians in (2.10) and in (2.12) to (2.18) are the same, the essential point being the relative growth-orders of the ϕ_i 's. From the point of view of asymptotic expansions the correct numbering is that adopted by us irrespective of the limiting process.

The above results substantiate the following definition of special asymptotic scales wherein we merely fix the neighborhood of b left undefined in Proposition 2.2 whose part (I) grants the existence of such scales whereas part (II) implies a lot of useful properties even for scales with less regularity. From now on the interval will be denoted as in the two-term theory [1].

Definition 2.1 (Chebyshev asymptotic scales). *The ordered n -tuple of real-valued functions (ϕ_1, \dots, ϕ_n) $n \geq 2$, is termed a "Chebyshev asymptotic scale" (T.A.S. for short) on the half-open interval $[T, x_0[$, $T \in \mathbb{R}$, $x_0 \leq +\infty$, provided the following properties are satisfied:*

$$\phi_i \in C^{n-1}[T, x_0[, 1 \leq i \leq n; \quad (2.21)$$

$$\phi_i(x) \neq 0 \text{ on some left deleted neighborhood of } x_0, 1 \leq i \leq n; \quad (2.22)$$

$$\phi_1(x) \gg \phi_2(x) \gg \dots \gg \phi_n(x), x \rightarrow x_0^-; \quad (2.23)$$

$$W(\phi_1(x), \dots, \phi_i(x)) \neq 0 \text{ on } [T, x_0[, 1 \leq i \leq n. \quad (2.24)$$

Whenever the ϕ_i 's satisfy the stronger regularity condition

$$\phi_i \in AC^{n-1}[T, x_0[, 1 \leq i \leq n, \quad (2.25)$$

they remain associated to the operator

$$L_{\phi_1, \dots, \phi_n} u := W(\phi_1(x), \dots, \phi_n(x), u) / W(\phi_1(x), \dots, \phi_n(x)), \quad (2.26)$$

which is the unique linear ordinary differential operator of type (2.1)_{1,2}, acting on the space $AC^{n-1}[T, x_0[$ and such that $\ker L_{\phi_1, \dots, \phi_n} = \text{span}(\phi_1, \dots, \phi_n)$.

Remarks. 1) Condition (2.21) is the usual regularity assumption in approximation theory (Chebyshev systems and the like), whereas in matters involving differential equations/inequalities it is natural to assume (2.25).

2) Choosing an half-open interval in this definition is a matter of convenience: the point x_0 involved in the asymptotic relations is characterized as the endpoint not belonging to the interval, possibly $x_0 = +\infty$, whereas the other endpoint marks off an interval whereon the inequalities involving the Wronskians are satisfied and these in turn allow certain integral representations valid on the whole given interval and essential to our theory. These remarks make evident the analogous definition for an interval $]x_0, T]$ where: $-\infty \leq x_0$, and $T \in \mathbb{R}$.

3) In the above definition we have merely supposed the nonvanishingness of various functions instead of specifying their signs as in Proposition 2.2; this avoids restrictions that are immaterial in asymptotic investigations. If the ϕ_i 's are strictly positive near x_0 then Levin's theorem provides the exact signs of certain Wronskians.

4) As concrete examples of such asymptotic scales on $[T, +\infty)$ the reader may think of scales whose non-identically zero and infinitely-differentiable functions are represented by linear combinations, products, ratios and compositions of a finite number of powers, exponentials and logarithms. As a rule such functions and their Wronskians have a principal part at $+\infty$ which can be expressed by products of similar functions, hence they do not vanish on a neighborhood of $+\infty$.

When comparing our notations with other authors' results the reader must carefully notice the numbering of the ϕ_i 's in the asymptotic scale (2.23) and in the Wronskians (2.24); the next proposition contains various additional properties of a T.A.S. and, in particular, it claims that conditions (2.21)-(2.24) imply the nonvanishingness of the reversed Wronskians:

$$W(\phi_n(x), \phi_{n-1}(x), \dots, \phi_i(x)) \neq 0 \text{ on } [T, x_0[, 1 \leq i \leq n, \quad (2.27)$$

though the converse generally fails as it may be easily checked for the scale

$$1 \gg cx + x^2 \gg x^2, x \rightarrow 0^-, (c > 0), \text{ on } (-\infty, 0[, \quad (2.28)$$

which satisfies (2.27) on $(-\infty, 0[$ whereas

$$\begin{cases} \phi_1 \text{ and } W(\phi_1, \phi_2, \phi_3) \neq 0 \text{ on } (-\infty, 0[; \\ W(\phi_1, \phi_2) \equiv W(1, cx + x^2) = c + 2x \neq 0 \text{ on }]-c/2, 0[\text{ but not on } (-\infty, 0[. \end{cases}$$

Proposition 2.3 (Several characterizations and additional properties of T.A.S.'s). *Let the ordered n -tuple of real-valued functions $(\phi_1, \dots, \phi_n), n \geq 2$, satisfy conditions (2.21)-(2.22)-(2.23).*

(I) *The following are equivalent properties:*

- 1) (ϕ_1, \dots, ϕ_n) is a T.A.S. on $[T, x_0[$, i.e. (2.24) hold true.
- 2) Both sets of inequalities (2.24) and (2.27) hold true.
- 3) The ordered n -tuple $(\epsilon_1 \phi_1(x), \dots, \epsilon_n \phi_n(x))$, with proper choices of the constants $\epsilon_i = \pm 1$, is an extended complete Chebyshev system on $[T, x_0[$.
- 4) The n -tuple (ϕ_1, \dots, ϕ_n) admits of an integral representation of the form

$$\phi_1(x) = w_0(x); \phi_2(x) = w_0(x) \cdot \int_x^{x_0} w_1; \phi_i(x) = w_0(x) \cdot \int_x^{x_0} w_1 \cdots \int_{x_{i-2}}^{x_0} w_{i-1}, 2 \leq i \leq n, x \in [T, x_0[; \tag{2.29}$$

with suitable functions w_i subjected to the following regularity conditions

$$w_i(x) \neq 0 \quad \forall x \in [T, x_0[; w_i \in C^{n-1-i}[T, x_0[, 0 \leq i \leq n-1; \int_x^{x_0} |w_i| < +\infty, 1 \leq i \leq n-1. \tag{2.30}$$

If this is the case the w_i 's are unique and may be expressed in terms of the ϕ_i 's on $[T, x_0[$ by the formulas

$$\begin{cases} w_0(x) := \phi_1(x); w_1 := -(\phi_2(x)/\phi_1(x))' = -(\phi_1(x))^{-2} W(\phi_1(x), \phi_2(x)); \\ w_i(x) := -\left[\frac{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_{i+1}(x))}{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_i(x))} \right]' \\ \equiv -[W(\phi_1, \dots, \phi_{i-1}) \cdot W(\phi_1, \dots, \phi_{i+1})] / [W(\phi_1, \dots, \phi_i)]^2, 2 \leq i \leq n-1. \end{cases} \tag{2.31}$$

Conversely we have the following formulas for the Wronskians of the ϕ_i 's

$$W(\phi_1, \dots, \phi_i) = (-1)^{i(i-1)/2} w_0^i w_1^{i-1} w_2^{i-2} \cdots w_{i-1} \text{ on } [T, x_0[, 2 \leq i \leq n. \tag{2.32}$$

(II) For (ϕ_1, \dots, ϕ_n) a T.A.S. on $[T, x_0[$ we have the inequalities

$$\phi_i(x) \neq 0 \text{ on } [T, x_0[, 1 \leq i \leq n, \text{ (implied by (2.29)-(2.30));} \tag{2.33}$$

$$W(\phi_{i_1}(x), \dots, \phi_{i_k}(x)) \neq 0 \text{ near } x_0, \tag{2.34}$$

for any set of indexes satisfying (2.11) and we also have the hierarchies between the Wronskians stated in Proposition 2.2-4) and referred to $x \rightarrow x_0^-$ in the present context. Whenever the ϕ_i 's are strictly positive then all the Wronskians in (2.27) are strictly positive on $[T, x_0[$ by (2.10), but not necessarily all the Wronskians in (2.24); in this case the inverted n -tuple (ϕ_n, \dots, ϕ_1) is an extended complete Chebyshev system on $[T, x_0[$. On the contrary, if the given n -tuple (ϕ_1, \dots, ϕ_n) is an extended complete Chebyshev system on $[T, x_0[$ i.e. all the Wronskians in (2.24) are strictly positive on $[T, x_0[$, then (2.29) and (2.31) imply that the ϕ_i 's have alternating signs, namely: $\text{sign } \phi_i = (-1)^{i-1}$ on $[T, x_0[$.

Part (I) of Proposition 2.3 generalizes a classical result, ([12], Ch. XI, Th. 1.2, p. 379), which characterizes those special asymptotic scales formed by functions with zeros of increasing multiplicities (namely $0, 1, \dots, n-1$) at an endpoint of a compact interval; also refer to ([12], Ch. I) and to [13] for locutions and facts about Chebyshev systems. Notice that formulas (2.31) in themselves are well defined if the n -tuple (ϕ_1, \dots, ϕ_n) satisfies (2.21) and (2.24); under the additional assumption (2.23) they establish a one-to-one correspondence between the ϕ_i 's and the w_i 's. For a T.A.S. on $]x_0, T]$ the integrals $\int_x^{x_0}$ in (2.29) are obviously replaced by $\int_x^{x_0}$, the w_i 's in (2.31) for $i \geq 1$ are defined without the minus sign and the coefficient $(-1)^{\dots}$ is absent in (2.32). If all

the Wronskians in (2.24) are strictly positive on $]x_0, T]$ then the same is true for all the ϕ_i 's.

Under condition (2.25) formulas in Proposition 2.3-2) are related to C.F.'s of type (II) at x_0 . We collect in the next proposition all the facts essential to develop our theory of asymptotic expansions focusing on C.F.'s rather than on integral representations of the given scale because we need both types of C.F.'s and the layout of Proposition 2.3 does not suit a C.F. of type (I).

Proposition 2.4 (Formulas concerning T.A.S.'s linked to differential operators). *Let the ordered n -tuple (ϕ_1, \dots, ϕ_n) satisfy conditions (2.21) to (2.25), hence the operator in (2.26) is disconjugate on the open interval $]T, x_0[$ and enjoys the properties in Propositions 2.1 and 2.2-(I). Moreover, as an operator acting on $AC^{n-1}[T, x_0[$, it has the following further properties:*

1) Define the following $(n+1)$ functions on $]T, x_0[$:

$$\begin{cases} q_0 := 1/|\phi_1|; & q_1 := (\phi_1)^2/|W(\phi_1, \phi_2)|; \\ q_i := [W(\phi_1, \dots, \phi_i)]^2/|W(\phi_1, \dots, \phi_{i-1}) \cdot W(\phi_1, \dots, \phi_{i+1})|, & 2 \leq i \leq n-1; \\ q_n := |q_0 q_1 \dots q_{n-1}|^{-1} \equiv |W(\phi_1, \dots, \phi_n)/W(\phi_1, \dots, \phi_{n-1})|. \end{cases} \quad (2.35)$$

Then, the q_i 's satisfy the following regularity conditions:

$$q_i(x) > 0 \quad \forall x \in]T, x_0[; q_i \in AC^{n-1-i}[T, x_0[, \quad 0 \leq i \leq n-1; q_n \in AC^0[T, x_0[. \quad (2.36)$$

Their reciprocals, left apart q_0 and q_n , may be expressed as derivatives of certain ratios

$$1/q_1(x) = \left| (\phi_2(x)/\phi_1(x))' \right|, \quad 1/q_i(x) = \left[\frac{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_{i+1}(x))}{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_i(x))} \right]', \quad 2 \leq i \leq n-1, \quad (2.37)$$

on the interval $]T, x_0[$, and

$$\int_T^{x_0} (1/q_i) < +\infty, \quad 1 \leq i \leq n-1. \quad (2.38)$$

Our operator admits of the following factorization on $]T, x_0[$:

$$L_{\phi_1, \dots, \phi_n} u \equiv q_n \left[q_{n-1} \left(\dots (q_0 u)' \dots \right)' \right]', \quad (2.39)$$

which is a global C.F. of type (II) at both endpoints T and x_0 .

2) Our T.A.S. (apart from the signs) admits of the following integral representation in terms of the q_i 's:

$$\left| \phi_1(x) \right| = \frac{1}{q_0(x)}; \quad \left| \phi_i(x) \right| = \frac{1}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \dots \int_{i-2}^{x_0} \frac{1}{q_{i-1}}, \quad 2 \leq i \leq n, \quad x \in]T, x_0[; \quad (2.40)$$

hence the ϕ_i 's, besides being everywhere non-zero on $]T, x_0[$, have the same order of growth at T , namely

$$\lim_{x \rightarrow T^+} \phi_i(x)/\phi_j(x) = c_{ij} \in \mathbb{R} \setminus \{0\} \quad \forall i \neq j. \quad (2.41)$$

In the special case where all the Wronskians in (2.24) are strictly positive, i.e. when (ϕ_1, \dots, ϕ_n) is an extended complete Chebyshev system on $]T, x_0[$, then the ϕ_i 's have alternating signs, namely

$$\text{sign } \phi_i = (-1)^{i-1} \text{ on }]T, x_0[. \quad (2.42)$$

3) Analogously we define the following $(n+1)$ functions on $]T, x_0[$:

$$\begin{cases} p_0 := 1/|\phi_n|; & p_1 := (\phi_n)^2/|W(\phi_n, \phi_{n-1})|; \\ p_i := [W(\phi_n, \phi_{n-1}, \dots, \phi_{n-i+1})]^2 \cdot |W(\phi_n, \phi_{n-1}, \dots, \phi_{n-i+2}) \cdot W(\phi_n, \phi_{n-1}, \dots, \phi_{n-i})|^{-1}, & 2 \leq i \leq n-1; \\ p_n := |p_0 p_1 \dots p_{n-1}|^{-1} \equiv |W(\phi_n, \phi_{n-1}, \dots, \phi_1)/W(\phi_n, \phi_{n-1}, \dots, \phi_2)|. \end{cases} \quad (2.43)$$

They satisfy the same regularity conditions on the half-open interval $[T, x_0[$ as the q_i 's do in (2.36) and their reciprocals may be expressed as derivatives of the following ratios analogous to those in (2.37):

$$1/p_1(x) = \left| \left(\phi_{n-1}(x)/\phi_n(x) \right)' \right|, \quad 1/p_i(x) = \left| \frac{W(\phi_n(x), \dots, \phi_{n-i+2}(x), \phi_{n-i}(x))}{W(\phi_n(x), \dots, \phi_{n-i+2}(x), \phi_{n-i+1}(x))} \right|', \quad 2 \leq i \leq n-1; \quad (2.44)$$

on the interval $[T, x_0[$. Moreover:

$$\int^{x_0} (1/p_i) = +\infty, \quad 1 \leq i \leq n-1, \quad (2.45)$$

$$\int_T (1/p_i) < +\infty, \quad 1 \leq i \leq n-1, \quad (2.46)$$

hence the associated factorization

$$L_{\phi_1, \dots, \phi_n} u \equiv p_n \left[p_{n-1} \left(\dots (p_0 u)' \dots \right)' \right], \quad (2.47)$$

is (up to constant factors) “the” global C.F. of type (I) at x_0 and it turns out to be of type (II) at T .

4) The special fundamental system of solutions to $L_{\phi_1, \dots, \phi_n} u = 0$ defined by

$$P_0(x) := \frac{1}{p_0(x)}; \quad P_i(x) := \frac{1}{p_0(x)} \int_T^x \frac{1}{p_1} \dots \int_T^{t_{i-1}} \frac{1}{p_i}, \quad 1 \leq i \leq n-1, \quad (2.48)$$

satisfies the asymptotic relations:

$$\begin{cases} P_0(x) \gg P_1(x) \gg \dots \gg P_{n-2}(x) \gg P_{n-1}(x), & x \rightarrow T^+, \\ P_{n-1}(x) \gg P_{n-2}(x) \gg \dots \gg P_1(x) \gg P_0(x), & x \rightarrow x_0^-. \end{cases} \quad (2.49)$$

Relations (2.49) uniquely determine the fundamental system (P_0, \dots, P_{n-1}) up to multiplicative constants. (In the terminology used by the author [10], [11] the n-tuple (P_0, \dots, P_{n-1}) is a “mixed hierarchical system” on $]T, x_0[$ whereas Levin ([7], p. 80) would call it a “doubly hierarchical system” because he uses different arrangements for asymptotic scales at the left or right endpoints ([7], p. 59).) Whenever the ϕ_i 's are strictly positive then the same is true for all the Wronskians appearing in (2.43) hence the absolute values are redundant; in this case it is the inverted n-tuple (ϕ_n, \dots, ϕ_1) which forms an extended complete Chebyshev system on $[T, x_0[$.

The construction of the two above factorizations starting from the given expressions of the coefficients q_i or p_i is the classical procedure by Pólya [14]. Notice that the functions p_i 's in (2.47), which are unique (constant factors apart) by a mentioned result by Trench, may be recovered from many different asymptotic scales and not just from one! The main feature of the above proposition is that we can express all the properties of our basic operator (at least those needed in our theory) in terms of the a-priori given Chebyshev asymptotic scale. The use of absolute values in the definitions of the q_i 's and p_i 's has the advantage of avoiding their use in the everywhere-present integral representations; and we must use them in at least one of the definitions as the two sets of Wronskians cannot have one and the same sign.

A quick proof of the existence of C.F.'s. The global existence of C.F.'s of type (I) was for the first time proved by Trench [9] by an original procedure which was subsequently adapted by the author [10] to show the local existence of C.F.'s of type (II). Trench's result played a historical role as it had a great impact on the asymptotic theory of ordinary differential equations. Levin's theorem easily implies both Trench's result about global existence (but not uniqueness) and the existence of a particular local C.F.'s of type (II) in the case of disconjugate operators: see the proof of Proposition 2.4. However we must point out that Trench's procedure, independent of properties of Wronskians, applies to a larger class of operators ([9], §1). As far as C.F.'s of type (II) are concerned the present quick approach does not yield a C.F. of type (II) at b for each interval $]a + \epsilon, b[$, as asserted in Proposition 2.1-5.

3. The Special Operators Associated to Canonical Factorizations

In this section, we collect some facts concerning those special operators associated to canonical factorizations: properties and formulas which our theory is constructed upon. We do not report the heuristic considerations which justify our approach and show how “natural” the obtained results are; we refer the reader to ([5], §3) for the heuristic approach and the related conjectures which will be proved in this paper.

Referring to the factorization of type (I) in (2.47), with the p_i ’s in (2.43), we define the differential operators acting on $AC^{n-1}[T, x_0[$:

$$L_0u := p_0(x)u; L_ku := p_k(x) \left[p_{k-1}(x) \left(\cdots (p_0(x)u)' \cdots \right)' \right], 1 \leq k \leq n; L_nu \equiv L_{\phi_1, \dots, \phi_n}u, \quad (3.1)$$

which satisfy the recursive formula

$$L_ku := p_k(x)(L_{k-1}u)', 1 \leq k \leq n. \quad (3.2)$$

And referring to the factorization of type (II) in (2.39), with the q_i ’s in (2.35), we define the differential operators acting on $AC^{n-1}[T, x_0[$:

$$M_0u := q_0(x)u; M_ku := q_k(x) \left[q_{k-1}(x) \left(\cdots (q_0(x)u)' \cdots \right)' \right], 1 \leq k \leq n; M_nu \equiv L_{\phi_1, \dots, \phi_n}u, \quad (3.3)$$

which satisfy the recursive formula

$$M_ku := q_k(x)(M_{k-1}u)', 1 \leq k \leq n. \quad (3.4)$$

We call L_k [respectively M_k] “the weighted derivative of order k with respect to the weights (p_0, p_1, \dots, p_k) , [respectively (q_0, q_1, \dots, q_k)],” in preference to the (some-times used) generic locutions of “quasi-derivatives or generalized derivatives” with no reference to the n -tuples of weights. The operator of order zero is included for convenience. Now representations (2.40) and (2.47) imply that:

$$\ker L_k = \text{span}(\phi_n, \phi_{n-1}, \dots, \phi_{n-k+1}); \ker M_k = \text{span}(\phi_1, \dots, \phi_k), 1 \leq k \leq n-1; \quad (3.5)$$

hence there exist never-vanishing functions $\widetilde{p}_k, \widetilde{q}_k$ such that:

$$L_ku = \widetilde{p}_k \cdot W(\phi_n, \phi_{n-1}, \dots, \phi_{n-k+1}, u); M_ku = \widetilde{q}_k \cdot W(\phi_1, \dots, \phi_k, u), 1 \leq k \leq n-1. \quad (3.6)$$

It follows that L_k and M_k preserve the hierarchy (2.23), namely we have the following asymptotic scales:

$$L_k[\phi_1(x)] \gg L_k[\phi_2(x)] \gg \cdots \gg L_k[\phi_{n-k}(x)], x \rightarrow x_0^-, \quad (3.7)$$

$$M_k[\phi_{k+1}(x)] \gg M_k[\phi_{k+2}(x)] \gg \cdots \gg M_k[\phi_n(x)], x \rightarrow x_0^-, \quad (3.8)$$

for each fixed k , $0 \leq k \leq n-2$. For $k=0$ they respectively reduce to

$$p_0(x)\phi_1(x) \gg p_0(x)\phi_2(x) \gg \cdots \gg p_0(x)\phi_n(x), x \rightarrow x_0^-, \quad (3.9)$$

$$q_0(x)\phi_1(x) \gg q_0(x)\phi_2(x) \gg \cdots \gg q_0(x)\phi_n(x), x \rightarrow x_0^-, \quad (3.10)$$

both equivalent to (2.23). Hence if we apply any n -tuple of operators L_k and M_k , $0 \leq k \leq n-1$, to an asymptotic expansion with an identically-zero remainder *i.e.* to a linear combination

$$f(x) = a_1\phi_1(x) + \cdots + a_n\phi_n, \quad (3.11)$$

we get again an asymptotic expansion with a zero remainder and in this sense we may say that “the asymptotic expansion (3.11) is formally differentiable $(n-1)$ times with respect to the n -tuples of weights (p_0, \dots, p_{n-1}) and (q_0, \dots, q_{n-1}) ” neglecting the n th-order weighted derivatives which yield identically-zero expressions. Beside this the operators M_k have a remarkable asymptotic link with the coefficients in (3.11) as claimed in the

following

Proposition 3.1 (The coefficients of an asymptotic expansion with zero remainder). *Referring to the T.A.S. in Proposition 2.4 and to the special factorization (2.39) the following facts hold true for the differential operators M_k in (3.3):*

(I) *The M_k 's satisfy the following relations:*

$$\ker M_k = \text{span}(\phi_1, \dots, \phi_k), 1 \leq k \leq n; \tag{3.12}$$

$$M_k [\phi_{k+1}(x)] \equiv \epsilon_k = \text{constant} = \pm 1, 1 \leq k \leq n-1; \tag{3.13}$$

$$\begin{cases} M_k [\phi_h(x)] = \epsilon_{h,k} \cdot \int_x^{x_0} \frac{1}{q_{k+1}} \dots \int_x^{x_0} \frac{1}{q_{h-1}} = o(1), x \rightarrow x_0^-; \\ \epsilon_{h,k} = \text{constant} = \pm 1, 1 \leq k \leq h-2, h \leq n. \end{cases} \tag{3.14}$$

$$M_k u \equiv \epsilon_k \frac{W(\phi_1, \dots, \phi_k, u)}{W(\phi_1, \dots, \phi_k, \phi_{k+1})}, 1 \leq k \leq n-1. \tag{3.15}$$

(II) *For a fixed k , $1 \leq k \leq n$, we have the logical equivalence:*

$$M_{k-1} [f(x)] \equiv \epsilon_{k-1} \cdot a_k = \text{constant on some interval } J \tag{3.16}$$

if and only if

$$f(x) = a_1 \phi_1(x) + \dots + a_k \phi_k(x) \text{ on } J \text{ for some constants } a_i, \tag{3.17}$$

a_k being the same as in (3.16) and ϵ_{k-1} as in (3.13). If (3.16)-(3.17) hold true on a left neighborhood of x_0 then the following limits exist as finite numbers and

$$\epsilon_{h-1} \cdot a_h = \lim_{x \rightarrow x_0^-} M_{h-1} [f(x)], 1 \leq h \leq k, \tag{3.18}$$

where, for $h = k$, (3.18) is the identity (3.16).

(III) *In the special case wherein all the Wronskians in (2.24) are strictly positive then the constants in (3.13)-(3.14) have the values:*

$$\epsilon_k = 1, \epsilon_{h,k} = (-1)^{h+k-1}. \tag{3.19}$$

We stress that the equivalence “(3.16) \Leftrightarrow (3.17)” is an algebraic fact based on (3.12)-(3.13) whereas the inference “(3.16)-(3.17) \Rightarrow (3.18)” is an asymptotic property whose validity requires that (ϕ_1, \dots, ϕ_k) be an asymptotic scale at x_0 and that the operators M_k be defined as specified.

4. The First Factorizational Approach

We start from the “unique” C.F. of our operator $L_{\phi_1, \dots, \phi_n}$ on the interval $[T, x_0[$ of type (I) at x_0 , *i.e.* identity (2.47) with conditions (2.45)-(2.46) and the p_i 's satisfying the same conditions as do the q_i 's in (2.36). We consider the fundamental system (2.48). By (2.49) the ordered n -tuple (P_{n-1}, \dots, P_0) is an asymptotic scale at x_0^- but it cannot coincide (constant factors apart) with the given scale (ϕ_1, \dots, ϕ_n) as (2.41) and (2.49) are incompatible. However (2.23) and (2.49) imply that the two scales are linked by the following relations

$$\phi_i(x) \sim b_i P_{n-i}(x), x \rightarrow x_0^-, 1 \leq i \leq n, \tag{4.1}$$

with suitable nonzero constants b_i , hence

$$\phi_i(x) = b_i P_{n-i}(x) + \sum_{j=i+1}^n \beta_{i,j} P_{n-j}(x), 1 \leq i \leq n-1; \phi_n(x) = b_n P_0(x), \tag{4.2}$$

and viceversa

$$P_0(x) = \frac{1}{b_n} \phi_n(x), P_i(x) = \frac{1}{b_{n-i}} \phi_{n-i}(x) + \sum_{j=n-i+1}^n \tilde{\beta}_{i,j} \phi_j(x), 1 \leq i \leq n-1, \tag{4.3}$$

with suitable constants $\beta_{i,j}, \tilde{\beta}_{i,j}$. In this approach the appropriate differential operators to be used are the L_k 's defined in (3.1) and here are some elementary properties of these operators.

Lemma 4.1. *The following relations are checked at once:*

$$\ker L_k = \text{span}(P_0, P_1, \dots, P_{k-1}) = \text{span}(\phi_n, \phi_{n-1}, \dots, \phi_{n-k+1}), 1 \leq k \leq n; \tag{4.4}$$

$$L_k [P_k(x)] \equiv 1, 0 \leq k \leq n-1; \tag{4.5}$$

$$L_k [P_i(x)] \equiv \int_T^x \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \dots \int_T^{t_{i-1}} \frac{dt_i}{p_i(t_i)}, 0 \leq k < i \leq n-1; \tag{4.6}$$

$$L_k [P_i(x)] \ll L_k [P_{i+1}(x)], x \rightarrow x_0^-, 0 \leq k \leq i \leq n-2. \tag{4.7}$$

Hence, we have the following chains of asymptotic relations:

$$\left\{ \begin{array}{l} L_0 [P_0(x)] \ll L_0 [P_1(x)] \ll \dots \ll L_0 [P_{n-1}(x)], \\ L_1 [P_1(x)] \ll L_1 [P_2(x)] \ll \dots \ll L_1 [P_{n-1}(x)], \\ L_2 [P_2(x)] \ll L_2 [P_3(x)] \ll \dots \ll L_2 [P_{n-1}(x)], \quad x \rightarrow x_0^-, \\ \vdots \\ L_{n-2} [P_{n-2}(x)] \ll L_{n-2} [P_{n-1}(x)]. \end{array} \right. \tag{4.8}$$

The first chain in (4.8) coincides with the second chain in (2.49) apart from the ordering and the factor $p_0(x)$. As the first term in each chain is the constant "1" all the other terms diverge to $+\infty$.

Lemma 4.2. *If a solution ϕ of $L_{\phi_1, \dots, \phi_n} u = 0$ satisfies the asymptotic relation*

$$\phi(x) \sim cP_i(x), x \rightarrow x_0^-, \tag{4.9}$$

for some $i \in \{0, 1, \dots, n-1\}$ and some nonzero constant c then the following relations hold true:

$$L_k [\phi(x)] \sim cL_k [P_i(x)], x \rightarrow x_0^-, 0 \leq k \leq i \leq n-1; \tag{4.10}$$

$$L_k [\phi(x)] \equiv 0, i+1 \leq k \leq n. \tag{4.11}$$

Moreover,

$$L_k [\phi_{n-i}(x)] \equiv \begin{cases} b_{n-k}, & 0 \leq i = k \leq n-1, \\ 0, & 0 \leq i < k, \end{cases} \tag{4.12}$$

with the b_i 's defined in (4.1). It follows from (4.1) and (4.10) that all relations in (4.8) hold true after replacing P_i by ϕ_{n-i} hence, consistently with (3.7), we have the asymptotic scales:

$$\left\{ \begin{array}{l} L_0 [\phi_1(x)] \gg L_0 [\phi_2(x)] \gg \dots \gg L_0 [\phi_n(x)], \\ L_1 [\phi_1(x)] \gg L_1 [\phi_2(x)] \gg \dots \gg L_1 [\phi_{n-1}(x)], \\ L_2 [\phi_1(x)] \gg L_2 [\phi_2(x)] \gg \dots \gg L_2 [\phi_{n-2}(x)], \quad x \rightarrow x_0^-, \\ \vdots \\ L_{n-2} [\phi_1(x)] \gg L_{n-2} [\phi_2(x)]. \end{array} \right. \tag{4.13}$$

Last, with the b_i 's defined in (4.1), we have the identity

$$L_k u \equiv b_{n-k} \frac{W(\phi_n, \phi_{n-1}, \dots, \phi_{n-k+1}, u)}{W(\phi_n, \phi_{n-1}, \dots, \phi_{n-k})}, 1 \leq k \leq n-1. \tag{4.14}$$

Lemma 4.3. *Any function $f \in AC^{n-1}[T, x_0[$ admits of a representation of type*

$$f(x) = c_1\phi_1(x) + \dots + c_n\phi_n(x) + \frac{1}{p_0(x)} \int_T^x \frac{1}{p_1} \dots \int_T^{t_{n-2}} \frac{1}{p_{n-1}} \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{p_n(t)} dt, \quad x \in [T, x_0], \quad (4.15)$$

with suitable constants c_i . From (4.6), (4.12) and (4.15) we infer at once the following representations of the weighted derivatives of f with respect to the weight functions (p_0, \dots, p_n) :

$$L_k[f(x)] = c_1 L_k[\phi_1(x)] + \dots + c_{n-k} L_k[\phi_{n-k}(x)] + \int_T^x \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \dots \int_T^{t_{n-2}} \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{p_n(t)} dt, \quad x \in [T, x_0]; \quad 0 \leq k \leq n-2; \quad (4.16)$$

$$L_{n-1}[f(x)] = c_1 L_{n-1}[\phi_1(x)] + \int_T^x \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{p_n(t)} dt \stackrel{(4.12)}{=} c_1 b_1 + \int_T^x \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{p_n(t)} dt, \quad x \in [T, x_0]. \quad (4.17)$$

By (4.13) the linear combination $\sum_{i=1}^{n-k} c_i L_k[\phi_i(x)]$ in the right-hand side of (4.16) is in itself an asymptotic expansion at x_0^- for each fixed k .

We shall now characterize various situations wherein relations (4.16)-(4.17) become asymptotic expansions. In the following two theorems we state separately three cases of a single claim lest a unified statement be obscure. The reader is referred to the first remark after Theorem 4.4 to grasp the meaning of the differentiated asymptotic expansions which exhibit a special non-common phenomenon.

Theorem 4.4 (Asymptotic expansions formally differentiable according to the C.F. of type (I)). *Let $f \in AC^{n-1}[T, x_0]$.*

(I) *The following are equivalent properties for a suitable constant a_1 :*

1) *The set of asymptotic relations*

$$L_k[f(x)] = a_1 L_k[\phi_1(x)] + o(L_k[\phi_1(x)]), \quad x \rightarrow x_0^-; \quad 0 \leq k \leq n-1. \quad (4.18)$$

2) *The single asymptotic relation*

$$L_{n-1}[f(x)] = a_1 b_1 + o(1), \quad x \rightarrow x_0^-, \quad \text{with } b_1 \text{ defined in (4.1),} \quad (4.19)$$

which is the explicit form of the relation in (4.18) for $k = n-1$.

3) *The improper integral*

$$\int_T^{x_0} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{p_n(t)} dt \quad \text{converges.} \quad (4.20)$$

Under condition (4.20) we have the representation formula

$$L_{n-1}[f(x)] = a_1 b_1 - \int_x^{x_0} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{p_n(t)} dt, \quad x \in [T, x_0]. \quad (4.21)$$

(II) *For a fixed $i \in \{2, \dots, n\}$ the following are equivalent properties for suitable constants a_i (the same in each set of conditions):*

4) *The set of asymptotic expansions as $x \rightarrow x_0^-$:*

$$\begin{cases} L_k[f(x)] = a_1 L_k[\phi_1(x)] + \dots + a_i L_k[\phi_i(x)] + o(L_k[\phi_i(x)]), \quad 0 \leq k \leq n-i; \\ L_{n-i+h}[f(x)] = a_1 L_{n-1+h}[\phi_1(x)] + \dots + a_{i-h} L_{n-i+h}[\phi_{i-h}(x)] + o(1); \quad 0 \leq h \leq i-1. \end{cases} \quad (4.22)$$

5) *The second group of asymptotic expansions in (4.22), i.e.*

$$L_{n-i+h}[f(x)] = a_1 L_{n-1+h}[\phi_1(x)] + \dots + a_{i-h} L_{n-i+h}[\phi_{i-h}(x)] + o(1), \quad x \rightarrow x_0^-; \quad 0 \leq h \leq i-1, \quad (4.23)$$

where we point out that the last meaningful term in the right-hand side is a constant.

6) *The following improper integral, involving “i” iterated integrations,*

$$\int_T^{x_0} \frac{1}{p_{n-i+1}} \int_{t_{n-i+1}}^{x_0} \frac{1}{p_{n-i+2}} \dots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{p_n(t)} dt \quad \text{converges.} \quad (4.24)$$

Under condition (4.24), we have the representation formula

$$L_{n-i} [f(x)] = a_1 L_{n-i} [\phi_1(x)] + \dots + a_i L_{n-i} [\phi_i(x)] + (-1)^i \int_x^{x_0} \frac{1}{p_{n-i+1}} \int_{t_{n-i+1}}^{x_0} \frac{1}{p_{n-i+2}} \dots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1, \dots, \phi_n} [f(t)]}{p_n(t)} dt, \tag{4.25}$$

for $x \in [T, x_0[$, as well as the corresponding formulas for the functions $L_{n-i+h} [f(x)]$ with $0 \leq h \leq i-1$, obtained by suitable differentiations of (4.25): see remark 3 below.

Remarks. 1) Relations in (4.22) may be read as follows. The first relation, involving L_0 , is equivalent to the asymptotic expansion

$$f(x) = a_1 \phi_1(x) + \dots + a_i \phi_i(x) + o(\phi_i(x)), \quad x \rightarrow x_0^-, \tag{4.26}$$

and the relations involving L_k , with $1 \leq k \leq n-i$, state that (4.26) can be formally differentiated $(n-i)$ times in the sense of formally applying the operators L_k to the remainder in (4.26). In so doing one arrives at the expansion

$$L_{n-i} [f(x)] = a_1 L_{n-i} [\phi_1(x)] + \dots + a_i L_{n-i} [\phi_i(x)] + o(1), \quad x \rightarrow x_0^-, \tag{4.27}$$

where $L_{n-i} [\phi_i(x)] \equiv \text{constant}$. The process of formal differentiation, from the order $(n-i+1)$ up to $(n-1)$, goes on according to the following rule: in (4.27) and in each expansion in (4.23) the last term is constant and is lost after one further weighted differentiation while the remainder preserves its simple growth-order estimate of “ $o(1)$ ”. So the first $(n-i+1)$ expansions, *i.e.* those involving L_0, L_1, \dots, L_{n-i} , have the same number of meaningful terms whereas each of the other $(i-1)$ expansions is deprived of the last meaningful term at each successive differentiation. We rewrite more explicitly the expansions in (4.22) to better highlight the dynamics of this process:

$$\left\{ \begin{array}{l} f(x) = a_1 \phi_1(x) + \dots + a_i \phi_i(x) + o(\phi_i(x)), \\ L_1 [f(x)] = a_1 L_1 [\phi_1(x)] + \dots + a_i L_1 [\phi_i(x)] + o(L_1 [\phi_i(x)]), \\ \vdots \\ L_{n-i} [f(x)] = a_1 L_{n-i} [\phi_1(x)] + \dots + a_i \underbrace{L_{n-i} [\phi_i(x)]}_{\text{constant}} + o(1), \\ L_{n-i+1} [f(x)] = a_1 L_{n-i+1} [\phi_1(x)] + \dots + a_{i-1} \underbrace{L_{n-i+1} [\phi_{i-1}(x)]}_{\text{constant}} + o(1), \\ \vdots \\ L_{n-2} [f(x)] = a_1 L_{n-2} [\phi_1(x)] + a_2 \underbrace{L_{n-2} [\phi_2(x)]}_{\text{constant}} + o(1), \\ L_{n-1} [f(x)] = a_1 \underbrace{L_{n-1} [\phi_1(x)]}_{\text{constant}} + o(1). \end{array} \right. \tag{4.28}$$

The loss of the last meaningful term, where it occurs, is caused by formula (4.12) for $i = k-1$ which, after renaming the indexes, reads

$$L_{n-i+h} [\phi_{i-h+1}(x)] \equiv 0. \tag{4.29}$$

Notice that in the second group of expansions in (4.28) *i.e.* those with remainder “ $o(1)$ ”, the meaningful terms disappear one after one in reversed order if compared with Taylor’s formula.

2) It is shown in §6, after the proof of Theorem 4.4, that the set (4.23) is not equivalent in general to the single relation

$$L_{n-i} [f(x)] = a_1 L_{n-i} [\phi_1(x)] + \dots + a_i L_{n-i} [\phi_i(x)] + o(1), \quad x \rightarrow x_0^-, \tag{4.30}$$

as in part (I) of the theorem (case $i = 1$).

3) Suitable weighted differentiations of (4.25) yield integral representations of the remainders in the differentiated expansions of orders greater than $(n - i)$ and these representations are numerically meaningful. On the contrary, if $i < n$, then successive integrations of (4.25) contain some constants not uniquely defined hence the corresponding representations are of no numerical use without additional information on f .

For $i = n$ the subset of (4.22) involving the operators $L_k, 1 \leq k \leq n - i$, is empty and here is an explicit and expanded statement.

Theorem 4.5 (The case $i = n$ in Theorem 4.4). *For $f \in AC^{n-1}[T, x_0[$ the following are equivalent properties:*

1) *The set of asymptotic expansions as $x \rightarrow x_0^-$ for suitable constants a_1, \dots, a_n :*

$$\begin{cases} f(x) = a_1\phi_1(x) + \dots + a_n\phi_n(x) + o(\phi_n(x)), \\ L_k[f(x)] = a_1L_k[\phi_1(x)] + \dots + a_{n-k} \underbrace{L_k[\phi_{n-k}(x)]}_{\text{constant}} + o(1), 1 \leq k \leq n-1, \end{cases} \quad (4.31)$$

where the last term in each expansion is lost in the successive expansion.

2) *The improper integral*

$$\int_T^{x_0} \frac{1}{p_1} \dots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{p_n(t)} dt \text{ converges.} \quad (4.32)$$

3) *There exist n real numbers a_1, \dots, a_n and a function Φ_n Lebesgue-summable on $[T, x_0[$ such that*

$$f(x) = a_1\phi_1(x) + \dots + a_n\phi_n(x) + \frac{(-1)^n}{p_0(x)} \int_x^{x_0} \frac{1}{p_1} \int_{t_1}^{x_0} \frac{1}{p_2} \dots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \Phi_n(t) dt, x \in [T, x_0[. \quad (4.33)$$

If this is the case Φ_n is determined up to a set of measure zero and

$$\Phi_n(x) = \frac{1}{p_n(x)} L_{\phi_1, \dots, \phi_n}[f(x)] \text{ a.e. on } [T, x_0[. \quad (4.34)$$

The phenomenon described in (4.28) and (4.31) is intrinsic in the theory; it occurs even in the seemingly elementary case of real-power expansions, ([4], Th. 4.2-(ii), p. 181, and formula (7.2), p. 195), where the asymptotic scale enjoys the most favourable algebraic properties. This type of formal differentiation of an asymptotic expansion does not frequently occur in the literature though the results in this section show that it is one of the possible natural situations. An instance (not inserted in a general theory) is to be found in a paper by Schoenberg ([15], Th. 3, p. 258) and refers to the asymptotic expansion

$$f(x) = a_1x^{-1} + a_2x^{-2} + \dots + a_nx^{-n} + O(x^{-n-1}), x \rightarrow +\infty. \quad (4.35)$$

5. The Second Factorizational Approach and Estimates of the Remainders

Now we face our problem starting from a C.F. of type (II) at x_0 . Referring to Proposition 2.4 the most natural choice is the special C.F. of $L_{\phi_1, \dots, \phi_n}$ in (2.39), with the q_i 's in (2.35) and satisfying conditions (2.36). According to the Conjectures formulated in ([5], §3) we shall characterize a set of asymptotic expansions, involving the operators M_k defined in (3.3), wherein each coefficient of the first expansion may be found by an independent limiting process instead of the recursive formulas (1.3), and the existence of the sole last coefficient implies the existence of all the preceding coefficients. In this new context a representation of the following type is appropriate for any function $f \in AC^{n-1}[T, x_0[$:

$$f(x) = c_1\phi_1(x) + \dots + c_n\phi_n(x) + \frac{1}{q_0(x)} \int_T^x \frac{1}{q_1} \dots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt, x \in [T, x_0[, \quad (5.1)$$

with suitable constants c_i . Applying the operators M_k to (5.1) we get the following representations of the

weighted derivatives of f with respect to the weight functions (q_0, \dots, q_n) :

$$M_k[f(x)] = c_{k+1}M_k[\phi_{k+1}(x)] + \dots + c_n M_k[\phi_n(x)] \\ + \int_T^x \frac{dt_{k+1}}{q_{k+1}(t_{k+1})} \dots \int_T^{t_{n-2}} \frac{dt_{n-1}}{q_{n-1}(t_{n-1})} \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt, \quad 0 \leq k \leq n-1. \quad (5.2)$$

Warning! To simplify formulas and to leave no ambiguity about the signs of the involved quantities we assume throughout this section that the Wronskians in (2.24) are strictly positive. Hence, by (3.19) $\epsilon_k = 1$, and the last relation in (5.2) explicitly is

$$M_{n-1}[f(x)] = c_n + \int_T^x \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt, \quad x \in [T, x_0[. \quad (5.3)$$

By (3.8), the ordered linear combination in (5.2),

$$\sum_{i=k+1}^n c_i M_k[\phi_i(x)], \quad (5.4)$$

is an asymptotic expansion at x_0^- for each fixed k , $0 \leq k \leq n-1$. Unlike §4 we first state here the result concerning a complete asymptotic expansion, *i.e.* of type (1.1), because it is the most expressive result in this paper and characterizes the simple circumstance that $M_{n-1}[f(x)] = a_n + o(1)$ via a set of n asymptotic expansions. Always refer to Proposition 3.2 for properties of the M_k 's.

Theorem 5.1 (Complete expansions formally differentiable according to a C.F. of type (II)). *Let our T.A.S. be such that all the Wronskians in (2.24) are strictly positive and let $f \in AC^{n-1}[T, x_0[$.*

(I) *The following are equivalent properties:*

1) *There exist n real numbers a_1, \dots, a_n such that:*

$$f(x) = a_1 \phi_1(x) + \dots + a_n \phi_n(x) + o(\phi_n(x)), \quad x \rightarrow x_0^-; \quad (5.5)$$

$$M_k[f(x)] = a_{k+1} M_k[\phi_{k+1}(x)] + \dots + a_n M_k[\phi_n(x)] + o(M_k[\phi_n(x)]), \quad x \rightarrow x_0^-, \quad 1 \leq k \leq n-1; \quad (5.6)$$

where the first term in each expansion is lost in the successive expansion, just the same phenomenon as in Taylor's formula. Notice that the relation that would be obtained in (5.6) for $k=0$ differs from relation in (5.5) by the common factor $q_0(x)$.

2) *All the following limits exist as finite numbers:*

$$\lim_{x \rightarrow x_0^-} M_k[f(x)] \equiv a_{k+1}, \quad 0 \leq k \leq n-1, \quad (5.7)$$

where the a_k 's coincide with those in (5.5).

3) *The single last limit in (5.7) exists as a finite number, *i.e.**

$$\lim_{x \rightarrow x_0^-} M_{n-1}[f(x)] \equiv a_n, \quad (5.8)$$

and (5.8) is nothing but the relation in (5.6) for $k=n-1$ which reads $M_{n-1}[f(x)] = a_n + o(1)$, $x \rightarrow x_0^-$.

4) *The improper integral*

$$\int_T^{x_0} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt \quad \text{converges,} \quad (5.9)$$

and automatically also the iterated improper integral

$$\int_T^{x_0} \frac{1}{q_1} \int_T^{x_0} \frac{1}{q_2} \dots \int_T^{x_0} \frac{1}{q_{n-1}} \int_T^{x_0} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt \quad \text{converges.} \quad (5.10)$$

5) *There exist n real numbers a_1, \dots, a_n and a function Ψ_n Lebesgue-summable on $[T, x_0[$ such that*

$$f(x) = \sum_{i=1}^n a_i \phi_i(x) + \frac{(-1)^n}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \dots \int_{t_{n-2}}^{x_0} \frac{1}{q_{n-1}} \int_{t_{n-1}}^{x_0} \Psi_n(t) dt, \quad x \in [T, x_0[\tag{5.11}$$

where, by (2.35), $1/q_0(x) = \phi_1(x)$. In this case Ψ_n is determined up to a set of measure zero and

$$\Psi_n(x) = \frac{1}{q_n(x)} L_{\phi_1, \dots, \phi_n} [f(x)] \text{ a.e. on } [T, x_0[. \tag{5.12}$$

(II) Whenever properties in part (I) hold true we have integral representation formulas for the remainders

$$R_0(x) := f(x) - \sum_{i=1}^n a_i \phi_i(x); \quad R_k(x) := M_k[f(x)] - \sum_{i=1}^{n-k} a_{k+i} M_k[\phi_{k+i}(x)], \quad 1 \leq k \leq n-1, \tag{5.13}$$

namely,

$$R_0(x) = \frac{(-1)^n}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \dots \int_{t_{n-2}}^{x_0} \frac{1}{q_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1, \dots, \phi_n} [f(t)]}{q_n(t)} dt, \tag{5.14}$$

$$R_k(x) = (-1)^{n+k} \int_x^{x_0} \frac{1}{q_{k+1}} \dots \int_{t_{n-2}}^{x_0} \frac{1}{q_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1, \dots, \phi_n} [f(t)]}{q_n(t)} dt, \tag{5.15}$$

for $x \in [T, x_0[$, $1 \leq k \leq n-1$. From (5.14) we get the following estimate of R_0 wherein the order of smallness with respect to ϕ_n is made more explicit than in Theorem 4.5 (formula in (2.40) for $i = n$ is used):

$$|R_0(x)| \leq |\phi_n(x)| \cdot \sup_{t \geq x} \left| \int_t^{x_0} \frac{L_{\phi_1, \dots, \phi_n} [f(\tau)]}{q_n(\tau)} d\tau \right|, \quad x \in [T, x_0[. \tag{5.16}$$

Under the stronger hypothesis of absolute convergence for the improper integral we get

$$|R_0(x)| \leq |\phi_n(x)| \cdot \int_x^{x_0} \frac{|L_{\phi_1, \dots, \phi_n} [f(\tau)]|}{|q_n(\tau)|} dt, \quad x \in [T, x_0[. \tag{5.17}$$

Similar estimates can be obtained for the R_k 's.

Remarks. 1) As noticed in ([4], Remark 1 after Th. 4.1, pp. 179-180) the remarkable inference “3) \Rightarrow 2)” is true for the special operator M_{n-1} stemming out from a C.F. of type (II) at x_0 but not for any $(n-1)$ -th-order differential operator originating from an arbitrary factorization of $L_{\phi_1, \dots, \phi_n}$.

2) Condition (5.9) involves the sole coefficient q_n which admits of the explicit expression in (2.35) in terms of ϕ_1, \dots, ϕ_n

$$q_n = W(\phi_1, \dots, \phi_n) / W(\phi_1, \dots, \phi_{n-1}); \tag{5.18}$$

hence (5.9) can be rewritten as

$$\int_T^{\rightarrow x_0} \frac{W(\phi_1(t), \dots, \phi_{n-1}(t))}{W(\phi_1(t), \dots, \phi_n(t))} L_{\phi_1, \dots, \phi_n} [f(t)] dt \text{ converges.} \tag{5.19}$$

For $n = 2$ the ratio inside the integral equals $\phi_1/W(\phi_1, \phi_2)$ and we reobtain the result in ([1]; condition (5.15), p. 265).

3) In Theorem 4.5, generally speaking, no such estimates as in (5.16)-(5.17) can be obtained due to the divergence of all the improper integrals in (4.33) if the innermost integral is factored out.

4) Theorem 5.1 changes the perspective of the elementary characterizations in (1.3) of the coefficients a_k : in (1.3) the a_k 's are defined recursively whereas in (5.7) each a_k has its own independent expression and, moreover, the existence of a_n , as the limit in (5.8), implies the existence of a_1, \dots, a_{n-1} .

In the following result about incomplete expansions formal differentiation is in general legitimate a number of

times less than the “length” of the expansion (see Remark 2 after the statement).

Theorem 5.2 (A result on incomplete asymptotic expansions). *Let our T.A.S. be such that all the Wronskians in (2.24) are strictly positive and let $f \in AC^{n-1}[T, x_0[$.*

(I) *For a fixed $i \in \{2, \dots, n-1\}$ the following are equivalent properties:*

1) *There exist i real numbers a_1, \dots, a_i such that:*

$$f(x) = a_1\phi_1(x) + \dots + a_i\phi_i(x) + o(\phi_i(x)), x \rightarrow x_0^-; \tag{5.20}$$

$$M_k[f(x)] = a_{k+1}M_k[\phi_{k+1}(x)] + \dots + a_iM_k[\phi_i(x)] + o(M_k[\phi_i(x)]), x \rightarrow x_0^-; 1 \leq k \leq i-1. \tag{5.21}$$

2) *All the following limits exist as finite numbers:*

$$\lim_{x \rightarrow x_0^-} M_k[f(x)] \equiv a_{k+1}, 0 \leq k \leq i-1, \tag{5.22}$$

where the a_k 's coincide with those in (5.20)-(5.21).

3) *The single last limit in (5.22) exists as a finite number, i.e.*

$$\lim_{x \rightarrow x_0^-} M_{i-1}[f(x)] \equiv a_i, \tag{5.23}$$

and (5.23) coincides with the relation in (5.21) for $k = i-1$.

4) *The improper integral (involving $n-i+1$ iterated integrations)*

$$\int_T^{\rightarrow x_0} \frac{dt_i}{q_i(t_i)} \int_T^{t_i} \frac{dt_{i+1}}{q_{i+1}(t_{i+1})} \dots \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt \text{ converges,} \tag{5.24}$$

and automatically also the iterated improper integral

$$\int_T^{x_0} \frac{1}{q_1} \dots \int_{t_{i-1}}^{x_0} \frac{1}{q_i} \int_T^{t_i} \frac{1}{q_{i+1}} \dots \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt \text{ converges.} \tag{5.25}$$

(II) *For $i = 1$ the theorem simply asserts that the asymptotic relation*

$$f(x) = a_1\phi_1(x) + o(\phi_1(x)), x \rightarrow x_0^-, \tag{5.26}$$

holds true for some real number a_1 iff the improper integral

$$\int_T^{\rightarrow x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \dots \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt \text{ converges.} \tag{5.27}$$

Remarks. 1) We shall see in the proof of Theorem 5.2, formula (7.44), that the representations of the quantities $M_k[f(x)]$, $0 \leq k \leq i-1$, contain some unspecified constants not determinable through the sole condition (5.24) which, for this reason, grants neither explicit representations nor numerical estimates of the remainders of the expansions in (5.20)-(5.21).

2) As concerns estimates of the quantities $M_k[f(x)]$ for $i \leq k \leq n-1$, the situation is as follows. By (3.13)-(3.14), the representation in (5.2) for $k = i$ has the form:

$$M_i[f(x)] = \underbrace{\int_T^x \frac{dt_{i+1}}{q_{i+1}(t_{i+1})} \dots \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt}_I + c + o(1), \tag{5.28}$$

for some constant c . If, as $x \rightarrow x_0^-$, $I(x)$ converges to a real number then we may apply Theorem 5.2 with i replaced by $i+1$; but if $I(x)$ is unbounded and oscillatory no asymptotic relation more expressive than (5.28) can be obtained generally speaking. On the contrary a favourable situation occurs when it is known a priori that $I(x)$ either converges or diverges to $\pm\infty$ and the corresponding estimates are reported in Part II-B of this paper, Theorems 8.3-8.4.

Theorem 5.3 (The analogue of Theorems 5.1-5.2 with “O”-estimates). *Let our T.A.S. be such that all the Wronskians in (2.24) are strictly positive, let $f \in AC^{n-1}[T, x_0[$ and let $i \in \{2, \dots, n\}$ be fixed. The following are equivalent properties:*

1) *There exist $(i-1)$ real numbers a_1, \dots, a_{i-1} such that:*

$$f(x) = a_1\phi_1(x) + \dots + a_{i-1}\phi_{i-1}(x) + O(\phi_i(x)), x \rightarrow x_0^-; \tag{5.29}$$

$$M_k[f(x)] = a_{k+1}M_k[\phi_{k+1}(x)] + \dots + a_{i-1}M_k[\phi_{i-1}(x)] + O(M_k[\phi_i(x)]), x \rightarrow x_0^-; 1 \leq k \leq i-1. \tag{5.30}$$

2) *All the following relations hold true:*

$$\lim_{x \rightarrow x_0^-} M_k[f(x)] \equiv a_{k+1}, 0 \leq k \leq i-2; M_{i-1}[f(x)] = O(1), x \rightarrow x_0^-; \tag{5.31}$$

where the a_k 's coincide with those in (5.29)-(5.30).

3) *It holds true the single last relation in (5.31) i.e.*

$$M_{i-1}[f(x)] = O(1), x \rightarrow x_0^-; \tag{5.32}$$

4) *We have the following estimate instead of condition (5.24):*

$$\int_T^x \frac{dt_i}{q_i(t_i)} \int_T^{t_i} \frac{dt_{i+1}}{q_{i+1}(t_{i+1})} \dots \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt = O(1), x \rightarrow x_0^-. \tag{5.33}$$

For $i = n$ condition (5.32) reads

$$\int_T^x \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt = O(1), x \rightarrow x_0^-, \tag{5.34}$$

and representation (5.11)-(5.12) must be replaced by

$$f(x) = \sum_{i=1}^{n-1} a_i \phi_i(x) + \frac{(-1)^{n-1}}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \int_{t_1}^{x_0} \frac{1}{q_2} \dots \int_{t_{n-2}}^{x_0} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt. \tag{5.35}$$

For $i = 1$ the theorem simply asserts that the asymptotic relation

$$f(x) = O(\phi_1(x)), x \rightarrow x_0^-, \tag{5.36}$$

holds true iff

$$\int_T^x \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \dots \int_T^{t_{n-1}} \frac{L_{\phi_1, \dots, \phi_n}[f(t)]}{q_n(t)} dt = O(1), x \rightarrow x_0^-. \tag{5.37}$$

6. Proofs

Proof of Proposition 2.1. For the equivalence of the two properties in 1), see Coppel ([8], Prop. 3, p. 82). “1) \Leftrightarrow 2)” is proved in Levin ([7], Th. 2.1, p. 66) where the interval I is explicitly stated to be open not in the statement of the cited theorem but at the outset of §2 on p. 58; “2) \Leftrightarrow 3)” is the classical result by Pólya [14]; “1) \Leftrightarrow 4)” is the fundamental result by Trench [9]; “1) \Rightarrow 5)” is to be found in ([10], Th. 2.2, p. 162) whereas the converse rests on the trivial fact that disconjugacy on $]a, b[$ is equivalent to disconjugacy on every compact subinterval of $]a, b[$.

Proof of Proposition 2.2. Part (I) is contained in ([12], Th. 2.1, p. 66) with reverse numbering of the ϕ_i 's whereas part (II) follows from ([12], Lemma 2.6, pp. 63-64, and Remarks on p. 67 concerning the hierarchies of the Wronskians), here again with reverse numbering of the ϕ_i 's. Levin's results are valid for an open interval and this is stated explicitly at the outset of §2 in ([7], p. 58); moreover, the tacit assumption of strict positivity of the functions forming the scale is agreed in a long list of notations and terminology in ([7], §1, p. 57, item 20).

Proof of Proposition 2.3. 1) \Rightarrow 2). Let $(\tilde{\phi}_1, \dots, \tilde{\phi}_n)$ be an extension of (ϕ_1, \dots, ϕ_n) of class $C^{n-1}]T - \epsilon, x_0[$, $\epsilon > 0$, such that

$$W(\tilde{\phi}_1, \dots, \tilde{\phi}_i) \neq 0 \text{ on }]T - \epsilon, x_0[, 1 \leq i \leq n. \tag{6.1}$$

In particular, we have

$$W(\tilde{\phi}_n, \tilde{\phi}_{n-1}, \dots, \tilde{\phi}_1) \neq 0 \text{ and } W(\tilde{\phi}_{n-1}, \dots, \tilde{\phi}_1) \neq 0 \text{ on }]T - \epsilon, x_0[, \tag{6.2}$$

and we may apply part (II) of Proposition 2.2 (regardless of the signs) because the second condition in (6.2) coincides with the condition in (2.16) for $r = 1$. So, we infer the inequalities:

$$W(\tilde{\phi}_n, \tilde{\phi}_{n-1}, \dots, \tilde{\phi}_i) \neq 0 \text{ on }]T - \epsilon, x_0[, 1 \leq i \leq n, \tag{6.3}$$

which imply (2.27). Proposition 2.2 also implies all the claims in part (II).

1) \Leftrightarrow 3). We refer to the standard definition of the concept of “extended complete Chebyshev system on a generic interval J ”, based on the maximum number of zeros for their linear combinations, see, e.g., ([12], Ch. I). A classical result states the equivalence between an ordered n -tuple (u_1, \dots, u_n) forming such a system on J and the strict positivity of the Wronskians $W(u_1, \dots, u_i), 1 \leq i \leq n$, on J . This is proved, e.g., in ([12], Ch. XI, Th. 1.1, p. 376) for a compact interval J , but the argument is valid for any interval as observed, e.g., by Mazure ([13], Prop. 2.6). This equivalence is a general fact involving only inequalities (2.24).

2) \Rightarrow 4). Here, we are retracing the steps of the proof in ([12], Ch. XI, Th 1.2, pp. 379-380) in a way that includes in one proof the expressions given in (2.31). First, inequalities (2.24) grant that the functions $w_i, 0 \leq i \leq n-1$, are well defined on $]T, x_0[$ and satisfy (2.29). The second expression for $w_i, i \geq 2$, in (2.30), is a classical identity, see ([8], Lemma 4, p. 87) for a synthetic proof under our regularity assumptions. Moreover, inequalities (2.24) and (2.27) together grant, by Proposition 2.2-(II), the asymptotic relations (2.14), hence

$$\phi_2(x)/\phi_1(x) = o(1); \quad \frac{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_{i+1}(x))}{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_i(x))} = o(1), x \rightarrow x_0^-, 2 \leq i \leq n-1. \tag{6.4}$$

This implies three facts: 1) the convergence of the improper integrals

$$\int^{x_0} (\phi_2/\phi_1)'; \quad \int \left[\frac{W(\phi_1(t), \dots, \phi_{i-1}(t), \phi_{i+1}(t))}{W(\phi_1(t), \dots, \phi_{i-1}(t), \phi_i(t))} \right]' dt, 2 \leq i \leq n-1; \tag{6.5}$$

2) the representations for ϕ_1, ϕ_2 ; 3) the identity

$$\frac{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_{i+1}(x))}{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_i(x))} = \int_x^{x_0} w_i(t) dt, 2 \leq i \leq n-1. \tag{6.6}$$

Before using induction, we prove the representation of ϕ_3 to highlight the role of (6.6). We have

$$W(\phi_1, \phi_3)/W(\phi_1, \phi_2) = \int_x^{x_0} w_2(t) dt; \quad W(\phi_1, \phi_2) = -(\phi_1)^2 w_1; \quad W(\phi_1, \phi_3) = (\phi_1)^2 (\phi_3/\phi_1)', \tag{6.7}$$

whence

$$\begin{cases} (\phi_3/\phi_1)'(x) = -w_1(x) \int_x^{x_0} w_2(t) dt, \\ \phi_3(x)/\phi_1(x) = c + \int_x^{x_0} w_1 \int_{t_1}^{x_0} w_2(t) dt \stackrel{\text{by (2.23)}}{=} \int_x^{x_0} w_1 \int_{t_1}^{x_0} w_2(t) dt, \end{cases} \tag{6.8}$$

which implies the representation of ϕ_3 in (2.28). To prove the representations of ϕ_i for $4 \leq i \leq n-1$, we proceed by induction supposing to have proved our inference 2) \Rightarrow 4) for any i -tuple forming a T.A.S. on $]T, x_0[$; hence, our representations hold true for ϕ_1, \dots, ϕ_i and we must prove it for ϕ_{i+1} . Putting

$$\psi_k(x) := (\phi_{k+1}/\phi_1)'(x), 1 \leq k \leq i, \tag{6.9}$$

we immediately infer from (1.5) and from (2.14) referred to $x \rightarrow x_0^-$ that

$$(\phi_1(x))^{k+1} W(\psi_1, \dots, \psi_k) \equiv W(\phi_1, \dots, \phi_{k+1}) \neq 0 \text{ on } [T, x_0]; \quad (6.10)$$

$$\begin{cases} \psi_k \equiv (\phi_1)^{-2} W(\phi_1, \phi_{k+1}) \gg (\phi_1)^{-2} W(\phi_1, \phi_{k+2}) \equiv \psi_{k+1}, & x \rightarrow x_0^-, \text{ i.e.} \\ \psi_1(x) \gg \psi_2(x) \gg \dots \gg \psi_i(x), & x \rightarrow x_0^-. \end{cases} \quad (6.11)$$

(The n -tuple (ψ_1, \dots, ψ_n) is sometimes called the “reduced system”). Moreover, (6.6) and (6.10) imply

$$\frac{W(\psi_1(x), \dots, \psi_{i-2}(x), \psi_i(x))}{W(\psi_1(x), \dots, \psi_{i-2}(x), \psi_{i-1}(x))} = \int_x^{x_0} w_i(t) dt. \quad (6.12)$$

We may now apply our inductive hypothesis inferring that

$$\psi_i(x) = \tilde{w}_0(x) \cdot \int_x^{x_0} \tilde{w}_1 \cdots \int_{i-2}^{x_0} \tilde{w}_{i-1}, \quad (6.13)$$

where the \tilde{w}_k 's are defined by the expressions on the right of (2.30) with the ϕ_k 's replaced by the ψ_k 's and (6.10) implies

$$\begin{cases} \tilde{w}_0 := \psi_1 \equiv (\phi_2/\phi_1)' = -w_1; \\ \tilde{w}_k := -\frac{W(\psi_1, \dots, \psi_{k-1}, \psi_{k+1})}{W(\psi_1, \dots, \psi_{k-1}, \psi_k)} = -\frac{W(\phi_1, \dots, \phi_k, \phi_{k+2})}{W(\phi_1, \dots, \phi_k, \phi_{k+1})} = w_{k+1}, & 2 \leq k \leq i-1; \end{cases} \quad (6.14)$$

and (6.13) becomes

$$(\phi_{i+1}/\phi_1)'(x) = -w_1(x) \cdot \int_x^{x_0} w_2 \cdots \int_{i-2}^{x_0} w_i, \quad (6.15)$$

which, by (2.23), gives the sought-for formula for ϕ_{i+1} . Formulas (2.32) may be proved quite simply, in alternative to the inductive argument suggested in ([12], p. 380), using the second expressions for the w_i 's given in (2.31); putting for brevity,

$$W_i := W(\phi_1, \dots, \phi_i), \quad (6.16)$$

we have as in ([8], p. 92):

$$\begin{cases} W_1 = w_0; & W_2/W_1 = -w_0 w_1; \\ W_{i+1}/W_i = -w_i \cdot W_i/W_{i-1} = +w_i w_{i-1} W_{i-1}/W_{i-2} = \dots = (-1)^i w_0 w_1 \cdots w_i, & 2 \leq i \leq n-1; \end{cases} \quad (6.17)$$

hence,

$$\begin{aligned} W_i &= (-1)^{i-1} w_0 w_1 \cdots w_{i-1} \cdot W_{i-1} = (-1)^{i-1} w_0 w_1 \cdots w_{i-1} \cdot (-1)^{i-2} w_0 w_1 \cdots w_{i-2} \cdot W_{i-2} = \dots \\ &= (-1)^{(i-1)+(i-2)+\dots+2+1} [w_0 w_1 \cdots w_{i-1}] [w_0 w_1 \cdots w_{i-2}] \cdots [w_0 w_1] w_0 = (2.32), \end{aligned} \quad (6.18)$$

and this shows the converse inference “4) \Rightarrow 2)”.

Proof of Proposition 2.4. 1)-2). Properties in (2.36) follow directly from the assumptions, and relations in (2.37) are a standard fact as remarked in the preceding proof. As concerns (2.38), the continuity of the q_i 's at the endpoint T implies $\int_T (1/q_i) < +\infty$, whereas from (2.37) we get

$$\begin{cases} \int_T^x \frac{1}{q_1} = \text{constant} + \frac{\phi_2(x)}{\phi_1(x)} \stackrel{\text{by (2.23)}}{\text{convergent as } x \rightarrow x_0^-}; \\ \int_T^x \frac{1}{q_i} = \text{constant} + \frac{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_{i+1}(x))}{W(\phi_1(x), \dots, \phi_{i-1}(x), \phi_i(x))} \stackrel{\text{by (2.14)}}{\text{convergent as } x \rightarrow x_0^-}, & 2 \leq i \leq n-1. \end{cases} \quad (6.19)$$

Factorization (2.39) is then the classical factorization arising from (2.35) and discovered for the first time by Pólya [14]. Representations (2.40) are contained in Proposition 2.3 with different notations. In general, by (2.12), the calculations in (6.19) prove the existence of a C.F. of type (II) at x_0 valid on a suitable left neighborhood of x_0 .

3) The very same reasonings prove the properties of the p_i 's; the proof of (2.45) is similar to that in (6.19)

$$\begin{cases} \int_T^x \frac{1}{p_1} = \text{constant} + \frac{\phi_{n-1}(x)}{\phi_n(x)} \stackrel{\text{by (2.23)}}{\text{divergent as } x \rightarrow x_0^-}; \\ \int_T^x \frac{1}{p_i} = \text{constant} + \frac{W(\phi_n(x), \dots, \phi_{n-i+2}(x), \phi_{n-i}(x))}{W(\phi_n(x), \dots, \phi_{n-i+2}(x), \phi_{n-i+1}(x))} \stackrel{\text{by (2.14)}}{\text{divergent as } x \rightarrow x_0^-, 2 \leq i \leq n-1}; \end{cases} \quad (6.20)$$

and in general, by (2.10), these calculations prove the existence of a C.F. of type (I) at x_0 valid on the whole open interval where the given operator is assumed disconjugate. The claims in 4) are trivial.

Proof of Proposition 3.2. Relations (3.12) to (3.14) are directly checked using representations (2.40). Relation (3.15) follows from the second relation in (3.6) replacing u by ϕ_{k+1} and using (3.13). If (3.16) holds true for some sufficiently regular f , then (3.4) implies $M_k[f(x)] \equiv 0$ and (3.17) follows from (3.12)-(3.13). The converse trivially follows again from (3.12)-(3.13). Now suppose (3.16)-(3.17) to be true on the left of x_0 ; relation (3.18) for $h=1$ is nothing but the obvious relation $a_1 = \lim_{x \rightarrow x_0^-} f(x)/\phi_1(x)$.

For $h \geq 2$, we use all relations (3.12), (3.13), (3.14) and get from (3.17)

$$M_{h-1}[f(x)] = \sum_{i=0}^{k-h} a_{h+i} M_{h-1}[\phi_{h+i}(x)] = \epsilon_{h-1} a_h + \sum_{i=1}^{k-h} a_{h+i} M_{h-1}[\phi_{h+i}(x)] = \epsilon_{h-1} a_h + o(1), \quad (6.21)$$

where the remainder “ $o(1)$ ” is $\equiv 0$ for $h=k$.

Proof of Lemma 4.2. From the chain $P_{n-1}(x) \gg \dots \gg P_0(x)$, $x \rightarrow x_0$, we get

$$\phi(x) = cP_i(x) + \alpha_{i-1}P_{i-1}(x) + \dots + \alpha_0P_0(x) \quad (6.22)$$

for suitable constants α_k , hence

$$L_k[\phi(x)] = cL_k[P_i(x)] + \alpha_{i-1}L_k[P_{i-1}(x)] + \dots + \alpha_0L_k[P_0(x)]; \quad (6.23)$$

now (4.10) follows from (4.7), and (4.11) follows from (4.4). If in (6.22) we replace ϕ by ϕ_{n-i} we have $c = b_{n-i}$ and the identities in (4.12) follow from (4.4) and (4.5). The identity in (4.12) for $i=k$, i.e. $L_k[\phi_{n-k}(x)] \equiv b_{n-k}$, together with the first relation in (3.5) imply (4.14).

Proof of Theorem 4.4. Part (I). From (4.12) and (4.17), with $c_1 = a_1$, we infer at once the equivalence “(ii) \Leftrightarrow (iii)” as well as representation in (4.21). The inference “1) \Rightarrow 2)” being obvious let us prove the converse simply denoting by L our operator $L_{\phi_1, \dots, \phi_n}$. We shall repeatedly use the recursive formula (3.2) in the form

$$L_{k-1}u = \int_T^x \frac{1}{p_k(t)} L_k[u(t)] dt + \text{constant}, 1 \leq k \leq n. \quad (6.24)$$

If (4.19) holds true we have (4.21), and representations in (4.16) can be rewritten as

$$\begin{aligned} L_k[f(x)] &= \{c_1 L_k[\phi_1(x)] + \dots + c_{n-k} L_k[\phi_{n-k}(x)]\} \\ &+ \left(\int_T^{x_0} \frac{L[f(t)]}{p_n(t)} dt \right) \cdot \int_T^x \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \dots \int_T^{t_{n-2}} \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} \\ &- \int_T^x \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \dots \int_T^{t_{n-2}} \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} \int_{t_{n-1}}^{x_0} \frac{L[f(t)]}{p_n(t)} dt; \quad 0 \leq k \leq n-2. \end{aligned} \quad (6.25)$$

Now we have

$$\int_T^x \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \cdots \int_T^{t_{n-2}} \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} \stackrel{\text{by (4.6)}}{=} L_k [P_{n-1}(x)] = \cdots \tag{6.26}$$

by (4.1) and (4.10) $\cdots = (1/b_1)L_k[\phi_1(x)] + o(L_k[\phi_1(x)])$. And after substituting into (6.25), we get

$$\begin{aligned} L_k[f(x)] &= c_1 L_k[\phi_1(x)] + o(L_k[\phi_1(x)]) + \bar{c} L_k[P_{n-1}(x)] + o(L_k[P_{n-1}(x)]) \\ &= \left(c_1 + \frac{\bar{c}}{b_1} \right) L_k[\phi_1(x)] + o(L_k[\phi_1(x)]), \quad 0 \leq k \leq n-2, \end{aligned} \tag{6.27}$$

where $\bar{c} := \int_T^{x_0} L[f(t)]/p_n(t) dt$, and the coefficient

$$c := c_1 + (\bar{c}/b_1) \tag{6.28}$$

is independent of k . To show that c coincides with the a_1 appearing in (4.19) we may suitably integrate (4.19) to obtain, by (3.1),

$$\begin{aligned} L_{n-2}[f(x)] &= \int_T^x \frac{L_{n-1}[f(t)]}{p_{n-1}(t)} dt + \text{constant} \stackrel{\text{by (4.19)}}{=} a_1 b_1 \int_T^x \frac{dt}{p_{n-1}(t)} + o\left(\int_T^x \frac{dt}{p_{n-1}(t)}\right) \\ &\stackrel{\text{by (4.6)}}{=} a_1 b_1 L_{n-2}[P_{n-1}(x)] + o(L_{n-2}[P_{n-1}(x)]) \\ &= \cdots \text{by (4.1) and (4.10)} \cdots = a_1 L_{n-2}[\phi_1(x)] + o(L_{n-2}[\phi_1(x)]). \end{aligned} \tag{6.29}$$

Part (II). Case $i = 2$. We must prove the equivalence of the following three contingencies:

$$\begin{cases} f(x) = a_1 \phi_1(x) + a_2 \phi_2(x) + o(\phi_2(x)), \\ L_k[f(x)] = a_1 L_k[\phi_1(x)] + a_2 L_k[\phi_2(x)] + o(L_k[\phi_2(x)]), \quad 1 \leq k \leq n-2, \\ L_{n-1}[f(x)] = a_1 L_{n-1}[\phi_1(x)] + o(L_{n-1}[\phi_1(x)]); \end{cases} \tag{6.30}$$

$$\begin{cases} L_{n-2}[f(x)] = a_1 L_{n-2}[\phi_1(x)] + a_2 L_{n-2}[\phi_2(x)] + o(L_{n-2}[\phi_2(x)]), \\ L_{n-1}[f(x)] = a_1 L_{n-1}[\phi_1(x)] + o(L_{n-1}[\phi_1(x)]); \end{cases} \tag{6.31}$$

$$\int_T^{x_0} \frac{dt}{p_{n-1}(t)} \int_t^{x_0} \frac{L[f(\tau)]}{p_n(\tau)} d\tau \text{ convergent.} \tag{6.32}$$

First, we prove “(6.31) \Leftrightarrow (6.32)”. If (6.32) holds true then, by part (I) of our theorem, we have all relations in (4.18) and in particular the second relation in (6.31). Moreover we can rewrite representation in (4.16) for $k = n-2$ in the form

$$L_{n-2}[f(x)] = a_1 L_{n-2}[\phi_1(x)] + a_2 L_{n-2}[\phi_2(x)] + \int_x^{x_0} \frac{dt}{p_{n-1}(t)} \int_t^{x_0} \frac{L[f(\tau)]}{p_n(\tau)} d\tau, \tag{6.33}$$

where a_1 is just the same as in the second relation in (6.31) and a_2 is a suitable constant. This yields the first relation in (6.31) because $L_{n-2}[\phi_2(x)]$ is a nonzero constant by (4.12). Viceversa if relations in (6.31) hold true then, by part (I), we have representation in (4.21) by which we replace the quantity $L_{n-1}[f(t)]$ in the first equality in (6.33). Denoting by c_{n-2}, \bar{c}_{n-2} suitable constants we get

$$\begin{aligned} L_{n-2}[f(x)] &= c_{n-2} + \int_T^x \frac{L_{n-1}[f(t)]}{p_{n-1}(t)} dt \\ &= c_{n-2} + \int_T^x \frac{1}{p_{n-1}(t)} \left[a_1 L_{n-1}[\phi_1(t)] - \int_t^{x_0} \frac{L[f(\tau)]}{p_n(\tau)} d\tau \right] dt \\ &= \bar{c}_{n-2} + a_1 L_{n-2}[\phi_1(x)] - \int_T^x \frac{dt}{p_{n-1}(t)} \int_t^{x_0} \frac{L[f(\tau)]}{p_n(\tau)} d\tau. \end{aligned} \tag{6.34}$$

By comparison with the first relation in (6.31) we get (6.32) because $L_{n-2}[\phi_2(x)]$ is a constant. As the inference “(6.30) \Rightarrow (6.31)” is obvious it remains to prove the converse. Using (6.24) and integrating the first relation in (6.31), we get (with suitable constants c_{n-3}, \bar{c}_{n-3})

$$L_{n-3}[f(x)] \equiv c_{n-3} + \int_T^x \frac{L_{n-2}[f(t)]}{p_{n-2}(t)} dt = c_{n-3} + a_1 \int_T^x \frac{L_{n-2}[\phi_1(t)]}{p_{n-2}(t)} dt + a_2 \int_T^x \frac{L_{n-2}[\phi_2(t)]}{p_{n-2}(t)} dt + \int_T^x o\left(\frac{L_{n-2}[\phi_2(t)]}{p_{n-2}(t)}\right) dt = \dots \tag{6.35}$$

as $L_{n-2}[\phi_2(x)]$ is a nonzero constant and $\int^{\rightarrow x_0} 1/p_{n-2}$ diverges

$$\dots = \bar{c}_{n-3} + a_1 L_{n-3}[\phi_1(x)] + a_2 L_{n-3}[\phi_2(x)] + o(L_{n-3}[\phi_2(x)]).$$

Here, the constant \bar{c}_{n-3} is meaningless as the comparison functions are divergent as $x \rightarrow x_0^-$. Iterating the procedure we get all relations in (6.30). By induction on i and the same kind of reasonings our theorem is proved for each $i \leq n$.

Proof of Theorem 5.1. 1) \Rightarrow 2). Relation (5.5) implies the existence of $a_1 \equiv \lim_{x \rightarrow x_0^-} f(x)/\phi_1(x) \equiv \lim_{x \rightarrow x_0^-} M_0[f(x)]$, and each relation in (5.6) implies the relation in (5.7) with the same value of k because of (3.13)-(3.14). 2) \Rightarrow 3) is obvious. 3) \Leftrightarrow 4). It follows from (5.3) that the limit in (5.8) exists in \mathbb{R} iff (5.9) holds true and, in this case, (5.3) can be written as

$$M_{n-1}[f(x)] = a_n - \int_x^{x_0} \frac{L[f(t)]}{q_n(t)} dt, \tag{6.36}$$

where, as above, $L \equiv L_{\phi_1, \dots, \phi_n}$.

4) \Rightarrow 1). We have already proved (6.36) which is (5.6) for $k = n - 1$ together with an integral representation of the remainder. For $k = n - 2$ the recursive formula (3.4) gives

$$(M_{n-2}[f(x)])' = \frac{1}{q_{n-1}(x)} M_{n-1}[f(x)], \tag{6.37}$$

whence, by (6.36) and (2.38), we get

$$M_{n-2}[f(x)] = a_{n-1} - a_n \int_x^{x_0} \frac{1}{q_{n-1}} + \int_x^{x_0} \frac{d\tau}{q_{n-1}(\tau)} \int_\tau^{x_0} \frac{L[f(t)]}{q_n(t)} dt, \tag{6.38}$$

for a suitable constant a_{n-1} . By (3.13)-(3.14) this is nothing but

$$M_{n-2}[f(x)] = a_{n-1} M_{n-2}[\phi_{n-1}(x)] + a_n M_{n-2}[\phi_n(x)] + \int_x^{x_0} \frac{d\tau}{q_{n-1}(\tau)} \int_\tau^{x_0} \frac{L[f(t)]}{q_n(t)} dt, \tag{6.39}$$

which is the relation in (5.6) for $k = n - 2$ with a representation of the remainder. In a similar way for $k = n - 3$ we start from

$$(M_{n-3}[f(x)])' = \frac{1}{q_{n-2}(x)} M_{n-2}[f(x)], \tag{6.40}$$

and integrate (6.38) after dividing by $1/q_{n-2}$, so getting

$$M_{n-3}[f(x)] = a_{n-2} - a_{n-1} \int_x^{x_0} \frac{1}{q_{n-2}} + a_n \int_x^{x_0} \frac{dt_{n-2}}{q_{n-2}} \int_{t_{n-1}}^{x_0} \frac{dt_{n-1}}{q_{n-1}} - \int_x^{x_0} \frac{dt_{n-2}}{q_{n-2}} \int_{t_{n-2}}^{x_0} \frac{dt_{n-1}}{q_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L[f(t)]}{q_n(t)} dt \tag{6.41}$$

for a suitable constant a_{n-2} . By (3.13)-(3.14), this can be rewritten as

$$\begin{aligned}
 M_{n-3}[f(x)] &= a_{n-2}M_{n-3}[\phi_{n-2}(x)] + a_{n-1}M_{n-3}[\phi_{n-1}(x)] \\
 &+ a_nM_{n-3}[\phi_n(x)] - \int_x^{x_0} \frac{1}{q_{n-2}} \int_{t_{n-2}}^{x_0} \frac{1}{q_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L[f(t)]}{q_n(t)} dt,
 \end{aligned}
 \tag{6.42}$$

with a suitable constant a_{n-2} . An iteration of the procedure gives all relations in (5.6) together with the representation formulas (5.11)-(5.12) for $R_0(x)$ and (5.15) for $R_k(x), k \geq 1$. “1) \Rightarrow 5)” has been proved. The last inference “5) \Rightarrow 1)” and (5.12) are trivially proved by applying the operators M_k to (5.11).

Proof of Theorem 5.2. 1) \Rightarrow 2) follows from (3.13)-(3.14). 2) \Rightarrow 3) is obvious. 3) \Leftrightarrow 4): by (3.13)-(3.14) the representation in (5.2) for $k = i - 1$ has the form

$$M_{i-1}[f(x)] = c_i + \int_T^x \frac{1}{q_i} \int_T^{t_i} \frac{1}{q_{i+1}} \dots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt + o(1),
 \tag{6.43}$$

whence our equivalence follows at once. If this is the case (5.2) can be rewritten as

$$M_{i-1}[f(x)] = a_i - \int_x^{x_0} \frac{1}{q_i} \int_T^{t_i} \frac{1}{q_{i+1}} \dots \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt + c_{i+1}M_{i-1}[\phi_{i+1}(x)] + \dots + c_nM_{i-1}[\phi_n(x)],
 \tag{6.44}$$

where a_i is uniquely determined by (5.23) but c_{i+1}, \dots, c_n are non-better specified constants not determinable by the sole condition (5.23).

4) \Rightarrow 1). As in the corresponding inference in Theorem 5.1 we integrate (6.44) starting from

$$(M_{i-2}[f(x)])' = \frac{1}{q_{i-1}(x)} M_{i-1}[f(x)],
 \tag{6.45}$$

whence, by (2.38), (3.4) and (3.14), we get

$$\begin{aligned}
 M_{i-2}[f(x)] &= a_{i-1} - a_i \int_x^{x_0} \frac{1}{q_{i-1}} + \int_x^{x_0} \frac{1}{q_{i-1}} \int_{t_{i-1}}^{x_0} \frac{1}{q_i} \int_T^{t_i} \frac{1}{q_{i+1}} \dots \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt \\
 &+ c_{i+1}M_{i-2}[\phi_{i+1}(x)] + \dots + c_nM_{i-2}[\phi_n(x)] \\
 &= \dots \text{ by (3.8) and (3.14)} \dots = a_{i-1} + a_iM_{i-2}[\phi_i(x)] + o(M_{i-2}[\phi_i(x)]),
 \end{aligned}
 \tag{6.46}$$

where the constant a_{i-1} , which includes all the constants from integration of the various terms, is uniquely determined by (5.22). By iteration of the procedure we get all relations in (5.20)-(5.21). Relation (5.28) easily follows from (6.44) by (3.13)-(3.14).

Proof of Theorem 5.3. This is almost a word-for word repetition of the proofs of Theorems 5.1-5.2. 1) \Rightarrow 2). For $0 \leq k \leq i - 2$ this is included in the same inference in Theorems 5.1-5.2; whereas the relation in (5.30) for $k = i - 1$ just reads $M_{i-1}[f(x)] = O(1)$. “2) \Rightarrow 3)” is obvious. “3) \Leftrightarrow 4)” follows from (6.43). To show “4) \Rightarrow 1)” we use (6.45) and the representation in (5.2) for $k = i - 1$ instead of (6.44) as in the proof of Theorem 5.2. Due to the convergence of $\int_x^{x_0} 1/q_{i-1}$ we may still apply the operator $\int_x^{x_0}$ so getting, instead of (6.46),

$$\begin{aligned}
 M_{i-2}[f(x)] &= a_{i-1} - \int_x^{x_0} \frac{1}{q_{i-1}} \int_T^{t_{i-1}} \frac{1}{q_i} \int_T^{t_i} \frac{1}{q_{i+1}} \dots \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt - \int_x^{x_0} \frac{1}{q_{i-1}} \left[\sum_{j=i+1}^n c_j M_{i-2}[\phi_j(x)] \right] \\
 &= a_{i-1} + O\left(\int_x^{x_0} \frac{1}{q_{i-1}}\right) + o\left(\int_x^{x_0} \frac{1}{q_{i-1}}\right) = a_{i-1} + O\left(\int_x^{x_0} \frac{1}{q_{i-1}}\right) \equiv a_{i-1} + O(M_{i-2}[\phi_i(x)]).
 \end{aligned}
 \tag{6.47}$$

By iteration, we get all relations in (5.28)-(5.29).

Acknowledgements

The author thanks the referees for their valuable suggestions.

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Corrections of Misprints in Previous Papers

In the above reference [1]:

- On p. 255, first line under title of §4: “gaphs” reads “graphs”.
- On p. 260: the reader may notice that $(4.29)_2$ is just a reformulation of (4.27).
- On p. 261, first line from above: delete “t” in locution “t limit position”.
- On p. 266: there is a redundant sign of absolute value “|” inside the integrals in (5.27) and (5.28).
- On p. 284: in the right-hand side of formula (8.26) the quantity $t^{\alpha_2 - \alpha_1 - 1}$ must be replaced by $t^{\alpha_2 - \alpha_1 + 1}$.
- On p. 286: in reference [7] the article pages are missing, namely, 173-218; and in reference [13] the correct article pages are 319-327.

In the above reference [5]:

- On p. 3: in the right-hand side of formula (2.3) the symbol u is missing in the innermost position so that the formula correctly reads:

$$L_n u \equiv r_n \left[r_{n-1} \left(\cdots \left(r_1 (r_0 u)' \right)' \cdots \right)' \right].$$

- On p. 19: in reference [7] the correct article pages are 319-327.